Set Theory

· 七大公理互推:

(1) Axiom of choice. Theorem: $\exists f.\ f: P(X) \longrightarrow X$. → f is called choose func.

st. $\forall S \in P(X)$, $f(S) \in S$ ⟺ $\prod A_i \neq \emptyset$. 计 $A_i \neq \emptyset$ → then we have a element!

(2) $(X, \leq)$ is a poset in which every well-ordered (in X) subset has a lub. then X has a maximal element.

(3) every poset $(X, \leq)$ has a maximal chain (极大原理)

(4) Zorn's Theorem (Lemma):

$(X, \leq)$ is a poset which every chain (in X) has an upper bound, X has maximal element. → 极大元! (并不唯一) $\forall a.\ x < a$. 不成立!

(5) Zernelo's Theorem: every poset has a well-order (well-ordering principle). corollary: exist a set well-ordered is unknown!

(6) surjective $f: X \longrightarrow Y$. $\exists g$. st. $g: Y \to X$

$f \circ g = 1$

(7) 取 A 为 $\bigcup_{\alpha \in A} S_\alpha$ 的指标集. 则 $\exists f$. → Index set: A.

$A \longmapsto \mathcal{S} = \{S_\alpha \mid \alpha \in A\}$. Surjection!

st. $f: A \to \bigcup S_\alpha$. $f(\alpha) \in S_\alpha$

Pf: (7) ⇒ (1) 取 $P(X) = A$. 即以 $P(X)$ 为 $\bigcup_{S \in P(X)} S$ 的指标集

(1) ⇒ (2) 反设 X 无 maximal element

$\forall a.\ X_a = \{x \in X \mid a < x\} \neq \emptyset$

而 $X_a \in P(X)$, $\therefore \exists f.\ f(X_a) \in X_a$.

取 g:

s.t. $g(a) = X_a$

$$\begin{tikzcd} P(X) \arrow[r, "f"] & X \\ X \arrow[u, "g"] \arrow[ur, "f\circ g"'] & \end{tikzcd}$$

$\therefore f \circ g = h$

$h(a) \in X_a$

$\therefore \forall a \in X,\ h(a) > a$

Lemma. Bourbaki's fixed point Theorem:

证: X is a poset which every well-ordered subset has a lub. and $\exists f: X \to X$.
st. $f(x) \geq x$. then $\exists a \in X$. st. $f(a) = a$.
use the lemma $\longrightarrow$ contradiction!

(2) ⇒ (3) consider $\mathcal{X} = \{C \mid C$ is a chain in $(X, \subseteq)\}$.

Prove: every chain has a lub.

$\forall T \in \mathcal{X}$: $\bigcup_{C \in T} C$ is its lub. (can be check)

(3) ⇒ (4) take C is the maximal chain in X.

its upper bound noted a.

Then a is the maximal element in X.

If not. $\exists x \in X.\ x > a$. then $\{x\} \cup C$ is the longer chain!

(4) $\Rightarrow$ (5) pick poset Y. then note:

$X = \{A \mid A = (S_A, \underset{A}{\leq}),\ S_A \subseteq Y$, which is well-ordered in $S_A\}$. → 取出所有良序集的合，设法找到最大为Y的良序集!

Def: $\leq$ on X: $A \leq A' \Leftrightarrow$

$A = A'$ or A is a section of A'.

namely, $\exists\, a \in A'$, st. $S_A = \{x \in S_{A'} \mid x < a\}$. And $\forall x_1, x_2 \in S_A$.

$x_1 \underset{A}{\leq} x_2 \Leftrightarrow x_1 \underset{A'}{\leq} x_2$

Then $\leq$ is a partial order //

Next. we prove: X has a maximal element:

Let C be a chain in X. note $A_0 = (S_{A_0}, \underset{A_0}{\leq})$

$S_{A_0} = \bigcup\limits_{A \in C} S_A$. and $\leq_{A_0}$: $\forall x_1, x_2 \in S_{A_0}$

Since C is a chain, where $A', A'' \in C$.

then $A' \leq A''$ or $A'' \leq A'$. $\therefore \exists\, A$. st.

$x_1, x_2 \in S_A$. we say $x_1 \underset{A_0}{\leq} x_2$ if $x_1 \underset{A}{\leq} x_2$.

But it's independent of the choice of A (arbitrary if $x_1, x_2 \in S_A$)

then "$\underset{A_0}{\leq}$" is a total order. //

Then we claim a fact: "$\underset{A_0}{\leq}$" is well-ordered in S_{A_0}.

Prove: $\forall T \subseteq S_{A_0}$. $T = T \cap S_{A_0} = T \cap (\bigcup_{A \in C} S_A)$

$= \bigcup_{A \in C} (T \cap S_A)$. then $\exists A$. $T \cap S_A \neq \emptyset$.

Since S_A is a section of some S_{A_0}.

which means $S_A = \{x \mid x \in S_{A_0}.\ x < a'\}$

Then T has a least element in $S_A \cap T$

which is also the least element in T.

Then $A_0 \in X$. which can be check

that it's a upper bound of C. // → $\forall A \in C$. is the section of A_0!

By Zorn's Lemma: The maximal element exists!

If it's not $(Y, \leq)$. Assume $(Y_0, \leq)$

But pick y from Y/Y_0. then

$\{y\} \cup Y_0 = Y_0'$ which make Y_0 is a

section. $\therefore Y_0' \geq Y_0$

$\therefore$ The maximal element is $(Y, \leq)$ in X

$\therefore (Y, \leq)$ is defined as a well-ordered set.

Application:

Ex1: Hahn-Banach Theorem:

X is a Vector Space on $K (=\mathbb{R})$. → Now we talk the linear function except "p"!

$p: X \to \mathbb{R}$, s.t. $\begin{cases} p(x_1+x_2) \le p(x_1)+p(x_2) \\ p(tx) = tp(x) \end{cases}$

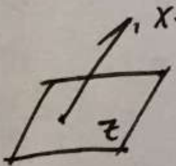
$\forall$ Y (subset of X). $\Lambda_0 : Y \to \mathbb{R}$.

s.t. $\Lambda_0(y) \le p(y)$ $(*)$. $\forall y \in Y$. → It says that we can extend a function's def field!

$\Rightarrow$ Then. $\exists \Lambda$ (Any linear): $X \to \mathbb{R}$. s.t. $\begin{cases} \Lambda|_Y = \Lambda_0 \\ \Lambda(x) \le p(x) \end{cases}$

pf: Lemma. Pick $H_0 : Z \to \mathbb{R}$. H_0 suffice to $(*)$, then $\forall x_0 \in X/Z$.

$\exists$ map $H : Z + \mathbb{R}x_0 \to \mathbb{R}$. s.t. $\begin{cases} H|_Z = H_0 \\ H(z+tx_0) \le p(z+tx_0) \end{cases}$



Pf: Def: $\begin{cases} H(x_0) = a \\ H(x) = H_0(x), x \in Z \end{cases}$ then $H(z \pm tx_0)$

$= H_0(z) \pm ta. \le p(z \pm tx_0)$. $\forall z$

$\therefore \dfrac{P(z_0 - tx_0) - H_0(z)}{-t} \le a \le \dfrac{P(z+t\cancel{x_0}) - H_0(z)}{t}$

Prove: max: $\dfrac{P(\cancel{z_0} - tx_0) - H_0(z)}{-t} \le$ min: $\dfrac{P(z+tx_0) - H_0(z)}{t}$

$\Leftarrow$ $\Rightarrow \dfrac{H_0(z') - P(z' - tx_0)}{t'} \le \dfrac{H_0(z') - P(z') + P(t'x_0)}{t'}$

$\le P(x_0)$.

Pf: To use Zorn's Lemma.

First. develop a poset:

Consider $P = \{W_\theta \xrightarrow{\theta} \mathbb{R}\}$. θ is the function suffices to (*).

Def: $\leq$: $\theta_1 \leq \theta_2 \Leftrightarrow W_{\theta_1} \subseteq W_{\theta_2}$.

$\therefore (P, \leq)$ is a poset //

Next we prove the existence of the maximal element:

$\forall$ C is a chain in $(P, \leq)$

Then $W = \bigcup_{\theta \in C} W_\theta$ is the upper bound in C, which is also the vector space since $\forall \theta', \theta'' \in C$. $W_{\theta'} \leq W_{\theta''}$ or $W_{\theta''} \leq W_{\theta'}$. //

Program:

A poset

$\downarrow$

Use Zorn's Lemma.

$\downarrow$

Check the maximal element.

$\longrightarrow$ Extend the Domain!

Now check it's well-defined?

pick $\pi: W \longrightarrow \mathbb{R}$
$w \longmapsto \theta(w)$. if $w \in W_\theta$

But it's independent of the choice of θ. Since it's in C (仅是定义域扩充!)

Remark: If choose $K = \mathbb{C}$. we can limit X on $\operatorname{Re} X$. then $\operatorname{Re} X \longrightarrow \mathbb{C}$. Extend $\operatorname{Re} X$ to X!

Ex. 2. $(X, \|\cdot\|)$ is a normed space

Def: $\Lambda: X \to \mathbb{R}$ is continous (*)
linear

$\Leftrightarrow \exists c, \forall x \in X, |\Lambda(x)| \leq c\|x\|$ ($\Leftrightarrow$ $x=0$. Λ is continous)

Then we can also extend a subset Y of X which suffices to (*) to X.

Pf: Let $\|\Lambda_0\| = \max \frac{|\Lambda_0 x|}{\|x\|}$. Λ_0 is contunous in Y.

then $\Lambda_0 \xrightarrow{\text{Extend}} \Lambda$, by Hanh Banach The.

Ex. 3. Let A, B be two sets.

Then there is an injection from A to B or from B to A.

Pf: Assume $B \supseteq A$. Consider $X = \{S_f \xrightarrow{f} B \mid f$ is an injection on a subset S_f on $A\}$.

Def: $f \leq f' \Leftrightarrow S_f \subseteq S_{f'}$. $f'|_{S_f} = f$.

Then "$\leq$" is a partial order.

$\forall$ c which is a chain in X.

$S_0 = \bigcup_{f \in c} S_f$ is its upper bound

Def: $S_0 \xrightarrow{\pi} B$, which is well-def by Ex. 1

Then $\exists$ a maximal element in X.

Assume it's g, if $S_g \neq A$:

Since $B \supseteq A$, g isn't surjection.

Pick b from B. Since $A/S_g \neq \emptyset$,

Pick y from A/S_g. Then,

the map: $h = \begin{cases} g(x), & x \in S_g \\ b, & x = y \end{cases}$ is the

maximal element in X! Condiction!

(5) $\Rightarrow$ (6) Def: $g(y)$ = the least element of $f^{-1}(y)$,

(6) $\Rightarrow$ (7) Consider $X = \{(s,\alpha) \in S\times A \mid s \in S_\alpha\}$.

then $X \subseteq S\times A$.

$$\begin{tikzcd} & A \\ X \arrow[ur, "q"] \arrow[dr, "p"'] & \\ & S \end{tikzcd} \qquad (s,\alpha) \mapsto \alpha, \quad (s,\alpha) \mapsto s$$

Note that q is the surjection, then $\exists\ g: A \to X$, st. $qg(a) = a$

then $h = pg(\alpha) = p(s,\alpha) = s \in S_\alpha$

which is the function needed!

· 集合运算法则：

Ref：$A-B=A/(B\cap A)$
$=\underline{A\cap B^c}$

① 分配律：$A\cap(\bigcup_{i\in I}B_i)=\bigcup_{i\in I}(A\cap B_i)$

$$A\cup(\bigcap_{i\in I}B_i)=\bigcap_{i\in I}(A\cup B_i)$$

② Demorgan：$\underline{A}-(B\cup C)=(A-B)\cap(A-C)$ / $(\bigcup_{t\in I}B_t)^c=\bigcap_{t\in I}B_t^c$

（并的补＝补的交，交的补＝补的并）$\underline{A}-(B\cap C)=(A-B)\cup(A-C)$ / $(\bigcap_{t\in I}B_t)^c=\bigcup_{t\in I}B_t^c$

Remark：补域可自行选择、且证明时常利用元素所在集合的性质（即 $\in$ or $\notin$、或 $\in$ and $\notin$！）

· 映射的集合关系（注：$A\subset B$ 在 f^{-1}, f 作用下保号，即 $\nearrow f^{-1}(A)\subset f^{-1}(B)$，$\searrow f(A)\subset f(B)$）

特别的，对 $\subseteq$, A 亦成立！$\underline{A\cup B}$ 可除掉！

(1) $f:A\to B$，且 $A_0\subseteq A$，$B_0\subseteq B$，

则 $A_0\subseteq f^{-1}(f(A_0))$，$B_0\supseteq f(f^{-1}(B_0))$

分别在单射与满射时等号成立、

Pf：$\forall a\in A_0$，$\exists f(a)\in B$，显然

$a\in f^{-1}(f(a))$，但 $f(a)$ 的逆可为一集合！

$\exists b\in B_0$，$f^{-1}(b)$ 可能不存在！

又 $\forall f^{-1}(b)\in f^{-1}(B_0)$，$f(f^{-1}(b))\in B_0$

(2) 补集关系：$f^{-1}(B_0)-f^{-1}(B_1)=f^{-1}(B_0-B_1)$

而 $f(A_0-A_1)\supset f(A_0)-f(A_1)$

Pf：注意 $f^{-1}(B_1)=f^{-1}((B_1-B_0)\cup(B_0\cap B_1))=f^{-1}(B_1-B_0)\cup f^{-1}(B_0\cap B_1)$

$\therefore LHS=f^{-1}(B_0-B_1)-f^{-1}(B_1-B_0)$

又 $f^{-1}(B_0-B_1)\cap f^{-1}(B_1-B_0)=\emptyset$ $\therefore LHS=f^{-1}(B_0-B_1)$

同理，当 $f(A_0-A_1)-f(A_1-A_0)$ 时，

$f(A_0-A_1)\cap f(A_1-A_0)$ 不一定也空集，非满射时！

$\forall \alpha \in f(A_0-A_1)-f(A_1-A_0)$，$\alpha \in f(A_0-A_1)$

$\therefore f(A_0-A_1)-f(A_1-A_0) \subset f(A_0-A_1)$

$\boxed{1.\text{关系}}$：

~~※"关系"为一个集合 $A\times A$，若在A上时，"等价类"也在关系R上的某某子集，给定"~"及x（等价），可产生由x决定的等价类.~~

※ 集合A上的关系为 $A\times A$ 的一个子集

（并非所有元素都存在关系）

⇒ 特殊的关系：等价关系 $\begin{cases}\text{reflective}\\ \text{symmetric}\\ \text{transitive}\end{cases}$

其与划分相同！

一个等价关系产生一个划分：等价类的族，→ 族：集合的集合

而一个划分又可产生一个等价关系：仅需在同一集合中的元素建立一个等价关系：$a\sim b \Leftrightarrow a,b\in A$

⇒ 等价关系拓展：congruence relation：$\begin{matrix} a_1Rb_1 \\ a_2Rb_2\end{matrix} \rightarrow a_1a_2Rb_1b_2$

即此时可把等价类内部元素的operation

拓展至 A/R 上，即等价类 $\bar{a}$ 之间的operation

namely：$\overline{a\circ b}=\bar{a}\circ\bar{b}$

Pf: Assume $\bar{a} = \bar{c}$, $\bar{b} = \bar{d}$, then $a \circ b = c \circ d$

$\therefore \bar{a} \circ \bar{b} = \overline{a \circ b} = \overline{c \circ d} = \bar{c} \circ \bar{d}$, which is

independent of the chooice of representive!

It's well-defined!

$\boxed{\cdot 序}$: Def 1: $\boxed{\text{Partial order}}$: $(X, \leq) \rightarrow$ named after po-set

i) $\forall x \in X$, $x \leq x$ (reflective)

ii) if $x, x' \in X$, $x' \leq x$, $x \leq x'$, then $x' = x$

iii) $a, b, c \in X$, $a \leq b$, $b \leq c$, then, $a \leq c$

Remark: (1) Strict Partial order: i) non-reflective
ii) transitive!

(2) Not every element in X can be compared!

Def. 2: $\boxed{\text{Simple / linear order}}$: $(X, <)$

i) comparable: $\forall a, b \in X$, aRb or bRa.

ii) non-reflective. iii) transitive.

① order type: ~~two~~ order types "$\underset{A}{\leq}$, $\underset{B}{\leq}$" which is simple order is equavilant

$\Leftrightarrow \exists$ order perserving map $f: A \rightarrow B$

$a \underset{A}{<} b \Rightarrow f(a) \underset{B}{<} f(b)$.

Remark: It depend on the existence of the largest element and smallest element, $\nearrow$ only exist in the simple order!

the immediate predeccessor and successor

(uniqueness!)

which decides the "chain":

$W_1 \to$ W_k is the element $(1 \le k \le n)$
⋮
$W_k \mapsto$ which has no im-pre!
⋮
$W_n \to$ Then number of chain is "n"

Denote: $\underbrace{W+W\cdots+W}_{n}$

Then we can define the lub and glb property on the subset A_0 of $(A, <)$, $A_0 \neq \emptyset$.

$\boxed{\text{Theorem}}$ The set $(A, <)$ which has the lub property, has the glb property too!

Pf: $A_0 \subseteq A$. then $X = \{x \mid x \le a, \forall a \in A_0\}$.
which is the set of all lower bound of A_0
Since X has a lub. that is the glb of A_0!

Def 3. Well-order: $(A, \underset{\sim}{<})$ is a well-ordered set. if $\forall A_0 \neq \emptyset \subseteq A$, there's a smallest element in A_0.

Remark: (1) Every finite well ordered set only has one order type: A_{n+1} (section of $\mathbb{Z}^+$)

Pf: By Induction: Firstly, prove it has a largest element "b" Then $A-\{b\} \cup \{b\}$ is easily to make a map perserve order!

$A - \{a\} \to$ largest a'
$\{a, a'\}$'s largest is of A!

(2) Every well-ordered set has lub property. → Let upper bound set be A. A has immed-pred.
Since the smallest element exists! ⇒ Cor. If $a \in X$, a isn't minimal, then $\exists$ immed-pred of a

(3) A set isn't a well-ordered set

$\Leftrightarrow \exists A_0 \subseteq A$, $A_0 \neq \emptyset$, A_0 has the same order type of $\mathbb{Z}^-$

Pf: By Axiom of Them. Use the choose func: $c(x)$

$f(-1) = c(A) = a$. Then, by inductive def:

$f(-n) = c(c\{x \mid x \in A, x < f(-n+1)\}) = a_n$

Since $\forall b \in A$, $\exists x < b$, $x \in A$.

Then $f: \mathbb{Z}^- \longrightarrow A$ Injection!

Corollary: If any subset of A is well-ordered, which is countable, then A is well-ordered!

⇒ 存在一个以Ω为最大元的不可数良序集 A，在Ω处的截 S_Ω 不可数，但其余处截可数。

称 S_Ω 为极小不可数良序集

Corollary: (1) $\forall$ section of S_Ω has a upper bound in S_Ω

Pf: $\bigcup_{a \in S_\Omega} S_a$ is decountable. Since S_a is decountable, $\therefore S_\Omega / \bigcup_{a \in S_\Omega} S_a \neq \emptyset$. Choose a element x from $S_\Omega / \bigcup S_a$, then $x > a$, $\forall a \in \bigcup_{a \in S_\Omega} S_a$

(2) S_Ω has no the largest element. → Use Disproof!

(3) $\forall a \in S_\Omega$, $\{x \mid x > a\}$ isn't decountable.

(4) X_0 is a subset of S_Ω, which's consist of the elements in S_Ω which has no im-pred.

Then X_0 isn't decountable (by (3))

1. 整数与归纳

Def. "A" set is inductive, if $1 \in A$, and $\forall x \in A$, $x+1 \in A$ // let $\mathcal{A}$ to be the family of all inductive sets in $\mathbb{R}$

Then: $\mathbb{Z}^+ = \bigcap_{A \in \mathcal{A}} A$

① Thm 1. $\mathbb{Z}^+$ is a well-ordered set.

Pf: (1) $\forall A_n$, A_n has a smallest element.

By Induction. $n=2$, trivial. $n \le k$. ✓

then: $n = k+1$ $A_{k+1} = A_k \cup \{k\}$.

By hypothesis, A_k has a smallest element which is the smallest of A_{k+1}

(2) $\forall C \subseteq \mathbb{Z}^+$, $C \cap A_n$, $\exists n \ne \emptyset$.

then the smallest element of it is the smallest of C!

② Principle of transfinite induction:

~~J is well-ordered, if $\forall \alpha \in J$, $J_0 \subseteq J$~~

when $S_\alpha \subset J_0 \Rightarrow \alpha \in J_0$, then $J_0 = J$

Pf: If $J_0 \neq J$, then $J/J_0 \neq \phi$. Assume "a" is the smallest element in J/J_0.
then $S_a \subset J_0 \Rightarrow a \in J_0$. contradiction!

In particular, Let $J = \mathbb{Z}^+$.

$\boxed{\cdot \text{ Cartesian product}}$:

投影函数: $\{A_i | i \in I\}$, $\exists$ set D, with
family of map $\{\pi_i : D \to A_i | i \in I\}$.
then: for any set C, with $\{\varphi_i : C \to A_i | i \in I\}$,
$\exists$ unique map $\varphi : C \to D$, such that $\pi_i \varphi = \varphi_i$. (*) Universary mapping Property!

Pf: Let $\underline{D = \prod A_i}^{(*)}$ then Def $\varphi : C \to D$
by: $c \longmapsto (\varphi_1(c), \varphi_2(c) \cdots \varphi_k(c) \cdots) \to$ uniqueness: fix c in C:
$\pi_i = (\varphi_1(c), \cdots) \xmapsto{\text{Onto!}} \varphi_i(c)$ $\quad \pi_i \varphi(c) = \varphi_i(c) = \pi_i \varphi'(c)$
Note that π_i is bijection! $\therefore \varphi = \varphi'$

$\boxed{\text{归纳定义原理}}$: A is a set, 则 $\exists$ 唯一一个
函数 h, $h(1) = a \in A$, $\forall i > 1$.
$h(i)$ 为 A 中唯一一个元素, 且仅与 i 相关.
即: $h(i) = C(A - h(\{1, 2, \cdots i\})) = \ell(h|\{1, 2 \cdots i\}) \downarrow$ 可作为确定的选择函数!

Pf: 从 A_{n+1} 开始 $\to$ Prove: Existence, Uniqueness.
We can def $C(A) = A$ 中最小元素
Then prove: $\mathbb{Z}^+ \times C$ 的 $(i, h(i))$ 唯一被
i 所确定, Since $f_n(i) = f_m(i)$, $n, m > i$.

Cardinal Number

① finite sets: $\exists n$. finite set $A \xrightarrow{onto} A_n$.

Lemma: $a_0 \in A$. $A \xrightarrow{onto} \{1,2,\cdots n+1\} \Leftrightarrow A/\{a_0\} \xrightarrow{onto} \{1,2\cdots n\}$.

Pf: by Induction: 计 $f(a_0)=m \neq n+1$.

then find $a_1 \in A$. $f(a_1)=n+1$. then def $h(x)=\begin{cases} n+1, & x=a_0 \\ m, & x=a_1 \\ f(x), & 其余. \end{cases}$

$\Rightarrow$ **Theorem**. $\exists n$. $A \xrightarrow{onto} A_{n+1}$. $B \subsetneq A$.

then $\exists g: B \xrightarrow{onto} A_{m+1}$. $m<n$.

Pf: take $a_0 \in A-B$. ($a \in B$) By Induction:

$A-\{a\} \longrightarrow \{1,\cdots n+1\}$. and

$B-\{a\} \subsetneq A-\{a\}$. since $a_0 \notin B$. $a_0 \in A$.

Corollary: (1) 不存在A与其真子集的一一对应.

$\Rightarrow \mathbb{Z}^+$非有限集: $f: \mathbb{Z}^+ \xrightarrow[onto]{} \mathbb{Z}^+/\{1\}$: $f(n)=n+1$

→ 无限集A有: 先取f.
$f: \mathbb{Z}^+ \to A$
设 $f(\mathbb{Z}^+)=B$
将可数集与不可数集
分离: $h(x)=\begin{cases} x_{n+1}, & x_n \in B \\ x_n, & x_n \in A-B \end{cases}$
归纳定义 x_n!

② Infinite Sets:

Def. countably infinite set: A.

$A \xrightarrow[onto]{} \mathbb{Z}^+$

Lemma. (1) $C \subsetneq \mathbb{Z}^+$. then $\exists f: \mathbb{Z}^+ \xrightarrow{onto} C$

Pf: 归纳定义 $f(n) = C - f(\{1,2,\cdots,n-1\})$的最小元

since $\mathbb{Z}^+$ is well-ordered!

Then f is obviously injective!

Then prove f is a surjection:

$\forall c \in C$. $h(\mathbb{Z}^+)$ 不被包含在 $\{1,2\cdots,c\}$ 中.

$\therefore \exists n$. s.t. $h(n) > c$. choose the minimal "m"

then $h(m) \geq c$. $\therefore \forall i < m$. $c > h(i)$

$\therefore c$ 不存在于 $h(\{1,2\cdots m-1\})$ 中.

$\therefore c \geq h(m)$ $\therefore c = h(m)$ $\Rightarrow$ corolary: 可数集子集仍可数!

(2) $\mathbb{Z}^+ \times \mathbb{Z}^+$ 可数无限: def: $f(n,m) \overset{\Delta}{=} 2^n \cdot 3^m$

(3) 可数集的可数个并仍可数. $|\bigcup_{n \in \mathbb{Z}^+} A^n| = S|A|$

可数集的有限个 cartesian 积仍可数. $|A^n| = |A|$、对 A infinite countable.

Pf: 1) 设 $J = A_{n+1}$ 或 $\mathbb{Z}^+$ 为 $\{S_n\}_{n \in J}$ 的指标集.

$\forall S_n$. $\exists f_n: \mathbb{Z}^+ \xrightarrow[\text{onto}]{} S_n$, 取 $g: \mathbb{Z}^+ \to J$.

则 $h: \mathbb{Z}^+ \times \mathbb{Z}^+ \longrightarrow \bigcup S_n$ by:

$h: (n,k) \longmapsto f_{g(n)}(k)$, surjection!

2) by Induction. Let $A_1 \times \cdots A_{n-1} \times A_n$

$\longmapsto (A_1 \times A_2 \cdots \times A_{n-1}) \times A_n$.

(4) X^ω, $X = \{0,1\}$ 为无限不可数集

pf: $g: \mathbb{Z}^+ \longrightarrow X^\omega$, by

$g(n) = (x_{n1}, x_{n2} \cdots x_{nn} \cdots x_{nm} \cdots)$

此时作 $\vec{y} = (y_1, y_2 \cdots y_n \cdots)$, $y_k = \begin{cases} 1, x_{kk} = 0 \\ 0, x_{kk} \neq 0 \end{cases}$ (选定 $g(k)$ 后!)

~~then $\vec{y}$ 没被 $\mathbb{Z}^+$ 映射!~~

③ Define <u>the same cardinarity</u>: → 设 f 为 injection then $|B| \geq |A|$.

若 $f: A \xrightarrow{\text{onto}} B$, then $|A|=|B|$

$\boxed{\text{Them 1}}$ 若 $B \subset A$, $\exists f: A \to B$, injection,

then: $|A| = |B|$

Pf: 设 $A_1 = A$, $B_1 = B$, 归纳定义 $f(A_n) = A_{n+1}$

, $f(B_n) = B_{n+1}$, then $A_1 \supset B_1 \supset A_2 \supset B_2 \cdots$

since $f(A_1) = A_2 \subset B = B_1$, $A_1 \supset B_1$. → Injection func. Develop:

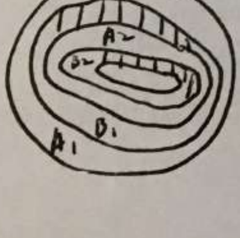
Def: $h(x) = \begin{cases} f(x), & \exists n \text{ s.t. } x \in A_n - B_n \\ x, & \text{其余情况} \end{cases}$

则 $h(x)$ is injective, can be check by 3

case: $\begin{cases} h(x) = x = h(y) = y \\ h(x) = f(x) = h(y) = y \to \text{obey the def of } h! \\ h(x) = f(x) = h(y) = f(y) \end{cases}$

And h is surjective, since:

$h|_{B_n - A_{n+1}}$ is surjective



when $x \in A_n - B_n$:

$A_n - B_n = f(A_{n-1}) - f(B_{n-1}) \subset f(A_{n-1} - B_{n-1})$

∴ $h|_{A_{n-1} - B_{n-1}}$ is surjective ∴ h is surjection!

$\boxed{\text{Them. 2}}$ Schroeder - Bernstein:

若 $\exists$ Injection $f: A \to B$ and $g: B \to A$.

then $|A| = |B|$

Pf: If $a \in A$. then $g^{-1}(a) = \begin{cases} \emptyset \text{ (say a is parentless)} \\ b.\text{ unique} \rightarrow \end{cases}$ → check which element exist f^{-1}, g^{-1}?

$f^{-1}(b) = \begin{cases} \emptyset \\ a', \text{unique} \end{cases}$ ... (parent)

By the way. we trace the ancestor of "a"

which mean it has no parent!

Then. 3 cases will happen:

$A_1 = \{x \mid x \in A, x$ has a parentless ancestor in $A\}$: $g^{-1}(x)$ exist? so as to $\rightarrow B_1$

$A_2 = \{x \mid x \in A, x$ has a parentless ancestor in $B\}$: $g^{-1}(x)$ exist exactly! $\rightarrow B_2$

$A_3 = \{x \mid x \in A, x$ has infinite ancestor$\}$ ✓ $\rightarrow B_3$

There're disjoint obviously, then $h(x) = \begin{cases} f(x), & \text{if } x \in A_2 \cup A_3 \\ g^{-1}(x), & \text{if } x \in A_1 \end{cases}$

Since $A_i \rightarrow B_i$, $i = 2, 3$), $B_1 \rightarrow A_1$ bijection!

④ Some inclusions:

(1) A is a set. then $f: A \rightarrow P(A)$ ☆.

f isn't a surjection (then we have: $|P(A)| > |A|$)

↓ use it to make the cardinality larger: $P(A_n) = A_{n+1}$)

Pf: make $B = \{a \mid a \in A - g(a)\}$ ($\Leftrightarrow a \notin g(a)$)

B is a subset of A. though it's

probably $\emptyset$.

However. there's a "x" $\rightarrow B$. $g(x) = B$

if $x \in B = g(x)$, then $x \notin g(x)$ contradiction!

if $x \notin B = g(x)$. then $x \in g(x)$

(2) $|P(\mathbb{Z}^+)| = |\mathbb{R}| = |X^\omega| = |\mathcal{B}|$

其中 $X = \{0,1\}$. $\mathcal{B}$ 为 X^ω 的所有可数子集的族.

1) $|P(\mathbb{Z}^+)| = |X^\omega|$: def $f: P(\mathbb{Z}^+) \to X^\omega$
$\{i_1, i_2 \cdots\} \mapsto (a_1, a_2 \cdots a_n \cdots)$

$\forall A \in P(\mathbb{Z}^+)$, $f(A) = (a_1, a_2 \cdots a_n \cdots)$

计 $n \in A$, $a_n = 1$, otherwise $a_n = 0$. f is bijection!

Remark: Let $\mathbb{Z}^+$ be finite A. then $|P(A)| = 2^{|A|}$

2) $|\mathbb{R}| = |P(\mathbb{Z}^+)|$,

$f: (0,1) \to \mathbb{R}$. bijection!
$x \mapsto \ln \tan \frac{\pi}{2} x$

Then Prove: $\exists g: P(\mathbb{Z}^+) \to (0,1)$ bijection

: $\forall A \in P(\mathbb{Z}^+)$ $g_1(A) = \sum_{k \in \mathbb{Z}^+} \frac{r_k}{3^k}$. 计 $k \in A$. $r_k = 1$. or $r_k = 0$!

g_1 is injection! $g_1: P(\mathbb{Z}^+) \to (0,1)$

$g_2(a)$. $\forall a \in (0,1)$. $a = \sum_{k=1} \frac{r_k}{2^k}$ $(r_k = 0/1)$

$g_2(a) = g(\sum_{k=1} \frac{r_k}{2^k}) = \{k \mid r_k = 1, k \in \mathbb{Z}^+\}$.

then $g_2: (0,1) \to P(\mathbb{Z}^+)$ is injection.

Then $|(0,1)| = |P(\mathbb{Z}^+)| = |\mathbb{R}|$

3°) $|X^\omega| = |\mathcal{B}|$:

$f: (x_1, x_2 \cdots x_n \cdots) \mapsto \{(x_1, x_2 \cdots x_n \cdots)\}$. Injection
$X^\omega \mapsto \mathcal{B}$

$f_2: \beta \longrightarrow (X^{w})^{w}$ injection!

$\{(X^{w})_1, (X^{w})_2, \cdots (X^{w})_n \cdots\} \longmapsto ((X^{w})_1, (X^{w})_2 \cdots, (X^{w})_n, \cdots)$

$\because (X^{w})^{w} = X^{w \times w}$. $w \times w \longrightarrow w$, bijection.

then $(X^{w})^{w} \cong X^{w}$ $\therefore f_2: \beta \to X^{w}$ injection!

(3) Ref the cardinarity of N is $\aleph_0$.

① $\forall$ infinite set has a denumberable subset

That's $|A| \geq \aleph_0$

Pf: By Axiom of Choice: Assume B is a finite subset of A. choose x_1 from $A-B (\neq \varnothing)$. $c(A-B) = x_1$, then $B_2 = B \cup \{x_1\}$.

inductive def: $c(A-B_n) = x_{n+1}$. $B_{n+1} = B_n \cup \{x_n\}$.

then $\exists g: Z^+ \longrightarrow$ subset of A. $g(n) = x_n$!

② $|A \cup F| = |A|$. A is infinite. $|F| = n$.

namely $\alpha + n = \alpha$.

Pf: Assume $A \cap F = \varnothing$. let $D = \{x_i | i \in Z^+, x_i \in A\} \subseteq A$. $F = \{b_i\}_1^n$

$$f(x) \begin{cases} b_i, & x = x_i, 1 \leq i \leq n \\ x_{i-n}, & x = x_i \quad n+1 \leq i \\ x, & x \in A-D \end{cases}$$ then $f: A \to A \cup F$ is bijection!

③ α, β are cardinarity. $\alpha \geq \beta$. α is infinite.

then $\alpha + \beta = \alpha$

Pf: Since: $\alpha \leq \alpha + \beta \leq \alpha + \alpha$. $\Leftrightarrow$ Prove: $\alpha + \alpha = \alpha$.

$X = \{ f_X \mid f_X : X \times \{0,1\} \to X,\ f_X \text{ is bijection} \}$.

Def "$\le$": $f_{X_1} \le f_{X_2} \iff X_1 = X_2$ or $X_1 \subseteq X_2$, $f_{X_2}|_{X_1} = f_{X_1}$

Then by Domain-Po-way: $\exists$ the maximal element

g_C in $(X, \le)$, if $|C| \ne \alpha$, then $A - C \ne \emptyset$.

choose a denumberable set $B \subset (A - C)$

(Note that $N \times \{0,1\} \to N$ by $f(n,0) = 2n$

$f(n,1) = 2n+1$, then f is bijection)

then $g_C \le h$. ($h: (B \cup C) \xrightarrow{\times\{0,1\}} B \cup C$)

④ $\beta \ne 0$, $\alpha \ge \beta$, α is infinite, then $\alpha\beta = \alpha$

pf: since $\alpha \le \alpha\beta \le \alpha\alpha$, $\iff$ Prove: $\alpha\alpha = \alpha$

Then the same as "③"

Note that $Z^+ \times Z^+ \to Z^+$ is bijection

Suppose f_B: domain: $|B| < |A - B|$ (By: $|(A-B) \cup B| = |A| = |A-B| + |B| > |B| = |B| + |B|$)

then choose C from $A - B$.

suffice that $|C| = |B|$, then

$|C| = |B| = |B \times B| = |B \times C| = |C \times B| = |C \times C|$

$\therefore |(B \cup C) \times (B \cup C)| = |(B \times B) \cup (B \times C) \cup (C \times B) \cup (C \times C)|$

$= (|B| + |B|) + (|C| + |C|) = |B| + |C| = |B \cup C|$

$\therefore h_{B \cup C}$ is larger than f_B !

Appendix:

Other Lemma.

(1) $\boxed{\text{Kuratowski Lemma}}$: $\mathcal{A}$ is the family of set.
if every simple order subset $\mathcal{B}$ of $\mathcal{A}$. st. $\mathcal{B} \subsetneqq \mathcal{A}$ and.
$\bigcup_{B\in\mathcal{B}} B \in \mathcal{A}$. then $\exists$ an element in $\mathcal{A}$. which isn't properly included in every element in $\mathcal{A}$. (which is the max)

Pf: $X = \{x \mid x \in \mathcal{A}$. x is simple order$\}$.
Def "$\leq$" in X by section-way
Then $\exists$ maximal element in $\mathcal{A}$ by Zorn's Lemma

(2) 关于有限特征: $\mathcal{A}$ 为 X 子集的族. X的子集 $B \in \mathcal{A}$.
$\Longleftrightarrow$ B的任一有限子集 $\in \mathcal{A}$.

Turkey Lemma: $\mathcal{A}$ 是一个集族. 若 $\mathcal{A}$ 有有限特征.
则 $\exists$ 一个元素不真包含于 $\mathcal{A}$ 的其它元素中.

Pf: 取 $\mathcal{B}$ 为 $\mathcal{A}$ 中任意的全序子族.
$\forall$ 有限 $C \subset (\bigcup_{B\in\mathcal{B}} B)$, $\forall c \in C$, $\exists B \in \mathcal{B}$
st. $c \in B$. then $C = \bigcup\{x \mid x \in B\cap C, B \in \mathcal{B}\} = \bigcup_{i\in I} M_i$, let $|M_i| = 1$
Since $\{x \mid x \in B \cap C, B \in \mathcal{B}\} \in \mathcal{A}$.
then $C \in \mathcal{A}$. then $(\bigcup_{B\in\mathcal{B}} B) \in \mathcal{A}$. by (1) ✓.

(3) Turkey lemma $\Rightarrow$ 极大原理.

Pf: $\mathcal{A} = \{B \mid B$ is simple order, $B \in P(A)\}$
$\forall$ finite $C \subseteq B$. C is simple order.
$\therefore C \in \mathcal{A}$.

conversely, if every simple subset of $B \in \mathcal{A}$. then B is simple order, too!
by Turkey Lemma!

i) $\left|\bigcup_{n\in N^*} A^n\right| = \aleph_0 |A|$

ii) $|F(A)| = |A|$. 对 A is infinite. and

$F(A)$ is the set of all finite subsets of A

Pf: i) 对 A is infinite $|A^n| = |A|$

$\Rightarrow \left|\bigcup_{n\in N^*} A^n\right| = |N^*| \times |A| = \aleph_0 |A|$

对 A is finite. $\exists f: \bigcup A^n \longrightarrow N^*$. injective

$\therefore |N^*| \le \left|\bigcup_{n\in N^*} A^n\right|$. but $|A^n| < |N|$

$\therefore |N^*| \le |\bigcup A^n| \le |N^* \times N^*| = |N^*|$

$\therefore |\bigcup A^n| = \aleph_0 = \aleph_0 |A|$. since A is finite.

ii) $f: A \longrightarrow F(A)$ injection. and

$a \longmapsto \{a\}$

$g: \bigcup A^n \longrightarrow F(A)$. surjection. $\therefore$

$|A| = |F(A)| \le |A| \aleph_0 = |A|$. by i)

Topo Space and continue function.

· Topo space

(1) $\boxed{\text{Def}}$. Topo on X is the family of subsets of X satisfies: (Denote τ) → strictly: (τ, X)

contains ← i) $\emptyset$ and $X \in \tau$

the property of ii) finite intersection and arbitrary union
open set!
⇕
element in τ!

of (elements in τ) belong to τ!

Some valuable examples: ① the finest topo:
discrete topo → contain all subsets of X

② the coarsest topo:
trivial topo: only contain X and $\emptyset$

③ finite complement topo: τ_f:
$\forall x \in \tau_f$. $X - x$ is finite set

④ countable complement topo: τ_c:
$\forall x \in \tau_c$. $X - x$ is countable set
(By De-Morgan Law → check)

⑤ $\{\tau_a | a \in A\}$ is the topo family.
then $\bigcap_{a \in A} \tau_a$ is the topo!

(2) $\boxed{\text{Basis}}$: the original define: On X,
topo "τ" generated by $\mathcal{B}$:

∵ $\mathcal{B}$ is the family of subsets of X, which is the

Basis satisfies
i) $\forall x \in X$. $\exists B \in \mathcal{B}$. $x \in B$.
ii) $\forall B_1, B_2 \in \mathcal{B}$. if $x \in B_1 \cap B_2$. then $\exists B_3 \in \mathcal{B}$, s.t. $x \in B_3 \subset B_1 \cap B_2$! → if $B_1 \cap B_2 \in \mathcal{B}$
let $B_3 = B_1 \cap B_2$ ✓

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判断U为开集 → 是否存在开集(关于x) $\subset U$!

Date.

No.

the presentation of U is not unique!

Thm. 设 $U \in \tau$. point a "x" $\in U$. then $\exists B \in \mathcal{B}$. st. $\underline{x \in B \subset U}$. then τ is generated by $\mathcal{B}$.

→ corollary: $\tau = \{\bigcup_{B \in \mathcal{B}'} B \mid \mathcal{B}' \subset \mathcal{B}\}$.

Note $U \in \tau$. $\underbrace{U = \bigcup_{x \in U} B_x}$

⇒ Check "τ" is actually a topo: i) $X \in \tau$. $x \in B$, since $B \subset X$. $\therefore x \in B \subset X$ $\therefore X \in \tau$.

ii) $\forall U_\alpha \in \tau$. $x \in \bigcup U_\alpha$ $\therefore x \in U_\alpha$, some α

$\therefore \exists B_\alpha \subset U_\alpha$. st. $x \in B_\alpha$ $\therefore B_\alpha \subset \bigcup U_\alpha$ $\therefore \bigcup U_\alpha \in \tau$!

By Induction, Merely prove: $U_1 \cap U_2 \in \tau$

设 $x \in U_1 \cap U_2$. $\exists B_1, B_2 \in \mathcal{B}$. $x \in B_1$. $x \in B_2$.

$x \in B_1 \cap B_2 \subset U_1 \cap U_2$. since $B_1 \subset U_1$, $B_2 \subset U_2$

then $\exists B_3 \in \mathcal{B}$. st. $x \in B_3 \subset B_1 \cap B_2 \subset U_1 \cap U_2$ ✓.

$\boxed{\text{Extend Criterion of Basis:}}$ $\mathcal{C}$ is the basis of τ. 若: $\underline{\mathcal{C} \subset \tau}$. $\forall U \in \tau$. then $\forall x \in U$. $\exists C \in \mathcal{C}$. st. $x \in C \subset U$

→ Easily check: τ generated by $\mathcal{C} \Leftrightarrow \tau = \{\bigcup_{B \in \mathcal{B}'} B \mid \mathcal{B}' \subset \mathcal{B}\}$.

⇒ subbasis: $\mathcal{S}$ is the family of subsets of X, and $\bigcup_{S \in \mathcal{S}} S = X$

the τ is generated by $\mathcal{S}$. is: $\tau = \{\bigcup_{B \in \mathcal{B}'} B \mid \mathcal{B}' \subset \mathcal{B}\}$.

where $\mathcal{B} = \{\bigcap_{k=1}^{s} S_{ik} \mid s$ is finite. $\forall S_{ik} \in \mathcal{S}\}$

Def: If τ' is finer than τ ($\tau' \supset \tau$). ⟺ $\mathcal{B}' \subset \tau'$. $\mathcal{B} \subset \tau$. $\forall x \in X$ (τ on X). $B \in \mathcal{B}$ st. $x \in B$. $\exists B' \in \mathcal{B}'$. st. $x \in B' \subset B$.

→ Prerequisite: τ', τ can be comparable. namely: $\tau' \subset \tau$ or $\tau' \supset \tau$ holds!

Cor. $\mathcal{A}$ is the Basis of (τ, X). 则 $\mathcal{A}$生成的拓扑为包含$\mathcal{A}$的X的所有拓扑的交!

→ Pf: $\tau_A \subseteq \bigcap_{\tau_A \subset \tau} \tau$. obviously!

Note $\tau_A \in \{\tau \mid \tau_A \subset \tau\}$. $\therefore \tau_A \supseteq \bigcap_{\tau_A \subset \tau} \tau$!

Def: subspace topo: Y is subset of X, then:

$\tau_Y = \{Y \cap U \mid U \in (\tau, X)\}$. is!

the Basis: $\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$.

(3) $\boxed{\text{close, open and limit:}}$

Lemma. Y is the subspace topo of X.

(criterion of open set in subspace topo)

A is closed in $Y \iff A = U \cap Y, U \underset{\text{closed}}{\subseteq} X$.

→ Cor. A is subset of Y, then $\bar{A}$ in X is the closure of A, $\bar{A} \cap Y$ is the closure of A in Y

⇒ Above the closed set: Y is the subspace of X.

(can be replaced by "open"!)
↓
Notice: $U \in \tau_Y \not\Rightarrow U \in (\tau, X)$!

if $Y \underset{\text{closed}(*)}{\subseteq} X$, $U \underset{\text{closed}(*)}{\subseteq} Y$, then $U \underset{\text{closed}(*)}{\subseteq} X$

Def: $\begin{cases} \text{closure of } A \in (X, \tau): \bigcap\limits_{\forall U \in \tau, A \subset U} U \\ \text{interior of } A \in \tau: \bigcup\limits_{\forall U \in \tau, U \subset A} U \end{cases}$

⇒ 判别 closed set A $\begin{cases} ① X - A \text{ is open?} \\ ② A = \bar{A}? \iff \forall U \in \tau, x \in U, \\ \text{then } U \cap A \neq \emptyset. \end{cases}$

Def: limit: x is the limit point in A

$\iff \forall U, x \in U, \underline{U / \{x\}} \cap A \neq \emptyset$

$\boxed{\text{Some conclusions}}$: ① $A \subset B \Rightarrow \bar{A} \subset \bar{B}$

② $\overline{A \cup B} = \bar{A} \cup \bar{B}$, however,

$\overline{\bigcup A_\alpha} \supset \bigcup \bar{A}_\alpha$ (chose $A_\alpha = \{[\frac{1}{\alpha}, 1 - \frac{1}{\alpha}] \mid \alpha \in \mathbb{Z}^+\}$.)

when $\{A_\alpha \mid \alpha \in I\}$ is locally finite.

$\overline{\bigcup A_\alpha} = \bigcup \bar{A}_\alpha$

1)
Pf: Note that $\bigcup A_i \supset A_i, \forall i$.

$\therefore \overline{\bigcup A_i} \supset \overline{A_i}$ $\therefore$ $\overline{\bigcup A_i} \supset \bigcup \overline{A_i}$.

2) 对 $\boxed{\text{Locally finite}}$: if $\forall x$ (point) $\in X$.

$\exists U$. $x \in U$. s.t. $\{\alpha \in I \mid U \cap A_\alpha \neq \emptyset\}$ is a finite set. then $\{A_\alpha \mid \alpha \in I\}$ is locally finite.

We prove: $\bigcup \overline{A_i} \supset \overline{\bigcup A_i}$ $\Leftrightarrow$ $X - \bigcup \overline{A_i} \subset X - \overline{\bigcup A_i}$ → 开拓展!

$\Leftrightarrow$ $\bigcap (X - \overline{A_i}) \subset X - \overline{\bigcup A_i}$, $\forall x \in \bigcap (X - \overline{A_i})$

$\exists U$. $x \in U$. s.t. finite i. $U \cap A_i \neq \emptyset$. Denote index set I'

then for every "i" (which is finite). Since $x \in X - \overline{A_i}$, $\forall i \in I'$

$\exists V_i$. $x \in V_i$. $V_i \cap A_i = \emptyset$. then exist.

$U \cap (\bigcap_{i \in I'} V_i)$. $x \in U \cap (\bigcap_{i \in I'} V_i) \cap A_i = \emptyset$. $\forall i \in I$.

$\therefore x \in X - \overline{\bigcup A_i}$

③ $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$ (By: $A \cap B \subset \overline{A} \cap \overline{B}$. $\overline{A \cap B} \subset \overline{\overline{A} \cap \overline{B}} = \overline{A} \cap \overline{B}$)

④ Kuratowski: operation: $\begin{cases} f: f(A) = \overline{A} \\ g: g(A) = A^c \end{cases}$

then give a set A. operate "f" "g" on X

Then at most produce 14 distinctive sets!

(*) 即需 fg… 交错作用才有效!

Pf: Note that $ff(A) = \overline{A}$, $gg(A) = A$ (*). However:

$gfg = X - \overline{X - A} = A^\circ$ (IntA). 而 $fgfgfgfg = fgfg$ 最多用 f, g 交错作用7次

$\therefore$ 以 f 或 g 为起始点.

$\therefore$ 共 $2 \times 7 = 14$ 种. e.g. $\{\frac{1}{n} \mid n \in \mathbb{Z}^+\} \cup (2,3) \cup (3,4) \cup \{\frac{9}{2}\} \cup [5,6] \cup \{x \mid x \in \mathbb{Q}\}$ $(7 \le x \le 8)$

(*) ($\Leftarrow$) $\Delta = \{x \times x \mid x \in X\}$ is closed in $X \times X$.

Pf: "$\Rightarrow$" Prove: Δ^c is open.

$\forall x, y \in \Delta^c$, $x \neq y$, then

~~$\exists\, U_y \cap U_x = \emptyset$, $x \in U_x$, $y \in U_y$~~

then $U_x \times U_y \subset \Delta^c$.

if $U_x \times U_y \cap \Delta \neq \emptyset$,

then $a = U_x \cap U_y \cap \Delta$

$\therefore a \in U_x, U_y$.

contradiction!

"$\Leftarrow$" is the same.

(4) $\boxed{\text{Some spaces}}$:

① Seperated Axioms:

T_0: (τ, X). $\forall x, y \in X$, $\exists\, U \underset{open}{\subseteq} X$.

s.t. $x \in U$, $y \notin U$ or $y \in U$, $x \notin U$. (Named Kolmogorov Space)

T_1: (τ, X). $\forall x, y \in X$, $\exists\, U, V \underset{open}{\subseteq} X$.

s.t. $x \in U$, $y \notin U$ and $x \notin V$, $y \in V$. $\Leftrightarrow$ Any finite point set is closed

T_2: (τ, X). $\forall x, y \in X$, $\exists\, U, V \underset{open}{\subseteq} X$.

s.t. $x \in \underbrace{U \cap V}_{=\emptyset} \ni y$. Named Hausdroff Space. (*)

T_3: T_1 holds. $\forall x \in X$, $C \underset{closed}{\subseteq} X$, s.t. $x \notin C$

then. $\exists\, U, V \underset{open}{\subseteq} X$. s.t. $x \in \underbrace{U \cap V}_{=\emptyset} \supset C$.

Named regular Space.

T_4: T_1 holds. $\forall\, C_1, C_2 \underset{closed}{\subseteq} X$. $C_1 \cap C_2 = \emptyset$. then

$\exists\, U, V \underset{open}{\subseteq} X$. s.t. $C_1 \subset \underbrace{U \cap V}_{=\emptyset} \supset C_2$. Named normal space.

② Countable Axioms:

1^{st} countable: $\forall\, x \in X$. $\exists$ local basis of x

is countable. ($\dot{x}$. local basis: cof x)

$W = \{U \mid x \in U,\ U \underset{open}{\subseteq} X\}$. $\forall V$, $x \in V$.

then $\exists\, U' \in W$. s.t. $x \in U' \subset V$)

2^{nd} countable: 设 topo space X has countable Basis 在 X. ($2^{nd} \Rightarrow 1^{st}$ holds!)

(Remark: ⓐ Note that $\mathcal{B}$ is the family of basis of X. then $\{B \in \mathcal{B} \mid x \in B\}$ is the countable local basis of "x")

Seperable topo: X has a countable dense subset

($\dot{X}$. dense: $A \subseteq X \Leftrightarrow \bar{A} = X \Leftrightarrow \forall U \subseteq X, U \cap A \neq \emptyset$)

ⓑ $\longrightarrow$ $2^{nd} \Rightarrow S$ holds: Pf: Assume $\mathcal{B}$ is the basis. then $\{x_B \in B \mid B \in \mathcal{B}\}$ is the dense 在 X. By 定义 check!

Lindelöf: If $U_\alpha \underset{open}{\subseteq} X$, $\alpha \in A$, $\bigcup U_\alpha \supset X$. $\exists$ countable set $A_0 \subset A$, s.t. $\bigcup_{\alpha \in A_0} U_\alpha \supset X$.

ⓒ $\longrightarrow$ $2^{nd} \Rightarrow L$ holds: Pf: If $X = \bigcup_{\alpha \in A} U_\alpha$. Note that $U_\alpha = \bigcup B_j$ $\therefore X = \bigcup_{\alpha \in A} \bigcup B_j$, $B_j \in \{B_j\}$ can be countable. $\therefore \bigcup_{\alpha \in A} \bigcup_{j} B_j = \bigcup_{j \in J} B_j$, then 对每 B_j, 找一个 $U_j \supset B_j$ $\therefore X = \bigcup_{j \in J} U_j$, 可数!

Now, we have:

$$2^{nd} \overset{ⓐ}{\Longrightarrow} 1^{st}$$

(If X is metricable!) $2^{nd} \underset{ⓑ}{\Longleftrightarrow}$ sperable topo ($\{O_r(x) \mid x, r \in \mathbb{Q}\}$ is the basis!)

$2^{nd} \underset{ⓒ}{\Longleftrightarrow}$ Lindelöf

(5) $\boxed{\text{Continue map:}}$

$\dot{X}$. continue: $V \underset{open}{\subseteq} Y$, $f: X \longrightarrow Y$. then $f^{-1}(V) \underset{open}{\subseteq} X$.

$\dot{X}$. Homeomorphism (同胚): $f: X \xrightarrow{onto} Y$. f, f^{-1} is continuous. $\Leftrightarrow \left(U \underset{open}{\subseteq} X \Leftrightarrow f(U) \underset{open}{\subseteq} Y\right)$ $\longrightarrow$ 诱导出的拓扑性质完全相同!

$\dot{X}$. imbedding map: 设 $f: X \xrightarrow{injection} Y$, restrict: $f: X \xrightarrow{onto} Z$ then f, f^{-1} is continuous!

(Remark: τ、τ' is the topo of X、X' respectively. then:

$f: X \to X'$. ① f conti $\Leftrightarrow$ X is finer than X'

② f home- $\Leftrightarrow$ $\tau = \tau'$ (Consider the Basis!))

let coarser or finer?

$\Rightarrow$ construct map (continue):

i) Recombination: f, g conti. $\Rightarrow$ fg conti

ii) 内射: A is the subspace of X. then $f: A \to X$. conti.

(Note that $U \underset{open}{\subseteq} X$. $f^{-1}(U) = U \cap A$)

universal property of subspace topology!

iii) Restrict the domain of f. by i). ii)

iv) Restrict or extend the co-domain of "$f: X \to Y$"

↓ 前提) (ob!)

($f': X \to Z$. s.t. $Z \supset f(X)$). then $f'^{-1}(V) = f^{-1}(V \cap Z) = f^{-1}(V) \cap f^{-1}(Z)$

$= f^{-1}(V)$ is open!)

若 $X = \bigcup U_\alpha$. $\alpha \in A$. $\forall U_\alpha$. $f|_{U_\alpha}$ is continue.

then $f: X \to Y$. is continuous!

(Pf: $f^{-1}(V) = \bigcup (f|_{U_\alpha})^{-1}(V) = \bigcup (f^{-1}(V) \cap U_\alpha)$)

連續性的局部表示!

A "tough" Qusetion:

If. $A \subseteq X$. $f: A \to Y$. (Y is the Hausdroff space.) can extend to another continuous map: $g: \bar{A} \to Y \Rightarrow g$ is determined by f uniquely!

Pf: If $\exists g, h$ satisfy the condition:

Note $g|_A = h|_A = f$. then if $g(x) \neq h(x)$. $x \in \bar{A}/A$. Pick a $x \in \bar{A}/A$. s.t. $g(x) \neq h(x)$.

then $\exists U, V$. s.t. $g(x) \subset U \cap V \supset h(x) = \emptyset$.

Since Y is the Hausdroff space.

$\therefore x \in g^{-1}(U)$. $h^{-1}(V)$. $\exists R_1, R_2$. $x \in R_1 \subset g^{-1}(U)$. $x \in R_2 \subset h^{-1}(V)$

Assume $R_1 \cap R_2 = R \supset \{x\}$. Since $R_1, R_2 \underset{\text{open}}{\subseteq} \bar{A}$

then $R \cap A = \emptyset$. Otherwise. $\exists y \in R \cap A$.

then $f(y) = h(y) \neq g(y) = f(y)$. contradict!

$\therefore x \in R \cap A = \emptyset$. contradict with $x \in \bar{A}$!

(6) $\boxed{\text{Kinds of Topo:}}$

universal property:

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[dr] & X \arrow[d, "f_\alpha"] \\ & Y_\alpha \end{tikzcd}$$

g is conti $\Leftrightarrow f_\alpha \circ g$ conti!

① Initial Topo: (which means X' is existent. (finer!) if $X' \xrightarrow{f_\alpha} Y_\alpha$ ✓. namely $X' \xrightarrow[g]{} X$. g conti)

Set $X \xrightarrow{f_\alpha} Y_\alpha$ (Topo space), $\alpha \in A$

Find the coarsest topo on X. s.t. f_α is continuous.

$\Rightarrow \langle S_{\{f_\alpha | \alpha \in A\}} = \{ f_\alpha^{-1}(U) \mid U \underset{\text{open}}{\subseteq} Y_\alpha \} \rangle$

e.g. ① order Topo: X is the linear order set.

the Basis = $\cup (a,b)$, $[a_0, b)$, $(b, a_1]$, if a_0, a_1 is the minimal or maximal!

(Or $(-\infty, b)$, $(b, +\infty)$ is the open set!)

② Product Topo:

Notice: π_β is continue!

i) Box Topo: $\prod X_\alpha$ generated by $\{\prod U_\alpha \mid U_\alpha \underset{open}{\subseteq} X_\alpha\}$, $\alpha \in A$

↓ finer!

ii) Product Topo: $\prod X_\alpha$ generated by $\{\pi_\alpha^{-1}(U_\alpha) \mid U_\alpha \underset{open}{\subseteq} X_\alpha\}$, $\alpha \in A$.

$\Rightarrow$ when $|A|$ is finite $\Rightarrow$ Box topo $\Leftrightarrow$ Product topo

When $|A|$ is infinite. Note that the Basis of Product topo is $\prod U_\alpha$, only finite α s.t. $U_\alpha \neq X_\alpha$

Then: the same properties: $\prod \overline{A_\alpha} = \overline{\prod A_\alpha}$

X_α is Hausdorff $\Rightarrow$ $\prod X_\alpha$ is Hausdorff

A_α is subspace $\Rightarrow$ $\prod A_\alpha$ is!

△ Map of the box/product topo:

$f: A \to \prod_{\alpha \in J} X_\alpha$, def by: $f(a) = (f_\alpha(a))_{\alpha \in J}$

If $|J|$ is finite, then f conti $\Leftrightarrow$ $\forall f_\alpha$ conti -- (a)

If $|J|$ is infinite, condition (a) only holds when product topo!

~~If $|J|$ is~~

$f: A \to B$, $g: C \to D$ are continue. ∵ By

$$\begin{array}{ccc} \prod X_i & \xrightarrow{\prod f_i} & \prod Y_i \\ \downarrow \pi_i & & \downarrow \pi_i \\ X_i & \xrightarrow{f_i} & Y_i \end{array}$$

then: $f \times g: A \times C \to B \times D$ is continue

② Final Topo:

(topo) $Y_\alpha \xrightarrow{f_\alpha} X$, $\alpha \in A$, then.

$\tau_{\{f_\alpha | \alpha \in A\}} = \{ U \subseteq X \mid f_\alpha^{-1}(U) \underset{\text{open}}{\subseteq} Y_\alpha \}$ is the finest topo on X. satisfies that $\forall f_\alpha$ is continue.

→ 不必在意 $Y_\alpha \cap f^{-1}(U)$ 的开集 找出, 甚至取 $\emptyset$, X 都可!

e.g. Quotient Topo:

※ R-saturated: X is a set. R is the relation on X. def by: $x', x \in X$. $x' R x \Leftrightarrow f(x') = f(x)$

$\Rightarrow$ If $A \subseteq X$ is R-saturated, then $\exists a \in A$. $\forall x$ s.t. $x R a$, then $x \in A$. namely, A is the union of some R-equivalances!

C is R-sat. then $f(f^{-1}(C)) = C$

※ quotient map: $X \xrightarrow{f} Y$. then. f is surjective and. $f^{-1}(U)$ is open $\Leftrightarrow$ U is open

(e.g. 连续且满的开/闭映射. 为商映射, 但存在其它商映射不满足开/闭映射. 因为此时只需在映射饱和集时为开/闭的映射且连续即可!)

一、<u>Quotient Topo</u>: $X \xrightarrow{p} Y$. <u>p is quotient map!</u>

then $\tau = \{V \subseteq Y \mid p^{-1}(V) \subseteq_{open} X\}$ on X

Remark: the product of quotient map isn't always quotient map! if X is hausdroff, then the quotient topo Y isn't always hausdroff $\longrightarrow$ e.g. Y 是将集合K黏在一起的R_K的商空间, $p: R_K \longrightarrow Y$ 为商映射.

则 i) Y 满足T_1公理, 但不为Hausdroff

ii) $p \times p: R_K \times R_K \longrightarrow Y \times Y$ 不为商映射.

Pf: i) 设B为Y中有限点集. 则$p^{-1}(B)$或为有限点集或同时并上K集合, 此时两者在R_K中都为闭集. $\therefore p^{-1}(B)$ close $\Rightarrow B$ closed!

Then. Pick $\bar{0}$ and $\bar{K}$, $\bar{0} \neq \bar{K}$ (0 & K集合)

if $\exists U, V$, $\bar{0} \in U$, $\bar{K} \in V$. $p^{-1}(U) \cap p^{-1}(V) = p^{-1}(U \cap V) = \emptyset$.

但 0 的任何邻域必有K集合. $\therefore Y$ 不是Hausdroff

ii) 注意 对角线集^{c}在Y中并非闭集 (Y isn't Hausdroff)

<u>但</u> $(p \times p)^{-1}(c) = c \cup (K \times K)$ 为闭集! (R_K为Hausdroff空间)

Also, the quotient map restrict on subspace isn't alway quotient map.

Criterion: Lemma:

$p: X \longrightarrow Y$. quotient map. A is the subspace of X, which is R-saturated about p. $q: A \longrightarrow p(A)$ is the restrict on p. (q is still continue!)

Then $\begin{cases} q^{-1}(V) = p^{-1}(V), \text{ if } V \subset p(A). \longrightarrow \text{obviously since } A \text{ saturated!} \\ p(U\cap A) = p(U)\cap p(A), \text{ if } U \subset X. \end{cases}$

Pf: Note that $p(U\cap A) \subset p(U)\cap p(A)$

And $\forall y \in p(U)\cap p(A)$, $y = p(x) = p(a)$

Note that $p^{-1}(p(a)) \subset A$. since A is saturated.

$\therefore x \in A$. $\therefore x \in U\cap A$. $\therefore y \in p(U\cap A)$

$\Rightarrow$ Then if $\begin{cases} A \underset{\text{open/closed}}{\subseteq} X \\ p \text{ is open/closed map} \end{cases} \Rightarrow q$ is quotient map!

Pf: If A is open $\underset{\text{open}}{\subseteq} X$. Prove: $q^{-1}(U) \underset{\text{open}}{\subseteq} A$. then $U \underset{\text{open}}{\subseteq} p(A)$

Note that $q^{-1}(U) = p^{-1}(U)$ $\therefore U$ is open in Y. so $p(A)$!

If p is open map. $q^{-1}(U) \underset{\text{open}}{\subseteq} A$. $\therefore q^{-1}(U) = A \cap V$

$V \underset{\text{open}}{\subseteq} X$. $\therefore p(q^{-1}(U)) = p(p^{-1}(U)) = U = p(A\cap V) = p(A)\cap p(V)$

$\therefore U \underset{\text{open}}{\subseteq} p(A)$ the circumstance of close is same!

X. The universal property:

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "g"] & \\ Y \arrow[r, "f"'] & Z \end{tikzcd}$$

p is quotient map.
g is a map st. $g(x)=g(x') \Leftrightarrow p(x)=p(x')$
then $\exists! f$. if g conti $\Rightarrow$ f conti
If g is quotient $\Rightarrow$ f quotient

Pf: continue is easy. By technique of converse.
$fp=g \Rightarrow g^{-1}=p^{-1}f^{-1}$! The same as quo-!
Note that compound of quotient map is quo-!

Cor. $g: X \to Z$. surjection. $X^* = \{g^{-1}(\{z\}) \mid z \in Z\}$.
Take quotient topo on X^*, then the f induced by g is the continue bijection
and f is home- $\Leftrightarrow$ g is quotient.

Pf:
$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "g"] & \\ X^* \arrow[r, "f"'] & Z \end{tikzcd}$$

f is bijection. Since X^* bond the point mapping to $\{z\}$ on g
And it's continuous since p.g quo-
(and quo-)

(7) $\boxed{\text{Metrizable Topo}}$: Def metric on $\underline{X}$.
should satisfy:
$$\begin{cases} d(x,y) \geq 0. \ x=y \Leftrightarrow d(x,y)=0 \\ d(x,y)=d(y,x) \\ d(x,y) \leq d(x,z)+d(z,y) \end{cases}$$

the Basis of Metric topo : $B_d(x,\varepsilon) = \{y \mid d(x,y) < \varepsilon\}$

定义了不同度量的topo，则d不同！

Some examples : $\begin{cases} \mathbb{R} = \text{standard topo. generated by "}(a,b)\text{"} \\ \mathbb{R}_l = \text{Lower limit topo : by "}[a,b)\text{"} \\ \mathbb{R}_K = K\text{-topo. by "}(a,b)-K\text{"} \end{cases}$

$\Rightarrow$ i) Standard Metric : $\mathbb{R}$上 $d(x,y) = |x-y|$ $\Leftrightarrow$ Basis Standard bounded metric : $\bar{d}(x,y) = \min\{d(x,y), 1\}$

ii) Euclidean Metric : $\mathbb{R}^n$上 $d(x,y) = \|x-y\| = \left(\sum_1^n (x_k-y_k)^2\right)^{\frac{1}{2}}$ $\Leftrightarrow d_1 = \|x-y\|_1 = \left(\sum_1^n (x_k-y_k)\right)^{\frac{1}{1}}$

iii) Square Metric : $\mathbb{R}^n$上 $\rho(x,y) = \max_{1\le i\le n}\{|x_i-y_i|\}$. In-duces a topo on $\mathbb{R}^n$. then equal to (*) the product topo!

To prove the metric topo is equavilant def by d_1, d_2 $\Leftrightarrow$ Prove : $c_1 d_1(x,y) \le d_2(x,y) \le c_2 d_1(x,y)$

iv) uniform metric : On $\mathbb{R}^J$. J为指标集

$\bar{\rho}(x,y) = \sup\{\bar{d}(x_\alpha, y_\alpha) \mid \alpha \in J\}$

Remark : (1) $U(x,\varepsilon)$ isn't always equavilant to $B_d(x,\varepsilon)$ $U(x,\varepsilon) = (x_1-\varepsilon_1, x_1+\varepsilon_1) \times (x_2-\varepsilon_2, x_2+\varepsilon_2) \cdots \times (x_n-\varepsilon_n, x_n+\varepsilon_n) \cdots$

e.g. Let $d = \bar{\rho}$, $U(x,\varepsilon) = (x_1-\varepsilon, x_1+\varepsilon) \times (x_2-\varepsilon, x_2+\varepsilon) \cdots$

Pick $\vec{z} = (x_0+\varepsilon-\frac{1}{n_0+1}, x_0+\varepsilon-\frac{1}{n_0+2} \cdots, x_n+\varepsilon-\frac{1}{n_0+n}, \cdots)$. $\frac{1}{n_0} < \varepsilon$

then $\vec{z} \in U(x,\varepsilon)$. But $\bar{\rho}(\vec{z},\vec{x}) = \varepsilon$ $\therefore \vec{z} \notin B_{\bar{\rho}}(x,\varepsilon)$

Actually. $B_{\bar{\rho}}(x,\varepsilon) = \bigcup_{\delta<\varepsilon} U(x,\delta)$, $U(x,\varepsilon) = \overline{B_{\bar{\rho}}(x,\varepsilon)}$

(2) $\mathbb{R}^J$ can be metrizable. when J is countable and take product topo. which Basis is equavilant to the topo induced by $D(x,y)$

$= \sup\left\{\frac{\bar{d}(x_i,y_i)}{i}\right\}$

Continue func : $f: X \to Y$. f is conti

on metrizable space. $\Leftrightarrow$ $\forall \varepsilon, \exists \delta$. $d_Y(f(x_1), f(x_2)) < \varepsilon$

st. $d_X(x_1, x_2) < \delta$.

($\Rightarrow$) X 中 $\{x_n\} \to x$ then $f(x_n) \to f(x)$. by prove: $f(\bar{A}) \subset \overline{f(A)}$

Metrizable ? $\Rightarrow$ sequence lemma: X can be metrizable, $A \subseteq X$

then $\forall x \in \bar{A}$, $\exists \{x_n\} \in A$, st. $\{x_n\} \to x$.

Pf: the condition can be weaken. Since

A metrizable space suffice to 1^{st} countable Axiom.

choose x_n from $B_n = \bigcap_1^n U_n$, or $B_d(x, \frac{1}{n})$ ✓.

Now, we know: metrizable $\to$ sequence lemma $\Updownarrow$ 1^{st} countable

We can prove: $\mathbb{R}^\omega$. box topo can't be metrizable. ($\Rightarrow$) it can't satisfy to sequence lemma

Choose $\vec{0}$. then $\vec{0} \in \bar{A}$.

choose A is $\{\vec{x}\}$, $x_i > 0$

计 $\vec{a_n} = (x_{n1}, x_{n2} \cdots x_{nk} \cdots)$ $\to \vec{0}$, then, Pick:

$B = (-x_{11}, x_{11}) \times (-x_{22}, x_{22}) \cdots \times (-x_{nn}, x_{nn}) \times \cdots$

which contains no one a_n. Since $x_{nn} \notin B$!

Also, 计 w is uncountable, then product topo can't be metrizable. Since $d: X^w \to Y$, is the initial topo. (box is!)

$\Rightarrow$ 取 A 为仅有限个 x 的下标 α 满足 $x_\alpha = 0$, 其余为1的集合, 再取 $\vec{0} \in \bar{A}$, 记 $\vec{x_n}$ 的下标集为 J_n. $\bigcup^\infty J_n$ 为可数个可数集的并, 而 J 不可数, $\therefore$ $\exists \beta \in J$, $\beta \notin \bigcup J_n$, 让其 $x_\beta = 0$ ✓

~~then~~ $\pi_\beta^{-1}((-1, 1))$ 无 x_n!

Topological group.

Def: A group G is a topological group if it satisfies:

i) G satisfies T_1 Axiom.

ii) $f: G\times G \to G$, $(x, y) \mapsto x\cdot y$, $g: G \to G$, $x \mapsto x^{-1}$. f, g are continue.

Remark: ii) $\Leftrightarrow \varphi: G\times G \to G$, $(x,y) \mapsto xy^{-1}$, φ is continue.

Pf: $(\Rightarrow)$. $id \times g$ is continue. then

$f\circ(id\times g) = \varphi$ is continue.

$(\Leftarrow)$ $\varphi|_{\langle e\rangle \times G}$ is continue.

$\therefore$ Note that $\pi_2|_{\langle e\rangle\times G}$ is homeo-

$\therefore$ $\varphi|_{\langle e\rangle\times G}\circ \pi_2^{-1}|_{\langle e\rangle\times G} = g$ is continue.

And $\varphi\circ(id\times g) = f$ is continue.

(1) Def: $f_\alpha: G \to G$, $x \mapsto \alpha\cdot x$, $g_\alpha: G\to G$, $x \mapsto x\cdot \alpha$ then

f_α, g_α is homoe- of G, and G is transititive.

Pf: f_α, g_α are bijection. clearly. and $\forall x\in G$

for some y. then $\alpha = yx^{-1}$ s.t. $f_\alpha(x) = y$.

(2) Subgroup: H is subspace of G. 若 $H < G$.

then $H, \bar{H}$ are both topo-group.

pf: $H, \bar{H}$ are both in T_1, since of G.

$i: H \to G$ continue. $\therefore i \times i$ conti.

$\therefore \varphi \circ (i \times i): H \times H \to H$. continue.

then H is topo-group.

Note that φ is continue. $\therefore \varphi(\bar{H} \times \bar{H})$
$= \varphi(\overline{H \times H}) \subset \overline{\varphi(H \times H)} = \bar{H}$. $\therefore \bar{H} < G$.

$\Rightarrow$ Coset: $H < G$. then $x \in G$. $xH = \{x \cdot h \mid h \in H\}$.

i) Extend f_a, g_a to G/H, which keeps the property. (check if well-define!)

ii) If $H \underset{\text{close}}{\subseteq} G \Rightarrow$ in G/H, single point is closed.

iii) $G \xrightarrow{p} G/H$. p is open

pf: ii) $p^{-1}(\{xH\}) = xH = f_x(H) \cup H$ $\therefore p^{-1}(\{xH\})$ is closed.

$\therefore \{xH\}$ is close in G/H

iii) $U \underset{\text{open}}{\subseteq} G$. $p(U)$ open $\Leftrightarrow p^{-1}(p(U))$ open.

$p^{-1}(p(U)) = \bigcup_{a \in H} g_a(U)$ open $\therefore p(U)$ open

Remark: ii) 若 H also is normal subgrp of G, then G/H is topo-group. Since it satisfies T_1.

and $G/H \times G/H \xrightarrow{\varphi'} G/H$

$(g_1H, g_2H) \longmapsto g_1Hg_2^{-1}H = g_1g_2^{-1}H$ $\quad \varphi' = p\circ\varphi\circ(p\times p)^{-1}$

$(p\times p)^{-1}$ conti., since p is open map!

Extend: i) If $A, B \subseteq G$. A is close. B is cpt. then $A\cdot B$ is closed in G.

ii) $H < G$. H is cpt. then $p: G \to G/H$ close.

iii) $H \underset{\text{cpt}}{<} G$. G/H is cpt. then G is cpt.

pf: i) $\forall x \in G - AB \Rightarrow x \cap AB = \emptyset$. $\therefore xB^{-1} \cap A = \emptyset$.

xB^{-1} is close. By Normality. $\forall a \in A$. $a \notin xB^{-1}$.

$\therefore \exists U_a, V_a$, $xB^{-1}V_a \cap aU_a = \emptyset$. Note that.

$\exists$ finite U_a cover A $\therefore \exists V = \cap V_a$

---

i) $\forall x \in G - AB \Rightarrow x \cap AB = \emptyset \Rightarrow x \cap (\cup \underset{\text{close}}{Ab}) = \emptyset$.

$\forall b \in B$. $\exists U_b, V_b$. $x \in U_b \cap V_b \ni Ab$, by regularity!

And $A^{-1}U_b \ni b$. $\therefore \exists$ finite $A^{-1}V_b$ cover B

Let $U = \bigcap_1^n U_b$, $x \in U$. $\xrightarrow{}$ open! $U \cap AB = U \cap A A^{-1} \bigcup_1^n V_b = \emptyset$.

ii) $U \underset{\text{close}}{\subseteq} \Rightarrow p(U) = U\cdot H$ close by i)

iii) By the perfect map!

(3) Def: $A\cdot B = \{a\cdot b \mid a\in A, b\in B\}$, $A^{-1} = \{a^{-1} \mid a\in A\}$, $A, B \subseteq G$.

A neighbour V of e is symmetric. if $V = V^{-1}$

G is a topological group.

then i) $\exists$ a symmetric neighbour of e: V.
s.t. $V\cdot V \subset U$. for U a neighbour of e.

ii) G satisfies T_2, moreover. T_3
and if $H \underset{\text{close}}{<} G$. then G/H keep T_3

Pf: i) $G\times G \to G$, $(x,y) \xrightarrow{f} xy$ open. $\therefore U \underset{\text{open}}{\subseteq} G$. $e\in U$

$f^{-1}(U)$ open. $\exists$ Basis: $U_1\times U_2 \subset f^{-1}(U)$,
Let $(e,e)\mapsto e$. then $\exists W_1$, $e\in W_1$
$W_1\times W_1 \to W_1 W_1 \subset U$. $G\times G \xrightarrow{g} G$, $(x,y)\mapsto xy^{-1}$
$\therefore W_2\times W_2 \to W_2 W_2^{-1} \subset W_1$ $\therefore V = W_2\cdot W_2^{-1} \subset U$

ii) T_2: Prove: $\exists U_1, U_2$, $x\in U_1 \cap U_2 \ni y = \emptyset$
$\Leftrightarrow$ $xU \cap yU = \emptyset$. $\exists U$ neighbour of e $\longrightarrow$ which keep the open set and $x\in \underline{xU}$! A neighbour
$\Leftrightarrow$ $x^{-1}y \cap UU^{-1} = \emptyset$. Since $[x^{-1}y]$ is close.
$\therefore \exists$ neighbour V of e. $V\cap x^{-1}y = \emptyset$.
$\exists \underset{\text{sym. open}}{UU^{-1}} \subset V$. $\therefore UU^{-1} \cap x^{-1}y = \emptyset$

T_3: $\Leftrightarrow$ Prove $\exists U$: $xU \cap AU = \emptyset$.
$A^{-1}x$ is close. $\Rightarrow \exists$ neighbour U of e.
$A^{-1}x \cap U = \emptyset \Rightarrow A^{-1}x \cap V^{-1}V = \emptyset$.

Inherit: $\Leftrightarrow$ $VxH \cap VAH = \emptyset \Leftrightarrow V^{-1}V \cap \overset{\subset U \text{ of } e}{\underline{p(A)p^{-1}(x)}} = p(Ax^{-1})$
$A' = Ax^{-1}$. A' is close in G/H and not contain H
$\therefore p^{-1}(A)$ close. and $p^{-1}(A) \cap H = \emptyset$
Since $H < G$. $e\in H$. $\therefore p^{-1}(A) \cap VV^{-1} = \emptyset$
$\therefore$ ~~$p(p^{-1}(A)V) \cap p(V) = \emptyset$~~

Compactness and Connectness

① Connecteness

(1) Def: Seperation of X: $X = U \cup V$, $U, V \neq \emptyset$. $U, V \underset{open}{\subseteq} X$. $U \cap V = \emptyset$

If the seperation of X doesn't exist $\Rightarrow$ X is connect

$\Leftrightarrow$ The clopen sets only $= \emptyset \cdot X \cdot$ in X.

Cr. $\Rightarrow$ For subspace Y, the seperation of Y is nonempty sets A, B. $Y = A \cup B$, s.t. $A \cap B = \emptyset$. the limit points of each one will not included in the other. $\rightarrow$ limit points should in Y. Such $Y = [-1, 0) \cup (0, 1]$ is a seperation

(Consider the clopen sets in Y!)

(2) Lemma:

1. If $C \cup D$ is seperation of X. If Y is a connected subspace $\Rightarrow Y \subset C$ or $Y \subset D$.

Pf: By connecteness, consider if seperation of Y exists?

$\Downarrow$

2. the union of $\{A_\alpha \mid \alpha \in I\}$, which has the public points and are connected, is connected. $\rightarrow$ If $A_n \cap A_{n+1} \neq \emptyset$. Also holds!

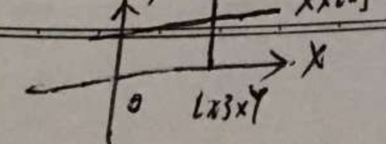
Pf: A_α will fall into A or B, if $A \cup B = \cup A_\alpha$ is the seperation. then all the A_α fall in it too.

3. A is connected subspace. if $A \subset B \subset \bar{A}$. then B is connected!

4: The finite product of connected subspace is connected.

Pf: By Induction ($\Rightarrow X \times Y$ holds. By 3. Fix $X \times \{b\}$.

$T_x = \{x\} \times Y \cup X \times \{b\}$ Connected



$\Rightarrow \cup T_x = X \times Y$ is connected!

(*) Moreover: $\cdot \mathbb{R}^{\omega}$ product topo is connected.

Since $\mathbb{R}^{\omega} = \bigcup \widetilde{\mathbb{R}^n}$, $\widetilde{\mathbb{R}^n} = \{(a_1 \cdots a_n, 0 \cdots 0)\}$

Date: $\mathbb{R}^{\omega}$, uniform topo is disconnected

No. Since the bounded seq is clopen

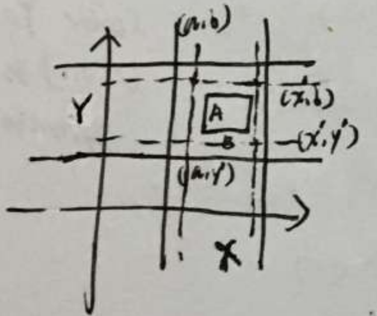
Remark: If the product is infinite, it needs not hold: e.g. the box topo in $\mathbb{R}^{\omega}$ (*)

has the seperation of {bounded seq} $\cup$ {Unbounded seq}.

5. The continue image of connected space is continue.

6. X, Y connected. If $A \subsetneq X$, $B \subsetneq Y$.

then $(X \times Y) - (A \times B)$ is conneced.



$(\{a\} \times Y) \cup (X \times \{y\})$ connected

$\Rightarrow \bigcup_{y \in Y-B} (\{a\} \times Y) \cup (X \times \{y\})$ connect

Also $\bigcup_{x \in X-A} (\{x\} \times Y) \cup (X \times \{b\})$ connect

$\Rightarrow$ Not disjoint!

7. Define: Totally disconnect: If the connect subspace are only the sets of single point

$\Rightarrow$ $\cdot$ Discrete topo is totally disconnected.

$\cdot$ $\mathbb{R}_l$ is, too. Since $\forall b \in \mathbb{R}$, $(-\infty, b) \cup [b, +\infty)$ is a seperation of $\mathbb{R}_l$

8. C is the connect subspace of X.

$A \subset X$. If $C \cap A \neq \emptyset$, $C \cap (X-A) \neq \emptyset \Rightarrow C \cap \mathrm{Bd} A \neq \emptyset$.

Pf: $X = \mathrm{Int} A \cup \mathrm{Int}(X-A) \cup \mathrm{Bd} A$. If $C \cap \mathrm{Bd} A = \emptyset$.

$\Rightarrow (C \cap \mathrm{Int} A) \cup (C \cap \mathrm{Int}(X-A))$ is seperation of C

9. $p: X \longrightarrow Y$. quotient map. If $p^{-1}(\{y\})$ is connected, Y is connected, then X is connected.

Pf: If $X = C \cup D$. Since $p^{-1}(p(\{x\}))$ connected.
$\therefore$ fall into C or D $\therefore C, D$ is saturated
depend on $x \in C$ or D? $\Rightarrow Y = p(C) \cup p(D)$ is the sepe!

10. $Y \subset X$. X, Y are connected. $A \cup B$ is the seperation of $X - Y$, then $Y \cup A$, $Y \cup B$ are connected.

Pf: $X - Y = A \cup B$ $\therefore X = Y \cup A \cup B$. If $C \cup D$ is the seperation of $Y \cup A$, then $X = C \cup D \cup B$
then $\bar{A} \cap \bar{B} = \bar{C} \cap \bar{D} = \emptyset$. Assume $Y \subset C$. Since Y connected.
Since X is connected. We want to contradict by it:
Prove $B \cup C$. D are both closed:
1°) $\bar{D} \subset X - C$. $D \subset C \cup D - C = Y \cup A - Y = A$
$\therefore \bar{D} \subset \bar{A} = X - B$ $\therefore \bar{D} \subset (X - B) \cap (X - C) = D$
2°) $\bar{B} \cup \bar{C} = \overline{B \cup C} \subset (X - A) \cup (X - D) = (B \cup Y) \cup (B \cup C)$
$= B \cup C$ $\therefore \overline{B \cup C} = B \cup C$. $\bar{D} = D$. $\square$

(3) The connected subspace in linear of R.

※. L is a linear order set with more than one element. if (1) L satisfies L-U-B property
(2) if $x < y$, $\exists z$ s.t. $x < z < y$.

$\Rightarrow L$ is a linear continuum.

$\Rightarrow$ Give L the order topo. then:

L is connected $\Leftrightarrow$ L is linear continuum.

Pf: ($\Leftarrow$) Prove If Y is a convex subspace of L
then Y is connected

If $A \cup B$ is seperation of Y. $a<b$
$[a,b] \subset Y$. then $(A \cap [a,b]) \cup (B \cap [a,b])$
is the seperation of $[a,b]$. But $\sup(A \cap [a,b])$
will not be contained in $(A \cap [a,b])$ or $(B \cap [a,b])$!

($\Rightarrow$) Lemma: Intermediate Value Theorem.
$f: X \longrightarrow Y$. X is connected space. Y is
linear order set with order topo. f continue
If $a, b \in X$. $f(a) < r < f(b)$, then $\exists c$
between a and b, s.t. $f(c) = r$

Then $X \xrightarrow[id]{f} X$. satisfies i).

If X doesn't satisfy i). then $A \subset X$. B is set of u-b of A
$\bigcap_{a \in A} [a, +\infty) = \bigcup_{b \in B} (b, +\infty)$, which is clopen!

(4) Def: path-connected: the path between x and y in X.
is conti- $f: [a,b] \longrightarrow X$, $f(a) = x$. $f(b) = y$.
Since $[a,b]$ is connected. If X has path between
arbitrary points x.y. then call X path-connected.
And so that X connected by $[a,b]$

Remark: connectedness 理解为不分离，而 path-connectedness 理解为有限长。因此可构构 connected but not path-connected 的 example: $\{x \times \sin(\frac{1}{x}) \mid x > 0\}$.

I_0^2 (注意: $x \times (0,1)$ 为 open sets)

Lemma.

1. Borsuk-Ulam Theorem: $f: S^n \longrightarrow \mathbb{R}^n$. conti.

then $\exists x \in S^n$. s.t. $f(x) = f(-x)$

Pf: Construct $g(x) = f(x) - f(-x)$

$\therefore g(-x) = -f(x) + f(-x) = -g(x)$. $\therefore \exists\, g(x_1) > 0, g(x_2) < 0$

$\therefore \exists s_0$. $g(s_0) = 0 \Rightarrow f(s_0) = f(-s_0)$

2. the produce of path-connected space is path-connected

Pf: For $\{A_j\}_{j \in J}$. $\gamma_j: [0,1] \longrightarrow A_j$.

then $[0,1] \xrightarrow{\gamma} \prod_{j \in J} A_j$, conti

$t \longmapsto \gamma(t) = (\gamma_j(t))_{j \in J}$

3. $\{A_j\}_{j \in J}$ is family of path-connected set, $\bigcap A_\alpha \neq \emptyset$

or $A_j \cap A_{j+1} \neq \emptyset$. the $\bigcup_{j \in J} A_j$ is path-connected

Pf: By $a \sim x$. $a \sim y$ then $\exists$ path $x \sim y$!

4. A is the countable subset of $\mathbb{R}^2$. then $\mathbb{R}^2 - A$ is path-connected.

Pf: $\forall x, y \in \mathbb{R}^2$. there're uncountable paths between x, y.

But A is countable. $\therefore \exists$ path: $x \sim y$.

5. A is connected $\not\Rightarrow$ $\mathrm{Int}A$. $\mathrm{Bd}A$ connected : $[0,1] \to \{0,1\} \cup (0,1)$

$\mathrm{Int}A \not\Rightarrow A$. $\mathrm{Bd}A$: $(0,1) \cup \{2\} = A$

$\mathrm{Bd}A \not\Rightarrow A$. $\mathrm{Int}A$: $\mathbb{Q} = A$. $\mathrm{Bd}A = \mathbb{R}$. $\mathrm{Int}\,\mathbb{Q} = \varnothing$.

(5) Component and Locally connected.

- $\dot{x}$. $X = \bigcup_{j \in J} C_j$. C_j is disjoint sets, which is connected/path-connected space.

$C_j = \bigcup_{x \in C_j} A_x$. A_x is the connected/path-connected subspace between x and x_0. $\Rightarrow C_j$ connected/p-c!

then it's closed!

(可理解为 x_0 连通/道路连通的最大空间!)

- $\dot{x}$. If $x \in X$, $\forall U$. $x \in U$. $\exists$ connected/path-c neighbour V. $x \in V \subset U$. then it's c or p-c at x. If $\forall x$. X is c/p-c at x then X is Locally connected/p-c

$\Rightarrow$ Theorem : X is Locally connected/p-c

$\Leftrightarrow \forall U \underset{\text{open}}{\subseteq} X$. $\forall$ component in U is open in X

Pf: ($\Leftarrow$) obviously it's the component

($\Rightarrow$) $\exists$ V open. $V \subseteq$ component $\therefore$ component open!

Remark: let $U = X$. then If L-c or L-p-c ~~then~~ components are open!

some examples:

i) R_l: component = $\{a\}$

ii) product topo in R^ω: R^ω since it's path-connected

iii) uniform topo in R^ω: $\vec{x}, \vec{y}$ in the same component $\Leftrightarrow$ $\vec{x}-\vec{y}$ is bounded.

iv) box topo in R^ω: $\vec{x}, \vec{y}$ in the same component $\Leftrightarrow$ $\vec{x}-\vec{y}$ is 0 eventually!

Pf: i) It's totally disconnected ii) $f = (f_i)_{i\in I}$ continue

iii) ($\Leftarrow$) $f: [0,1] \longrightarrow R^\omega$, $t \longmapsto x + t(x+y)$. the f is continue

($\Rightarrow$) If $x-y$ is unbounded, then $U = \{y \mid x-y$ is unbounded$\}$.

and $V = \{y \mid x-y$ is bound$\}$ is the seperation!

since $x \in V$, $y \in U$.

iv) ($\Leftarrow$) see it as product topo

($\Rightarrow$) h_n: $\begin{cases} h_n(x_n) = 0 \\ h_n(y_n) = n \end{cases}$ if $x_n \neq y_n$ $\begin{cases} h_n(x_n) = 0 \\ h_n(y_n) = 0 \end{cases}$ if $x_n = y_n$

the $h: R^\omega \longrightarrow R$, $h(x) = (h_n(x_n)) = (0)$

$h(y) = (h_n(y_n)) \in$ unbounded sequence

$\boxed{\text{Lemma}}$: 1. If X is locally path connected. $\forall$ connected open set U is path-connected

Pf: $x \in U$, let A is the set of the paths of x. $x \in A$, since X is l-pc. $\therefore A$ is open

note that if $x_0 \in U - A$. $\forall U$, $x_0 \in U$, $U \cap A \neq \emptyset$

then $\exists$ pathconnected V. $x_0 \in V \cap A = \emptyset$. $\therefore x_0 \in A$.

contradict! $\therefore$ the limit points should in A, then

A is clopen in U $\therefore A = U$

2. Ref: weakly locally connected at x:

If $\forall U$. $x \in U \underset{open}{\subset} X$. $\exists$ connected subspace A

$x \in A \subset U$. and $\exists$ neighbor V. $x \in V \subset A$.

$\Rightarrow$ If X is weakly locally connected at any x

then X is locally connected.

Pf: $x \in X$. $\forall U$. $x \in U \subset X$. s.t. $V \subset A$

$\therefore V$ is connected neighbor in U.

3. $p: X \longrightarrow Y$. quotient map. If X is locally connected / path-connected, then Y is locally connected / p-c.

Pf: $C \underset{open}{\subseteq} Y$. $p^{-1}(C) \underset{open}{\subseteq} X$. If D is component of C. we prove D is open.

Firstly. we claim: $p^{-1}(D)$ is union of components of $p^{-1}(C)$.

If A is component in X and $A \cap p^{-1}(D) \neq \emptyset$ $\Rightarrow$ $p(A)$ connected

and $p(A) \cap D \neq \emptyset$ $\therefore p(A) \subset D$ $\therefore A \subset p^{-1}(D)$

$\therefore \forall$ components in $X \subset p^{-1}(D)$ $\therefore \bigcup U_x \subset p^{-1}(D)$

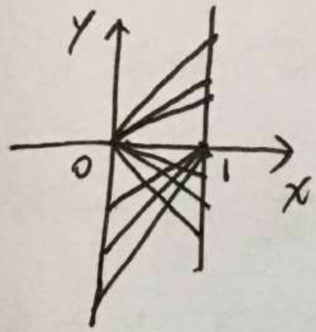
If $x \in p^{-1}(D)$ $\exists U_x$ is component of $x \subset p^{-1}(C)$

$U_x \cap p^{-1}(C) \neq \emptyset$. $\therefore p(U_x) \subset C$ $\overset{x \in}{\therefore} U_x \subset p^{-1}(C)$

$\therefore x \subset \bigcup U_x$ $p^{-1}(D) \subset \bigcup U_x$

$\therefore p^{-1}(D) = \bigcup U_x$ which is open!

4. The subspace of $\mathbb{R}^2$, which is path-connected, but $\forall$ point, not locally connected:



draw line from $(0,0)$ to $(1,q)$, $q \in \mathbb{Q}$
And $(1,0) \to (0,-q)$

② Compactness:

(1) $\boxed{\text{Lemma}}$ In Compact space, every close set is compact.

pf: $\bigcup_{\mathcal{A}} A$ cover X, if $Y \underset{\text{close}}{\subseteq} X$, $X-Y$ is open

$\therefore \exists$ finite sets $(\bigcup_{\mathcal{B}} B) \cup (X-Y)$ cover X.

$\Rightarrow \bigcup_{B \in \mathcal{B}} B \supseteq Y$

2. If in Hausdorff space, every compact subspace is close.

Pf: If X is Hausdorff, $Y \underset{cpt}{\subseteq} X$.
$\forall x \notin Y$, $\exists U_{x_y}$ for some $y \in Y$.
st. $\exists U_y$, $x \in U_{x_y} \cap U_y \ni y = \emptyset$.
Since $\exists$ finite U_y cover Y. then
$\bigcap_{y \in \mathcal{A}} U_{x_y}$ and $\bigcup_{y \in \mathcal{A}} U_y$ which cover Y
are disjoint $\therefore x \in \bigcap U_{x_y} (\underset{\text{open}}{\subseteq} X) \cap Y = \emptyset$!

$\longrightarrow$ Cor. If A, B cpt, then $\exists U, V$ open, $A \subset U \cap V \supset B$ $= \emptyset$

pf: $\forall x \in A$, $\exists W_x = \cap U_{x_y}$ $\cap (\cup U_y) = \emptyset$. then $\exists$ finite W_x cover A.
$\therefore \bigcup_i W_{x_i}$ and $\bigcap_i (\cup U_{y})_{x_i}$ is U and V

Cor. if $f: X(\text{cpt}) \longrightarrow Y(\text{Hausdorff})$, f conti and bijection.
then f is homeo!

3. The finite product of compact spaces is compact.

Pf: <u>Tube Lemma.</u> In $X\times Y$. if Y compact. N is open. covering $\{x\}\times Y$. then $\exists$ neighbour W of x. st. $\{x\}\times Y \subset W\times Y \subset N$

Pf: $\{x\}\times Y \cong Y$ $\therefore \exists$ finite $U_i\times V_i$ $(1\le i\le n)$ cover $\{x\}\times Y$. when $\bigcup_{i\in I}(U_i\times V_i) \supset \{x\}\times Y$

Then $W = \bigcap_1^n U_i$, then $U_i\times V_i$ cover $W\times Y$

and let $U_i\times V_i \subset N$, since N is open.

$\therefore$ $W\times Y \subseteq N$

$\Rightarrow$ since W is neighbour of x. $\therefore \bigcup(W_x\times Y) = X\times Y$

$= \bigcup_{\text{Finite}}(W_x\times Y)$.

If $\bigcup_{A\in\mathcal{A}} A$ cover $X\times Y$, since $\{x_0\}\times Y$ cpt.

$\therefore \exists$ $\bigcup_1^m A_i \supseteq \{x_0\}\times Y$. $\exists$ $\bigcup_1^m A_i \supset W_{x_0}\times Y$

$\therefore$ $X\times Y = \bigcup_{\text{Fin}}(\bigcup_1^m A_i) = \bigcup_{\text{Fin}} A$

Remark: the finite union of cpt space is cpt.

$\triangle$ A. Y is compact. then $\pi_1: X\times Y \to X$. is close map.

Pf: If $C \underset{\text{close}}{\subseteq} X\times Y$. Prove: $\pi_1(C)$ is close.

~~($\Rightarrow$) 对 $x\in X$. $x\notin \pi_1(C)$. then $\{x\}\times Y \cap C = \emptyset$~~

$\therefore \{x\} \times Y \subseteq X \times Y - C$ which is open.

By Tube lemma $\exists M$ open. $\{x\} \times Y \subseteq M \times Y \subseteq X \times Y - C$

$\therefore \exists \pi_1(M)$ is open. $x \in \pi_1(M) \cap \pi_1(C) = \emptyset$.

5. Corollary of Tube Lemma

$A \subseteq X$, $B \subseteq Y$. $A \times B \subseteq N \underset{open}{\subseteq} X \times Y$. 对 A, B cpt.

then $\exists U \underset{open}{\subseteq} X$, $V \underset{open}{\subseteq} Y$. s.t. $A \times B \subseteq U \times V \subseteq N$

pf: $\forall (a, b) \in A \times B$. $\exists a \in U_a$. $b \in V_b$. $U_a, V_b \underset{open}{\subseteq} X, Y$.

s.t. $U_a \times V_b \subseteq N$. since N is open.

$\Rightarrow \exists$ cover, $\bigcup (U_a \times V_b) \supseteq A \times B \xrightarrow{\text{Finite}} \bigcup_1^n (U_{ai} \times V_{bi})$

then for $\{a_j\} \times B$. $\forall a_j \in \{a_i\}_1^m$. $\exists \{ U_{aj} \times V_{bj} \mid b_j \in S \subset \{b_i\}_1^m$. $a_j \in U_{aj} \}$.

then $V_j = \bigcup_{b_j \in S} V_{bj}$ cover B.

for $A \times \{b_j\}$. $\forall b_j \in \{b_i\}_1^m$. $\exists \{ U_{aj} \times V_{bj} \mid b_j \in V_{bj}$. $a_i \in S' \subset \{a_i\}_1^m \}$.

then $U_j = \bigcup_{a_j \in S'} U_{aj}$ cover A

then $U = \bigcap_1^m U_j$. $V = \bigcap_1^m V_j$ since $U \times V \subset N$.

An easy way:

By Tube lemma: $\{a_i\} \times B \subseteq N$. $\exists W_i$ of a_i neighbor

s.t. $W_i \times B \subseteq N$. but B not open. We tweak a little.

$\exists U_i \times V_i$ cover $\{a_i\} \times B \Rightarrow \bigcup (U_i \times V_i)$ cover $\{a_i\} \times B$.

then $\bigcap_1^n U_i$ cover $\{a_i\}$. $\bigcup_1^n V_i$ cover B (Fin)

then $\{a_i\} \times B \subset (\bigcap_1^n U_i) \times (\bigcup_1^n V_i) \subset N$. for $\forall a_j$

for $(\bigcap_1^n U_i)$ is neighbor of $\{a_i\}$ $\therefore \exists$ Fin $\bigcup_{a_i} (\bigcap_1^n U_i)$ cover A

then Let $\bigcap_{a_i} (\bigcup_1^n V_i) = V$. $\bigcup_{a_i} (\bigcap_1^n U_i) = U$

6. $f: X \longrightarrow Y$: cpt Hausdorff, then

f is continuous $\Leftrightarrow$ $G_f = \{x \times f(x) \mid x \in X\}$ close in $X \times Y$

Pf: ($\Rightarrow$) $\forall \{x\} \times \{y\} \notin G_f$, then $y \neq f(x)$

By Hausdorff, $\exists U, V$ open. $y \in U \cap V \ni f(x) = \emptyset$.

Since $x \in f^{-1}(V)$, then $\{x\} \times \{y\} \in f^{-1}(V) \times U \cap G_f = \emptyset$.

($\Leftarrow$) Prove: $\forall V \underset{\text{open}}{\subseteq} Y$. $f^{-1}(V)$ is open $\Leftrightarrow \exists U$ open.

$U \subseteq f^{-1}(V) \Leftrightarrow f(U) \subseteq V$, $\Leftrightarrow f(U) \cap (Y - V) = \emptyset$

Since $G_f \cap (X \times (Y-V))$ close, $\therefore \pi_1(G_f \cap (X \times (Y-V)))$

is close. $\therefore X - \pi_1(G_f \cap (X \times (Y-V)))$ open.

$\exists U_1 \underset{\text{open}}{\subseteq} X - \pi_1(G_f \cap (X \times (Y-V)))$

then $\pi_1^{-1}(U_1) \cap (G_f \cap (X \times (Y-V))) = \emptyset$

$\pi_1^{-1}(U_1) = U_1 \times Y$ $\therefore f(U_1) \cap (Y-V) = \emptyset$

7. $f_n : X \to \mathbb{R}$, conti. $f_n \xrightarrow{\uparrow} f$. for fix x

if X is compact. then $f_n \longrightarrow f$ is uniform

Pf: $f(x) - f_n(x) \in (0, \varepsilon) \Rightarrow x \in (f - f_n)^{-1}(0, \varepsilon)$

$\exists$ finite $(f-f_n)^{-1}(0, \varepsilon)$ cover X. choose the

maximal n, denote N then $n > N \Rightarrow$

$f - f_n(x) < f - f_N(x) < \cdots < \varepsilon$

8. X is cpt Hausdorff space. $\mathcal{A}$ is family of

close connected subsets and is total order under

including relation. then $Y = \bigcap_{A \in \mathcal{A}} A$ is connected

$A \subseteq B$ or $B \subseteq A$

Pf: Y is close. Hausdroff $\therefore$ Y is cpt

if $\exists C, D \underset{\text{closed}}{\subseteq} Y$. $Y = C \cup D$ then C, D cpt

$\therefore \exists U, V \underset{\text{open}}{\subseteq} X$. $C \subset U \cap V \supset D = \emptyset$, since

A is close. since $C \cup D = \bigcap_{A \in \mathcal{A}} A$.

$\therefore C \cup D \subset A$. $\forall A \in \mathcal{A}$. If $A - U \cup V = \emptyset$

$\Rightarrow A \subset U \cup V \Rightarrow$ if $A \subset U$ or $A \subset V$. contradict!

But $U \cup V$ is seperation of itself. $\therefore A - U \cup V \neq \emptyset$

$\therefore \bigcap_{A \in \mathcal{A}} (A - U \cup V) \neq \emptyset$. since X cpt. has finite intersection

$\therefore \bigcap_{A \in \mathcal{A}} A \overset{\|}{-} U \cup V \neq \emptyset$. contradict! (By: $A_1 \subset A_2 \subset A_3 \cdots$)

9. $p: X \longrightarrow Y$. close. conti. surjection. $\forall y \in Y$.

$p^{-1}(\{y\})$ is cpt space. then if Y cpt $\Rightarrow$ X cpt

Pf: If $X = \bigcup U_i$, arbitrary cover.

$x \in U_i$. then $p^{-1}(p(x))$ is cpt space

We want $\forall U_i \xrightarrow{\text{correspond to}} W_i$ in Y

i.e. $\exists W_i$. $U_i \supset p^{-1}(W_i)$

since $X - U_i$ is close $\therefore p(X - U_i)$ close

$\Rightarrow Y - p(X - U_i)$ open

$\therefore$ If $p^{-1}(\{y\}) \subset U \Rightarrow p^{-1}(\{y\}) \cap (X - U) = \emptyset$

$\therefore \{y\} \cap p(X - U) = \emptyset$ $\therefore \exists W_y$. $y \in W_y \cap (p(X-U)) = \emptyset$

$\therefore p^{-1}(W_y) \cap p^{-1}(p(X - U)) = \emptyset$ $\therefore p^{-1}(W_y) \subset U$

Since $p^{-1}(\{y\})$ cpt $\therefore p^{-1}(\{y\}) \subset \bigcup^{n} U_k$

$\Rightarrow p^{-1}(W_y) \subset \bigcup^{n} U_k$

$\therefore \forall p^{-1}(W_y)$ can be covered with finite U_k

since Y can be covered by finite W_y $\therefore X$ can be finite covered!

(2) The compact space on linear of R

1. If X has L-U-B property, which is the total order set with order topo

then $\forall$ close set $[a,b]$ is cpt

Pf: By construction of L-U-B. Assume $\bigcup_{A \in \mathcal{A}} A = [a,b]$

1') if $x \in [a,b]$. then if $x \neq b$, $\exists \exists y \in [a,b]$

s.t. $[x,y]$ can be cover for some A. $A \in \mathcal{A}$.

Choose A. s.t. $x \in A$. Since A is open. $\therefore \exists c$

s.t. $[x,c) \subset A$. then choose $x < y < c$.

$\therefore [x,y] \subset A$

2') $[a,a]$ can be finitely covered. Assume "c" is the L-U-B of $[a,c]$ can be finitely covered.

then $[a,c]$ can be finitely cover

for $c \in$ some A_1, $\exists z < c$, $z \in A_1$, since A_1 open.

$\therefore [a,z] \cup [z,c]$. $[z,c] \subset A_1$. $[a,z]$ can be finitely covered $\therefore$ so $[a,c]$ is!

3') If $c \neq b$. construct by 1')!

$\Rightarrow$ Cor. $\forall$ close set in R is cpt.

and if X is path-connected. Note:

$[0,1] \xrightarrow{f} X$. f conti. $\therefore X$ is cpt!

2. Extreme Value Theorem.

$f: X \to Y$. conti. Y is totally order set with order topo. If X is cpt. $\exists c, d \in X$. s.t. $\forall x \in X$.

$f(c) \le f(x) \le f(d)$

Pf: If $A = f(X)$ cpt. doesn't have maximal element. then $\{(-\infty, a) \mid a \in A\}$ doesn't has finite cover!

3. Lebsesgue Lemma.

If $\mathcal{A}$ is the family of open sets which cover cpt. metric space (X, d). then $\exists \delta$. $\forall r < \delta$. $B_d(x_0, r) \in A$. for some $A \in \mathcal{A}$.

Pf: Assume $\{A_i\}_1^n \subset \mathcal{A}$. $\bigcup_1^n A_i = X$

Denote $D_i = X - A_i$. $f: X \to R$: $f(x) = \frac{\sum d(x, D_i)}{n}$

$\forall x \in X$. $x \in A_i$. for some i $\therefore \exists B_d(x, \varepsilon) \subset A_i$.

$\therefore f(x) > \frac{\varepsilon}{n}$. $\forall x$. Since X cpt. $\therefore \exists$ minimal $f(x_0)$

Denote $f(x_0) = \delta$. then $\delta \le f(x) \le \max_{1 \le i \le n}\{d(x, D_i)\} := d(x, D_k)$

$\therefore x \in B_d(x, \delta) \subset A_k$. $\forall x \in X$.

4. Uniform continue.

If $f: (X, d_X) \longrightarrow (Y, d_Y)$. conti. and X is cpt. then f is uniform continue.

Pf: $B_{d_Y}(y, \frac{\varepsilon}{2})$ is the ball of Y. then. $f^{-1}(B_{d_Y}(y, \frac{\varepsilon}{2}))_{y \in Y}$ ~~$\exists \delta$. s.t. $d(x_1, x_2) < \delta$. then~~ x_1, x_2 ~~$\in f^{-1}(B_{d_Y}(y, \frac{\varepsilon}{2}))$~~

is the open cover.

5. x. isolated point: $\{x\}$ is open in X ($\exists$ open sets $\cap X = \emptyset$)

If X is nonempty cpt Hausdorff and doesn't contain isolated point, then X is uncountable

Pf: 1°) Lemma: Give a point x and open set U.

$\exists$ $V \subset U$. s.t. $\bar{V} \cap \{x\} = \emptyset$.

Pf. We can choose $y \in U$. $y \neq x$. since no isolated-point

then $\exists$ W_1, W_2 $x \in W_1 \cap W_2 \ni y = \emptyset$. then $V = W_2 \cap U$.

2°) Let $U_1 = X$. then denote $V = V_1$, $\bar{V}_1 \cap x_1 = \emptyset$

Let $U_2 = V_1$. $x_2 \in V_1$. $\exists V_2 \subset U_2$. $\bar{V}_2 \cap x_2 = \emptyset$.

---- then $V_1 \supset V_2 \supset V_3 \cdots \supset V_n \cdots$

$\bigcap_{1}^{n} \bar{V}_i \neq \emptyset$, by compactness $\Rightarrow$ $\bigcap \bar{V}_i \neq \emptyset$.

then $\exists x \in \bigcap \bar{V}_i$. but $x \neq x_n$. $\forall n$.

$\therefore$ $Z^+ \longrightarrow X$, not surjection!

6. $d(x, A) = d(x, \bar{A})$. $x \notin \bar{A}$.

Pf: 1°) $d(x, A) = 0 \Leftrightarrow x \in \bar{A}$. Prove: $\forall \varepsilon$.

$d(x, a) < \varepsilon$. $\exists \delta$. $a \in B_d(x, \delta)$.

$\Rightarrow$ Since $\forall$ $B_d(x, r) \cap A = \emptyset$. Let $\delta = \varepsilon$.

2°) $d(x, A) = d(x, a)$. $a \in A$. if A is cpt

By Extreme Value Theorem.

3°) $d(x, A) \geq d(x, \bar{A})$. Since $\bar{A} \supset A$.

$d(x, \bar{A}) = d(x, a) + d(a, A) \geq d(x, A)$. $a \in \bar{A}$

7、The connected metric space contains more than one point. Then it's uncountable

Pf: Note that $d(a,z) \neq \lambda d(a,b)$, $\forall z$, $\lambda \neq 1$, $0<\lambda<1$ can't happen. Otherwise produce a separation:

$\{x \mid d(a,x) < \lambda d(a,b)\} \cup \{x \mid d(a,x) > \lambda d(a,b)\}$.

$\therefore f: X \to R \qquad X \to (0, d(a,b))$
$x \mapsto d(a,x)$ which is uncountable!

Remark: If metric space X is countable
Then X is discrete!

(3) limit point compactness

- X: limit point compact: X. for $\forall A \subset X$. ①
there exists (a) limit points in X. (A infinite)
- X: Sequentially compact: $\forall (x_n)$ is seq. ②
then $\exists$ sub-seq (x_{n_i}) converged!

③

compact

Pf: ③ $\to$ ① $\Leftrightarrow$ Prove: If A has no limit point. then A is finite.
(No limit point means $\forall a \in A$. $\exists U_a$. $U_a \cap A = \{a\}$)
① $\not\to$ ③: e.g. S_Ω. $N \times Y$. (Y is indiscrete topo)

If in a metric space X. ①、②、③ are equivalent
① $\to$ ② obviously. if only 1^{st} countable, it holds.

② → ③ 1°) If X sequentially compact, then it satisfies Lebesgue Lemma.

Suppose $\mathcal{A}$ is an open cover, and the "δ" doesn't exist

$\therefore \forall n, \exists r < \frac{1}{n}$. $B_d(x_n, \frac{1}{n}) \not\subset A$, $\forall A \in \mathcal{A}$. then

choose $y_n \in B_d(x_n, \frac{1}{n})$, $y_{n_i} \to y$. then $y \in A$, some $A \in \mathcal{A}$.

Since A is open, $\therefore \exists B_d(y, \varepsilon) \subset A$. But $\exists N$, $n_i > N$.

$d(y_{n_i}, y) < \frac{\varepsilon}{2}$ and $\frac{1}{n_i} < \frac{\varepsilon}{2}$ $\therefore$ $B_d(x_{n_i}, \frac{1}{n_i}) \subset A$, contradict!

2°) $\exists$ finite balls cover X

3°) balls $\xrightarrow{\text{corresponding}}$ the cover sets!

Remark: 1. Uniform topo in $[0,1]^{\omega}$ is not limit-point-compact

$A = \{(a_k) \mid a_k = 0 \text{ or } 1\}$ has no limit point

2. Subspace $[0,1]$ in R_ℓ isn't limit point compact

$\{1 - \frac{1}{n} \mid n \in \mathbb{Z}^+\}$. $\to$ $\{1\}$ open by $[1, +\infty) \cap [0,1]$

3. $f: X \to Y$, conti, X is L-P-C. But

$f(X)$ probably not. $\underset{\text{indiscrete}}{\mathbb{N} \times [0,1]} \xrightarrow{\pi_1} \mathbb{N}$ (not even T_1!)

But $A \underset{\text{close}}{\subseteq} X$, then A is L-P-C!

($\forall U \subseteq A$, $U \subseteq X$, the limit point of U in X

are all in A, since $\bar{U} \subseteq \bar{A} = A$)

4. Ref: countably compact: If $\forall \bigcup_1^\infty U_n = X$.

$\exists$ finite $\{k_i\}_1^n \subset \{k\}_1^\infty$, s.t. $\bigcup_1^n U_{k_i} = X$

Thm in T_1 space. Countably compact $\Leftrightarrow$ L-P-C

pf: ($\Rightarrow$) 1°) If $\exists A \subset X$. A has no limit point $\therefore$ A close

then $\forall \{x_i\}_1^\infty \subset A$. $\{x_i\}_1^\infty$ is close in A $\therefore$ close in X

$\therefore U_n = X - \{a_i\}_n^\infty$ open! $\cup U_n$ is cover. but not finite subcover!

($\Leftarrow$) If $\bigcup_1^\infty U_n = X$ has no finite subcover.

then choose $x_n \in X / \bigcup_1^n U_n$. Assume $\{x_{ik}\} \to x$.

then $x \in U_k$. some k. Since $\exists N$. $k > N$ $x_{in} \notin U_k$, then $\{x_{ik}\}$ fall in U_k finitely!

Contradict with x is limit point of $\{x_{ik}\}$

(Assume $\{x_{ik}\}_1^N$ fall in U_k. then $X - \{x_{ik}\}_1^N$ open.

$\therefore U_k \cap (X - \{x_{ik}\}_1^N)$ is neighbor of x not contain $\{x_{ik}\}$!)

☆ T_1: If x is limit point of $\{x_k\}$, then $\forall U$ neighbour of u. $U \cap \{x_k\} \neq \emptyset$. (Since if not T_1 then $\{x\}$ is neigh but not contain $\{x_k\}$)

(4) Locally compact:

- X. locally compact: If X is L-C at x. then $\exists$ compact subspace of x. s.t. contains a neighbour of x. $\longrightarrow$ Prerequisite: $\forall$ Basis B. $x \in B$ $\bar{B}$ is cpt Since it's contain in cpt space C.

<u>A if in Hausdorff</u>

($\Leftrightarrow$) $\forall$ U of x. $\forall$ a neighbour V of x. $x \in V \subset U \underset{op}{\subset} X$

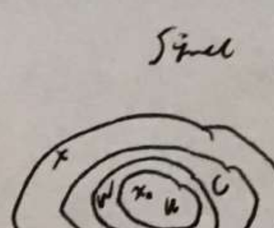
V is cpt and $\bar{V} \subset U$

pf: ($\Leftarrow$) V is the cpt subspace of x.

($\Rightarrow$) if u is an arbitrary neighbour of x.

Since $\exists$ cpt subspace of X. contains neighbour W

then $C \cap (X - U)$ is close in C, then cpt



$\therefore \exists$ A.B open in X. $x \in A \cap B \supset (C \cap (X-U)) = \emptyset$

then $A \cap (X-U) = \emptyset$ $\therefore A \subset U$. and $\bar{A} \subset U$

If X is L-C everywhere. then still need depend on Hausdorff

By classification $X \to Y$ choose A,B cover x, Y-C, which are disjoint

Cor. $\forall$ Totally order set with L-U-B property is Locally compact. Since a basis is contained in a close set of X, which is cpt.

$\boxed{\text{Theorem:}}$ X is a locally compact Hausdorff space if and only if it satisfies: $\exists Y$. st

i) $X \subseteq Y$ ii) $Y - X = \{\infty\}$, which is one point set.

iii) Y is compact Hausdorff Space; which is equivalent to the homeomorphism (restrict on X is id)

($\Rightarrow$)

Pf: 1°) $Y = Y'$: if Y' is another space. let $f(\{\infty\}) = \{\infty'\}$, $f|_Y = id$

Then use the compactness to prove f continue. (By duality)

2°) Let $Y = X \cup \{\infty\}$. give Y the topo:

1. open sets in X is open in Y 2. $Y - C$ is open in Y. C is cpt in X

$\Rightarrow$ 3°) check Y is a real topo. and X is subspace of Y

Use the topo structure of Y. $\leftarrow$

4°) Prove: Y is cpt (By $Y-C$ is open, must be contained in the open cover, since $U \underset{\text{open}}{\subseteq} X$. $\{\infty\} \notin U$.)

5°) Prove: Y is Hausdorff (obviously)

($\Leftarrow$) X is Hausdorff since it's subspace.

$\forall x \in X$. by the proof above. choose an open sets cover $Y - X$ and neighbor of x

$\Rightarrow$ If Y is compact Hausdorff space, $X \subseteq Y$. $\bar{X} = Y$. the Y is the classification of X. If $Y - X = \{\infty\}$ called one-point-classification.

e.g. the one-point-classification of R^n is S^n.

$$f: R^n \longrightarrow \prod(-1,1) \xrightarrow[g]{\text{homeo}} S^n - \{\infty\}.$$

$$(x_\alpha) \longmapsto \left(\frac{|x_\alpha|}{1-x_\alpha}\right) \qquad (\text{like } (-1,1) \to S^1)$$

$\boxed{\text{Conclusions:}}$

1. If X is L-C, $A \underset{\text{close}}{\subseteq} X$. then A is L-C.
   If Y is L-C Hausdorff, $A \underset{\text{open}}{\subseteq} Y$, then A is L-C Hausdorff

Pf: 1°) $\forall x \in A$. $x \in X$, $\exists C$ cpt subspace of X. contain U.
Since $C \cap A$ close in $C \Rightarrow$ cpt. $U \cap A \Rightarrow$ open neigh of x!

2°) By the equivalent proposition of L-C in Hausdorff.
$\exists U$, $U \subset A$. $\Rightarrow \exists V$ cpt neigh, $\bar{V} \subset U$.

2. Only the open map keep the L-C.

3. Lemma: $p: X \to Y$. quotient map, Z is L-C Hausdorff.
then $\pi = p \times id: X \times Z \longrightarrow Y \times Z$ is quo-map.

Pf: By the continue of p. $id \Rightarrow p \times id$ is continue
$\therefore \forall U \underset{\text{open}}{\subseteq} Y \times Z$. $\pi^{-1}(U)$ is open. Now 证 $\pi^{-1}(U)$ open
$\forall x \times y \in U$. $\forall W, V$ open.

~~We prove U is open $\Leftrightarrow$~~

s.t. $x \times y \in W \times V \subset U$. $\Leftarrow$ $\pi^{-1}(W \times V) \subset \pi^{-1}(U)$

$\pi^{-1}(W \times V) = p^{-1}(W) \times V$. Note that $\pi^{-1}(U)$ is open

If we want open set $p^{-1}(W) \times V \subset \pi^{-1}(U)$. then

$p^{-1}(W)$ is open $\Leftrightarrow$ W is open, But we can't ressure it.

So we just construct $p^{-1}(W)$ which is saturated and open.

Pf: $\exists$ $R \times V$ open $\subset \pi^{-1}(U)$. $\bar{V}$ is close then cpt in Y.

$\forall a \in \pi^{-1}(\pi(R))$, choose a neighbour U_i^a s.t. → Note $\pi^{-1}(U)$ is saturated $\therefore \forall a \in \pi^{-1}(\pi(R_i))$ $a \in \pi^{-1}(U)$!

$U_i^a \times V \subset \pi^{-1}(U)$. Let $R_1 = \bigcup_{a \in \pi^{-1}(\pi(R))} U_i^a$. then $R \subseteq R_1$. By Induction

$\Rightarrow$ choose U_i^a for $\forall a \in \pi^{-1}(\pi(R_n))$, s.t. $U_i^a \times V \subset \pi^{-1}(U)$

$\Rightarrow$ $R_i = \bigcup_{a \in \pi^{-1}(\pi(R_n))} U_i^a$, then $\bigcup R_i$ is open and saturated.

since $\forall x \in \bigcup R_i$. $x \in R_i$, for some i, then $p^{-1}(p(x)) \subset R_{i+1}$

Net (A generalized sequence)

(1) 定义: A direct set J: If $(J, \le)$ is a set with a relation "$\le$", satisfies:

i) $\forall \alpha \in J$. $\alpha \le \alpha$. ii) If $\alpha' \le \alpha''$, $\alpha'' \le \alpha'''$, then $\alpha' \le \alpha'''$

iii) $\forall \alpha, \beta \in J$. then $\exists \gamma$. s.t. $\alpha \le \gamma$, $\beta \le \gamma$.

(2) 定义: Net: X is a topo space. A <u>net</u> in X is the <u>map "f"</u>: J: direct set $\xrightarrow{f} X$, i.e. $\alpha \in J$. $f(\alpha) = x_\alpha \in X$.

~~Denote $f = (x_\alpha)_{\alpha \in J}$, If $(x_\alpha) \xrightarrow{conv} x$. then $\forall U$ of x.~~

$\exists \alpha$. if $\alpha \le \beta$. then $x_\beta \in U$.

We say (x_α) lies in A ($A \subseteq X$) eventually, if $\exists \delta \in D$, $\forall \alpha \geq \delta$, then $x_\alpha \in A$. Or "frequently": $\forall \delta \in D$, $\exists \alpha \geq \delta$, s.t. $x_\alpha \in A$.

$\Rightarrow$ Then, (x_α) converage to x $\Leftrightarrow$ (x_α) lies in arbitrary neighbour of x eventually

Remark: The outside of U (a neighbour of x) may be also contains infinite x_α!

(3) 1. Theorem: In Hausdorff space X $\Leftrightarrow$ $\forall$ net has at most one limit point in topo space X.

Pf: ($\Rightarrow$) If (x_α) converage to x_1 and x_2, use U_1, U_2 seperate x_1, x_2, (x_α) will not lie in U_1 or U_2 eventually at the same time!

($\Leftarrow$) If X isn't Hausdorff, $\exists$ x, y can't be seperated, choose neighbour $\{U_{x_i}\}$, $\{V_{y_i}\}$ of x, y. s.t. $x \in U_{x_i} \cap V_{y_i} \ni y \neq \emptyset$. Let $D = \{(U_{x_i}, V_{y_i}) \mid i \in I\}$ → Not countable! Point depend on countability!

with order "$\leq$": If $(U_{x_i}, V_{y_i}) \leq (U_{x_j}, V_{y_j}) \Leftrightarrow$ $U_{x_i} \supseteq U_{x_j}$, $V_{y_i} \supseteq V_{y_j}$. then $D \xrightarrow{f = (x_\alpha)} X$.

Def by $f(U_{x_i}, V_{y_i}) = C(U_{x_i} \cap V_{y_i})$ (choose a point in $U_{x_i} \cap V_{y_i}$)

But (x_α) converage to x and y at the same time!

2. Proposition: X is topo space, $A \subseteq X$, then $\bar{A} = \{x \in X \mid \exists (a_\alpha) \to x\}$ $= \{$all limit points of A in $X\}$

Pf: Use the neighbour U_i, $i \in I$, which needn't countability.

. then give "$\leq$". $U_1 \leq U_2 \Leftrightarrow U_1 \supseteq U_2$. $D = \{U_i \mid i \in I\}$.

And $D \xrightarrow{f} X$. $f(U_i) = C(U_i / U_{i+1})$

3、 X = topo space $\xrightarrow{f}$ Y = topo space. f conti.

$\Leftrightarrow$ Net $(x_\alpha) \to x$. then $f(x_\alpha) \to f(x)$

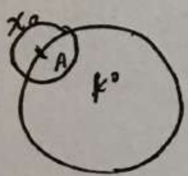
Pf: $(\Rightarrow)$ obvious $(\Leftarrow)$ Suppose f isn't conti. at x_0

i.e. $\exists$ open nbd V of $f(x_0)$. $f^{-1}(V)$ isn't open.

i.e. $x_0 \in f^{-1}(V) / (f^{-1}(V))^\circ$. Def $(D, \leq)$: D is

the set of all nbd of x_0. "$\leq$" is contain order

Note that $\forall A \in D$. $A \not\subseteq K^\circ$ (Assume $K = f^{-1}(V)$)



Then choose $x_A \in A - K$.

$$D \xrightarrow{(x_A)_{A\in D}} X, \quad (x_A)_{A \in D} \xrightarrow{\text{converge}} x_0.$$
$$A \longmapsto x_A$$

But $f(x_A) \not\longrightarrow f(x_0)$. since all $f(x_A) \notin V$

which is one of nbd of $f(x_0)$!

(4) Def: <u>final map</u> h : $(D, \geq) \xrightarrow{h} (D', \geq')$ $\longrightarrow$ Imitate the def of subsequence

between direct sets, if $\forall \delta \in D'$. : $\{k_i\} \to N \xrightarrow{(x_0)} X$

$\exists\, \varepsilon \in D$. st. $\forall \alpha \geq \varepsilon$. $h(\alpha) \geq \delta$

-x: Let D'' be the subset of D'. h = id.

then $\forall \delta \in D'$. $\exists \varepsilon \in D''$ $\varepsilon \geq \delta$. then

D'' is cofinal with D. D'' is also direct set

一. Universal Net: (x_α) in X. 若 $\forall A \subseteq X$.

(x_α) lies in A eventually or

(x_α) lies in X/A eventually.

Lemma 1. (x_α) net in X. $S \subseteq X$. (x_α) lies in S frequently $\Leftrightarrow$ $\exists$ subnet of (x_α) lies in S eventually.

2. (x_α) doesn't lie in A eventually $\Leftrightarrow$ (x_α) lies in X/A frequently!

Proposition. (1) $X \xrightarrow{f} Y$. map. (x_α) is the universal net in X. then $(f(x_\alpha))$ is universal.

(2) subnet of universal net is universal.

Pf: (1) $\forall B \subset Y$. $f^{-1}(B) \subset X$. then (x_α) lies in $f^{-1}(B)$ or $X/f^{-1}(B) = f^{-1}(Y/B)$ frequently. then $f(x_\alpha)$ is!

(2) It's obvious since it's just fraction!

$\boxed{\text{Theorem:}}$ Every net has a universal subnet

Pf: $(x_\alpha)_{\alpha\in D}$ is a net in X. D is direct set

(1) Consider $Y = \{\mathcal{A} \subseteq P(X) \mid \mathcal{A}$ is family of subsets of X which satisfies i) $\forall A \in \mathcal{A}$. (x_α) lies in A frequently

ii) $\forall A_1, A_2 \in \mathcal{A}$. $A_1 \cap A_2 \in \mathcal{A}$. $\}$.

Since $\{X\} \in Y$. $\therefore Y \neq \emptyset$. give the including order "$\leq$"

then $(Y, \leq)$ is a poset. Apply Zorn's lemma.
to get the maximal element A_0 (By union way!)

2°) $D_0 = \{(A, \alpha) \in \mathcal{A}_0 \times D \mid x_\alpha \in A\}$ with order "$\geq_D$"

$(A', \alpha') \geq_D (A, \alpha) \iff A' \subseteq A,\ \alpha' \geq \alpha$. which
is also a direct order set! (By Intersection)

We claim: $D_0 \xrightarrow{h} D$ is final map.
$(A, \alpha) \longmapsto \alpha$

$\forall \delta \in D$, need a (A, α), s.t. $(A', \alpha') \geq_D (A, \alpha)$
then $h(A', \alpha') = \alpha' \geq \delta$, choose α is an index
$\geq \delta$. since $x_\alpha \in A$, $(x_\alpha)_{\alpha \in D}$ lies in A frequently.
$\therefore \exists \varepsilon$ s.t. $\alpha \geq \varepsilon$, x_α lies in A $\therefore \alpha$ exists!
Then 对$\forall (A', \alpha') \geq_D (A, \alpha) \Rightarrow A' \subseteq A,\ \alpha' \geq \alpha \geq \delta$.
$\therefore h$ is a final map!

3°) Prove subnet (y_0): $D_0 \xrightarrow{(y_0) = (x_\alpha) \circ h} X$ is universal.

Note that $\forall S \subset X$. $\Rightarrow$ (y_0 lies in X/S frequently)
$(\Rightarrow) \neg$(y_0 lies in S eventually) generally.

But we claim: If "y_0 lies in S frequently" holds
$\Rightarrow S \in \mathcal{A}_0$

4°) $\forall A \in \mathcal{A}_0$, (x_0) lies in $S \cap A$ frequently

Since $A \in \mathcal{A}_0$. $\therefore \forall \delta \in D$, $\exists \alpha$, $\alpha \geq \delta$ s.t. $x_\alpha \in A$.

$\therefore (A,\alpha) \in D_0$. Since y_0 lies in S frequently.

$\Rightarrow \exists (A_1,\alpha_1) \geq (A,\alpha)$. s.t. $y_0(A_1,\alpha_1) \in S$.

By $A_1 \subseteq A$. $\alpha_1 \geq \alpha$. $\therefore y_0(A_1,\alpha_1) \in S \cap A_1 \subset S \cap A$

$\therefore \forall y_0(A,\delta)$, $\exists y_0(A_1,\alpha_1) \in S \cap A$. $(A_1,\alpha_1) \geq (A,\delta)$

5°) $\mathcal{A}_0 \cup \{S \cap A \mid A \in \mathcal{A}_0\} \cup S \in \Upsilon$. By 4°)

and $\mathcal{A}_0 \cup \{S \cap A \mid A \in \mathcal{A}_0\} \cup S \supseteq \mathcal{A}_0$ $\therefore S \in \mathcal{A}_0$

6°) So If $\neg$(y_0 lies in X/S eventually) $\Rightarrow$ (y_0 lies in S frequently)

If y_0 doesn't lie in S eventually $\Rightarrow$ y_0 lies in X/S frequently

$\therefore X/S$ also $\in \mathcal{A}_0$. then $X/S \cap S = \emptyset$

But (x_α) doesn't lie in $\emptyset$ eventually!

(5) Describe compactness by net.

① X is cpt

② $\forall$ Family $\mathcal{F}$ of close subsets of X.

$\mathcal{F}$ is FIP $\Rightarrow \bigcap \mathcal{F} \neq \emptyset$

③ $\forall$ universal net in X converges

④ $\forall$ net in X has convergent subnet

Pf by:
$$\begin{array}{ccc} ① & \rightarrow & ③ \\ \Uparrow & & \Downarrow \\ ② & \Leftarrow & ④ \end{array}$$

pf: ① $\Rightarrow$ ③ ; suppose ③ doesn't hold.

$\exists U_x$ open nbd of $\forall x$, $\neg$(x_α lies in U_x eventually) $\Rightarrow$ (x_α lies in X/U_x eventually)

$\bigcup U_x \supset X \quad \therefore X = \bigcup_1^n U_x \quad \therefore \bigcap_1^n (X/U_x) = \emptyset$

But (x_α) lies in $\forall X/U_x$ eventually. contradict!

③ $\Rightarrow$ ④ By $\exists$ universal subnet

② $\Rightarrow$ ① By close sets

④ $\Rightarrow$ ② $\mathcal{F}$ is the family of closed sets of X which satisfies FIP.

Let $\mathcal{F}' = \{\bigcap_1^n F_i \mid F_i \in \mathcal{F}\}$ which also satisfies FIP. give "$\leq$" (including order)

$\forall C \in \mathcal{F}'$, choose $x_c \in C$.

Then net: $(x_\alpha) = \mathcal{F}' \longrightarrow X$. $\exists$ subnet (y_α) converges $\to x_0$

$D \xrightarrow{y_\alpha} X$. by $D \xrightarrow{h} \mathcal{F}'$ h is final map.

$\alpha \longrightarrow x_{h(\alpha)} \qquad \alpha \longmapsto h(\alpha)$

$\therefore \forall C \in \mathcal{F}'$, $\exists \delta \in D$, s.t. $\forall \alpha \geq \delta$.

$h(\alpha) \geq C$, then $\forall x_C$. $\exists h(\delta) \geq C$.

s.t. $x_{h(\alpha)} \in C$, if $\alpha \geq \delta$.

$\therefore y_\alpha$ lies in any C eventually.

$\therefore x \in C$. $\forall C \in \mathcal{F}'$. $\therefore x \in \bigcap_{C \in \mathcal{F}'} C$

(b) Tychonoff Theorem proof by Net.

Lemma. X_j $(j \in J)$ is family of topo space.

Net: $D \xrightarrow{(x_\alpha)} \prod_{j\in J} X_j$, $D \xrightarrow{x_{\alpha j}} X_j$

$\alpha \longmapsto (x_{\alpha_j})_{j\in J}$ $\quad \alpha \longmapsto x_{\alpha j}$

Then (x_α) converges $\Leftrightarrow \forall j\in J$. $(x_{\alpha j})$ converges!

Pf: $(\Rightarrow)$ by the continue map π_K. $\pi_K(x_\alpha) = x_{\alpha K}$ converges!

$(\Leftarrow)$ Recall that basis in product topo is

$\{\prod_{j\in J} Y_j \mid Y_j \neq X_j, j$ is finite$\}$. So we consider

finite set J_0. If $x_{\alpha j} \to x_j$. then claim

$(x_{\alpha j}) \to (x_j)$. since $\forall$ neighbor of (x_j) is

the form $\prod_{j\in J_0} Y_j \prod X_j$. since $\forall j\in J_0$, $\exists \alpha_j$ s.t.

$\varepsilon \geq \alpha_j$. $x_{\varepsilon j} \in Y_j$. choose $\alpha = \max\{\alpha_j\}_{j\in J_0}$.

$\therefore$ when $\varepsilon \geq \alpha$, then $(x_{\varepsilon j}) \in \prod_{j\in J_0} Y_j$!

Tychonoff Theorem:

Under product topo. If X_j is cpt. $\forall j\in J$.

Then $\prod_{j\in J} X_j$ is cpt.

pf: $\forall$ universal net (x_α) in $X = \prod_{j\in J} X_j$

then $\pi_j(x_\alpha)$ is also universal net in X_j.

$\therefore \pi_j(x_\alpha)$ converges by compactness of X_j

$\therefore (x_\alpha)$ converges $\Rightarrow X$ is cpt!

The other way:

1') Consider use the "finite intersection"

In $X = \prod_{\alpha \in A} X_\alpha$. 记 $\mathcal{A}$ is a family of sets which satisfies FIP. Then we find $\mathcal{D}$. s.t. $\mathcal{A} \subset \mathcal{D}$. $\mathcal{D}$ is the maximal family satisfies FIP. This step can complish by Zorn Lemma.

Then we prove: $\emptyset \neq \bigcap_{D \in \mathcal{D}} \bar{D} \subseteq \bigcap_{A \in \mathcal{A}} \bar{A}$

2') The property of family $\mathcal{D}$.

(a) $\forall$ finite intersection of sets in $\mathcal{D}$ belongs to $\mathcal{D}$

(b) If $A \subseteq X$. $A \cap D \neq \emptyset$. $\forall D \in \mathcal{D}$. then $A \in \mathcal{D}$.

(c) $\forall D \in \mathcal{D}$. $x \in \bar{D} \Leftrightarrow \forall$ nbd of x belongs to $\mathcal{D}$. then 记 X satisfies T_1. then $\bigcap_{D \in \mathcal{D}} \bar{D}$ only contains a point

记 $y \in \bigcap_{D \in \mathcal{D}} \bar{D} \Rightarrow y \in U_x$. $\forall U_x$ nbd of x.

$\therefore y \in \overline{\{x\}} = \{x\}$. $\therefore x = y$.

3') choose $x_\alpha \in \bigcap_{D \in \mathcal{D}} \pi_\alpha(D)$, then $x = (x_\alpha)_{\alpha \in A} \in \bigcap_{D \in \mathcal{D}} \bar{D}$.

$\Leftarrow \pi_\beta^{-1}(U_\beta) \in \mathcal{D}$. $\forall U_\beta$ of $x_\beta \Leftarrow \forall D$. $U_\beta \cap \pi_\beta(D) \neq \emptyset$.

Countable and seperable Axiom.

(1) ☐ C_1. C_2 will contain under the subspace of the countable product (under product topo) → open conti map will also contain!

☐ If X is metric. then separable or Lindelöf $\Rightarrow C_2$, moreover. separable $\Leftrightarrow$ Lindelöf.

pf: 1) Assume $A=\{a_i\}_1^\infty$ is the dense countable set.

$A_n = \{B_\rho(a_n, \frac{1}{k}) \mid k\in\mathbb{Z}^+\}$. $\bigcup A_n$ is countable Basis.

($\forall x$. U_x. $U_x\cap A\neq\emptyset$. choose $a_n\in U_x\cap A$. Control $d(a_n,x)$)

2) Choose $\bigcup\limits_{x\in X} U(x;r) \Rightarrow \exists$ countable set $A=\{x_i\}$.

s.t. $X=\bigcup\limits_{x_i\in A} B_\rho(x_i,r)$. denote $A_{x_i}=\{B_\rho(x_i,r) \mid r\in\mathbb{Q}\}$.

$\Rightarrow \bigcup A_{x_i}$ is countable basis.

Remark: Lindelöf $\Rightarrow$ separable: choose countable set A_r st $\bigcup\limits_{a\in A_r} D(a,r)=X$. then $\bigcup\limits_{r\in\mathbb{Q}} A_r = A$ is countable dense.

$\forall B(x,\varepsilon)$. choose $0<r<\varepsilon \Rightarrow A_r\cap B(x,\varepsilon)\neq\emptyset$.

☐ Something about C_2:

1. If X is C_2, then $\forall$ Basis $\mathcal{C}$ contains countable bases of X.

pf: choose $C_{n,m}\in\mathcal{C}$. st. $B_n\subset C_{n,m}\subset B_m$. $B_k\in\mathcal{B}$ countable basis.

then $\{C_{n,m}\}\subset\mathcal{C}$ is countable bases.

2. X is C_2. If $A \subseteq X$. A is uncountable then there're uncountable points in A. which are limit points.

Pf: If A' is countable $\Rightarrow$ $A - A'$ is set of isolated points, which is uncountable. choose U_x of $x \in A-A'$. s.t. $U_x \cap \{x\} = \{x\}$. $\Rightarrow$ contradict with C_2!

3. Any compact metric space have C_2.

Pf: compactness is much stronger than Lindelöf. We can also construct dense countable set!

□ Something about Lindelöf

1. close subspace A of X or continue map $f: X \longrightarrow f(X)$. then A and $f(X)$ is Lindelöf

Pf:
1) the close sets of A is close in X. By countable intersection definition of Lindelöf. $\Rightarrow$ close subspace also satisfies CIP!

2) It's obvious by $f^{-1}(U)$ is open!

2. X is Lindelöf. Y is compact. then $X \times Y$ is Lindelöf.

Pf: $\bigcup U_i$ is the open cover of X. then choose finite cover $\{x\}\times Y$. say $\bigcup\limits_i^{\tilde{n}} U_i \Rightarrow$ By tube Lemma. $\exists\ U_x \times Y \subset \bigcup\limits_i^{\hat{n}} U_i$. cover $\{x\}\times Y$. $\Rightarrow$ countable $\bigcup\limits_i^{\hat{n}} U_i$ will cover $X\times Y$!

⊡ something about separable.

1. continue map $f: X \to f(X)$ and product (countable) contain separability.

Pf: i) by $f(\bar{A}) \subset \overline{f(A)}$

ii) Recall the basis in product topo.

construct: fix y_0. $E_m = \{\vec{y} \in \prod X_i \mid$ when $i \geqslant m$, $y_i = y_0\}$. then $\forall B \in \prod X_i$. $B \cap (\bigcup\limits_{m\in\mathbb{Z}^+} E_m) \neq \emptyset$. $\bigcup E_m$ is countdense!

2. If X is separable $\Rightarrow$ The disjoint open sets are countable. (Denote $\mathcal{A}$)

Pf: $U \cap A \neq \emptyset$. choose $a_i \in A$. construct map:

$f: \overset{\mathcal{A}}{U} \longmapsto \overset{A}{a_i}$. Since $U \cap V = \emptyset \Rightarrow a_i \neq a_j$

$\therefore$ f is injection!

(2) equivalent statement of regularity and normality.

i) Regularity $\Leftrightarrow$ $\forall x \in X$. $\forall U_x$ of x. $\exists V$. s.t.

$x \in V \subset \bar{V} \subset U_x$.

ii) Normality $\Leftrightarrow$ $A \subset V \subset \bar{V} \subset U_A$. A is close in X.

Cor. If X is regular. $\forall x, y \in X$. $\exists U, V$ open nbd

s.t. $x \in U \cap V \ni y = \emptyset$. and $\bar{U} \cap \bar{V} = \emptyset$.

(If it's normal. replace "x, y" by close set "A, B")

$\boxed{\cdot}$ Something about regular.

1. product/subspace of regular space is regular.

Pf: 1) $x \in U \Rightarrow \exists \prod U_\alpha$ s.t. $x \in \prod U_\alpha \subset U$.

choose V_α. s.t. $x_\alpha \in V_\alpha \subset \bar{V}_\alpha \subset U_\alpha$.

Then $\prod V_\alpha$ is what we need!

2) Let $U \cap Y$. $V \cap Y$.

2. Locally compact Hausdorff space is regular.

Pf: $A \underset{close}{\subseteq} X$. $x \in X$. $\exists$ compact subspace C

$x \in U \subset C$. And C is close by Hausdorff

then Note compact space is regular.

for $\bar{U}$ close $\therefore \bar{U}$ cpt in C. $\bar{U} \cap A$ close in C, so cpt.

$\therefore \exists$ open set U', V' st. $x \in U' \wedge V' \supset \bar{U} \cap A = \emptyset$.

then $U' \cap U$ and $V' \cup (X / \bar{U})$ is what we need.

Remark: Actually. X is completely regular.

Let Y be one-point-compactification $\Rightarrow Y$ is normal

$\Rightarrow Y$ is completely regular $\Rightarrow X$ is subspace $\Rightarrow X$ is com-regular

□ Something about normal:

1. i) regular space which has C_2
ii) Hausdorff compact space.
iii) metric space
iv) Totally order set under order topo
v) regular Lindelöf space.

$\Rightarrow$ There're normal space!

Pf: Note that relation: i) $\Rightarrow$ v)

v) $A, B \underset{close}{\subseteq} X$. choose V_0, U_b s.t. $b \in B$. $b \in U_b \cap V_0 \supset A = \emptyset$. $\to$ then $\overline{U_b} \cap A = \emptyset$. Since $\bigcup_{b \in B} U_b = B$ < do it again. Also $\exists \bigcup_1^\infty V_i$ cover A. s.t. $\overline{V_i} \cap U_0 = \emptyset$. $\therefore \overline{V_i} \cap B = \emptyset$.

$\therefore \exists$ countable cover $\bigcup_1^\infty U_i = B$. then let: $U_i' = U_i - \bigcup_1^i \overline{V_k}$. $V_i' = V_i - \bigcup_1^i \overline{U_k}$. $\bigcup U_i'$. $\bigcup V_i'$ cover B and A. and there're disjoint!

(*) Use A, B close!

ii) $\Rightarrow$ regular. $A, B \underset{close}{\subseteq} X$. then A, B cpt.
$\therefore$ We can choose open sets disjoint.

iii) choose $B_d(a, \varepsilon_a)$. open nbd of a. s.t. $B_d(a, \varepsilon_a) \cap B = \emptyset$. Also $B_d(b, \varepsilon_b) \cap A = \emptyset$.
then $\bigcup_{a \in A} B_d(a, \frac{\varepsilon_a}{2})$. $\bigcup_{b \in B} B_d(b, \frac{\varepsilon_b}{2})$

iv) Note that every totally order set has immediate successor $\therefore (a, b]$ is an open set of X. where b is the immediate successor $\Rightarrow \bigcup_{a \in A} (x_a, a]$. $\bigcup_{b \in B} (y_b, b]$

x_a, y_b should satisfy: $(x_a, a] \cap B = \emptyset$. $(y_b, b] \cap A = \emptyset$.

2、The close subset contains the normality.

3、Completely normal: If $\forall$ subspace of X is normal $\Leftrightarrow$ $\exists U, V$ open. seperates the separable sets pair A, B (s.t. $\bar{A} \cap B = A \cap \bar{B} = \emptyset$)

Pf: ($\Rightarrow$) $X - \bar{A} \cap \bar{B}$ is subspace. then

$$\emptyset = \begin{array}{l} U_1 \supset \bar{A} \cap (X - \bar{A} \cap \bar{B}) \supset A \\ \cap \\ U_2 \supset \bar{B} \cap (X - \bar{A} \cap \bar{B}) \supset B. \end{array}$$

(only consider A, B are open. Or it's trival)

($\Leftarrow$) $\forall Y$ is subspace of X. $A, B \underset{close}{\subseteq} Y$.

then $A = Y \cap V_1$, $B = Y \cap V_2$. V_1, V_2 close.

$\therefore \bar{V_1} \cap \bar{V_2} = \emptyset$. Sin $A \cap B = \emptyset$.

then (A, B) is actully separable set pair!

Remark: Metric space and totally order space in order topo are completely normal!

(3) Urysohn Lemma:

X is normal space. $A, B \underset{close}{\subseteq} X$. disjoint.

$\exists$ continue map $f: X \longrightarrow [a, b]$ s.t. $\left[\begin{array}{l} f(A) = \{a\} \\ f(B) = \{b\}. \end{array}\right.$

$\Downarrow$

Urysohn Theorem: $\forall$ C_2, regular space X is ~~metrizable~~.

(~~weak-metric~~ - criterion)

Pf: 1°) Lemma: $[a,b] \xrightarrow{\text{homeo}} [0,1]$. Denote $U_0 \supseteq A$. $U_1 = X - B$.

construct family of open sets: $\{U_q \mid q \in \mathbb{Q}\}$. by Induction.

s.t. $U_q \subset \overline{U_q} \subset U_p$. iff $p > q$. by normality!

$\Rightarrow f(x) = \inf\{t \mid x \in U_t\}$. Verify it's conti.

Remark: 1. properey of f: $f^{-1}(r) = \bigcap_{p>r} U_p - \bigcup_{q<r} U_q$

then $f^{-1}(0) = \bigcap_{p>0} U_p$ = G_δ set (countable set intersections)

(By $x \in \overline{U_r} \Rightarrow f(x) \le r$. $x \notin U_r$. $f(x) \ge r$)

$\Rightarrow$ Def: f vanishes precisely on A: X is normal. $f: X \to [0,1]$

conti. st. $x \in A$. $f(x) = 0$. $x \notin A$ $f(x) > 0$ iff A is close G_δ.

Pf: ($\Rightarrow$) the construction on Lemma is! By $f^{-1}(\{0\})$ close.

$= \bigcap_{p>0} U_p$. by continue.

($\Leftarrow$) $A = \bigcap_{n \in \mathbb{Z}^+} V_n$. def $U_1 = V_1$. $A^c U_{\frac{1}{n+1}} \subset \overline{U_{\frac{1}{n+1}}} \subset U_{\frac{1}{n}} \cap V_{n+1}$ (Inductive)

relabel $\{U_{\frac{1}{k}} \mid k \in \mathbb{Z}^+\} \to \{U_q \mid q \in \mathbb{Q}\}$. By Lemma ✓

$\Downarrow$

Strengthen the Lemma: X is normal. f continue: $X \to [0,1]$. s.t.

$f(A) = \{0\}$. $f(B) = \{1\}$. $x \notin A, B$. $f(x) \in (0,1)$ $\Leftrightarrow$

A, B are close G_δ sets.

Pf: choose $f^{-1}(\{1\})$. $g^{-1}(\{0\})$ as A, B. f, g are

continue func. above. let $f/f+g = h$.

then h is what we need!

Application: The regular space contains more than 1 point

is uncountable

~~Pf: If it's countable~~ $\Rightarrow$ ~~Lindelöf~~ $\Rightarrow$ ~~normal~~

$\Rightarrow$ Use Urysohn Lemma on $\{x\}, \{y\} \longrightarrow \{1\}, \{1\}$.

X connected $\Rightarrow f(X)$ connected and contains $\{0\},\{1\}$.

$\therefore f(X)=[0,1]$ $\therefore X$ is uncountable!

2. If X is metric. directly construct:

$$f(x)=\frac{d(x,A)}{d(x,A)+d(x,B)} \quad (\text{check})$$

2') Consider: $X \xrightarrow{\text{imbed}} [0,1]^{\omega}$ produce topo

$X \xrightarrow[\text{imbed}]{} [0,1]^{\omega}$ uniform topo.

$\Rightarrow$ Construct $\{f_i \mid i \in I\}$. I is countable. which will seperate $\forall x$ and U_x of x. s.t. $f(x)>0$. $f(X-U_x)=0$.

Pf: By using the countable bases. $\{B_n\}$.

if $\overline{B}_n \subset B_m$. then By Urysohn Lemma. $\exists g_{n,m}$ s.t. $g_{n,m}(\overline{B}_n)=\{1\}$. $g_{n,m}(X-B_m)=\{0\}$. then $\forall x$ and U.

choose $x \in B_m \subset U$. By regularity, $\exists B_n$. $x \in B_n \subset \overline{B}_n \subset B_m$

$\Rightarrow$ Now relabel $\{g_{n,m}\} \Rightarrow \{f_n\}$. $f=(f_n)_{n\in\mathbb{Z}^+}$ is the imbedding we need.
$\begin{cases} \text{continue } \checkmark \text{ produce topo} \\ \text{injective: point set close!} \\ \text{pre-continue: vanish on } X-U \end{cases}$

Cor. Tietze Extension:

i) X is normal. $A \underset{\text{close}}{\subseteq} X$. $f: A \to [a,b]$. conti.

can be extended $= \tilde{f}: X \to [a,b]$. conti

ii) $f: A \to R \underset{\text{Extend}}{\Rightarrow} \tilde{f}: X \to R \underset{\text{homeo}}{\to} (0,1)$

Actually. use Tietze Extension, we can prove Urysohn Lemma. conversely: $A, B \underset{close}{\subseteq} X$. and

$f(x) = \begin{cases} 1, & x \in A \\ 0, & x \in B \end{cases}$ then f conti. $\Rightarrow$ extend $A \cup B \to X$.

Some conclusions:

1. Def: completely regular: In T_1 space X. $\forall x_0$ and $\forall U^{close}$ which doesn't contains x_0. $\exists$ conti g: $g(x_0) = 1$, $g(U) = \{0\}$. → By Urysohn. it can be imbedded into $[0,1]^?$. then it's metrizable!

Then: $\begin{cases} \text{i) subspace} \\ \text{ii) product} \end{cases} \Rightarrow$ retain the complete regularity.

Pf: i) Restrict on the subspace Y.

ii) choose $\forall \prod U_\alpha.(\vec{b})$. only $\alpha = \alpha_1, \cdots, \alpha_n$, $U_\alpha \neq X_\alpha$.

then for $\{\alpha_i\}_1^n$. $f_{\alpha_i}: X_{\alpha_i} \to [0,1]$ by

$f_{\alpha_i}(b_{\alpha_i}) = \{1\}$. $f_{\alpha_i}(X - U_{\alpha_i}) = \{0\}$. Let $\phi_i = f_{\alpha_i}(\pi_{\alpha_i}(\vec{x}))$

$\Rightarrow f = \prod_1^n \phi_i$ is what we need!

2. Def: Perfectly normal: $\forall$ close sets in X is G_δ.

Then Perfectly normal $\Rightarrow$ completely normal.

Pf: If A, B is a pair of separable sets.

$\Rightarrow$ Let f, g be the continuous functions. st.

$f(\bar{A}) = 0$. $f(X - \bar{A}) = (0,1]$. $g(\bar{B}) = 0$, $g(X - \bar{B}) = (0,1]$

Then $h = f - g$, $\Rightarrow h^{-1}[-1,0)$ and $h^{-1}(0,1]$ seperate A and B!

3. X is completely regular. If $A, B \underset{\text{close}}{\subseteq} X$. $\rightarrow$ If $\exists$ cpt set, can be seperated from another set!

$A \cap B = \emptyset$. A is cpt $\Rightarrow \exists f$ conti. st.

$f: X \longrightarrow [0,1]$. $f(A) = \{0\}$. $f(B) = \{1\}$.

Pf: $\forall$ $a \in A$. $\exists f_a$ seperate a and B. $f_a(B) = \{1\}$. $f_a(a) = \{0\}$

And $f_a^{-1}[0, \frac{1}{2})$ is an open nbd of a.

$\exists$ finite $f_{a_i}^{-1}[0, \frac{1}{2})$ cover A. $1 \le i \le n$

Then let $f = \min_{1 \le i \le n} \{f_{a_i}\}$. $g = \max\{f - \frac{1}{2}, 0\} \cdot 2$

$g(B) = 1$, $g(A) = 0$. Since $\forall a \in A$. $\exists$ $a \in f_{a_i}^{-1}[0, \frac{1}{2})$

4. If X is metrizable, then the propositions is equivalent following.

i) The metric induced by X is bounded

ii) $\forall$ conti $\phi: X \longrightarrow \mathbb{R}$ bounded

iii) X is limit point compact

trival.

Pf: i) $\Rightarrow$ ii) $\phi': X \longrightarrow X \times \mathbb{R}$ (metric!)

$x \longmapsto (x, \phi(x))$

Def metric in $X \times \mathbb{R}$: $d((x, \phi(x)), (y, \phi(y))) = \sqrt{(x-y)^2 + (\phi(x) - \phi(y))^2}$

$\Rightarrow d$ is bounded $\Rightarrow \phi$ is bounded!

ii) $\Rightarrow$ iii) If A is the infinite set which doesn't have limit point $\Rightarrow A$ is set of isolated point

then close and discrete. $\therefore \exists$ conti $f: A \rightarrow \mathbb{Z}^+$

which is contradiction!

5. Def: Retract: Z is a topo space. Y is subspace

若 $\exists$ continue $r: Z \to Y$, s.t. $\forall y \in Y$. $r(y) = y$.

Then say Y is a retract of Z

Proposition: If Z is Hausdorff, then the retract Y is closed

pf: $f: Z \longrightarrow Z \times Z$, conti. then $A = \{(y, r(y)) \mid y \in Y\}$.

$z \longmapsto (z, r(z))$ close in Hausdorff $Z \times Z$

$\therefore$ $f^{-1}(\Delta)$ is close by continue!

6. Def: locally metrizable = $\forall x \in X$. 若 $\exists U$ nbd of x.

s.t. U can be metrizable on subspace.

$\Rightarrow$ If X is locally metrizable. And $\begin{cases} \text{i) } X \text{ is regular Lindelöf} \\ \text{ii) } X \text{ is compact Hausdorff} \end{cases}$

Then X is metrizable (Hint: Therefore normal, prove C_2!)

pf: i) $\forall x \in X$. $\exists$ metrizable nbd U. then choose $B_x \in \mathcal{B}$ (countable basis). s.t. $x \in B_x \subset U$. then $\overline{B_x}$ is Lindelöf and metric $\Rightarrow$ has countable basis, so B_x is.

Then by $\mathbb{Z} \times \mathbb{Z} \cong \mathbb{Z}$ $\Rightarrow$ X is C_2

ii) Cover X by the metrizable nbd $U_x \subset U$

$\overline{U_x}$ is metric cpt $\Rightarrow C_2$, so U_x is!

(4) Inclusion of Manifold

Def: <u>m-manifold</u>: C_2, Hausdorff X. $\forall x \in X$.

$\exists U_x$ nbd of x, s.t. U_x is homeo- to an open set of $\mathbb{R}^m$

Remark: every manifold is regular. then it's metrizable

Pf: locally Euclid is locally compact
And by the Hausdorff $\Rightarrow$ Regular.

$\Rightarrow$ If X Hausdorff cpt. And locally Euclid to R^m. then X is m-manifold.

Pf: choose $\forall x \in X$. $\exists U_x$ of $x \cong$ open set of R^m.
choose close set $U \subset U_x$, then U is cpt. metric
$\therefore U$ is C_2. $\exists U_x' \subset U$. then $\bigcup_i U_{x_i}' = X$.
$\Rightarrow X$ is C_2 $\therefore X$ is m-manifold!

Def: support of ϕ: $\overline{\phi^{-1}(0,1]}$
Partition of unity: $\{U_\alpha\}_{\alpha \in A}$ is the cover of X.
$\{\phi_\alpha\}$ is the set of continue func. st. $\phi_\alpha : X \to [0,1]$
sf: i) $\forall \alpha \in A$. $\sup \phi_\alpha \subset U_\alpha$
ii) $\forall x \in X$. $\sum \phi_\alpha = 1$
iii) $\{\sup \phi_\alpha\}$ is locally finite.
$\Rightarrow$ say $\{\phi_\alpha\}$ is P of U controled by $\{U_\alpha\}$

Remark: existence of "P of U" in normal space.

Pf: Retract Lemma: X is normal, $\{U_i\}$ is locally finite open cover of X. then $\exists$ open locally finite cover $\{V_i\}$. st. $\overline{V_i} \subset U_i$.

Pf: choose V_1 from $X - \bigcup_{i\ge 2} U_i$ (close) $\subset U_1$ by normality.

By Induction. choose V_n from $X - \bigcup_1^{n-1} V_i - \bigcup_{i\ge n+1} U_i$

Since $\forall x$ only falls into finite U_i. so $\{V_i\}$ is!

Now. If X is normal space. $\{U_i\}_1^n$ is the open cover.

$\exists$ "U of P" controlled by $\{U_i\}_1^n$

Pf: Retract $\{U_i\}_1^n \longrightarrow \{V_i\}_1^n \longrightarrow \{W_i\}_1^n$. By Urysohn.

$\exists\ f_i : X \to [0,1]$. $f_i(X - V_i) = \{0\}$. $f_i(\overline{W_i}) = \{1\}$.

then $\{f_k / \sum_1^n f_i\}_1^n$ is "P of U"!

Theorem: m-compact-manifold. then it can be included into R^N. N is some integral.

Pf: Let $\{U_i\}_1^n$ cover X. which can be imbedded into R^m. choose $\{\phi_i\}_1^n$ the "U of P"

and $g_i : U_i \longrightarrow R^m$. Def h_i:

$$\begin{cases} \phi_i(x) g_i(x). & x \in U_i \\ \vec{0}. & x \in X - \text{support}\,\phi_i \end{cases} \Rightarrow \{\phi_i\}_1^n \times \{h_i\}_1^n :$$

$$X \longrightarrow R \times \cdots R \times (R^m \times R^m \cdots \times R^m) \text{ is inclusion!}$$

Remark: Similarly. If X is cpt Hausdorff. $\exists$ nbd U_x of x. for every x. st homeo- to R^k. for some k. Then X can be imbedded into R^N. for some N.

Pf: Let $\{U_i\}_1^n$ cover X. $\Rightarrow$ Def $f \cdot g(x) = \begin{cases} f(x) g(x), & x \in A \cap B \\ 0, & x \in X / A \cap B \end{cases}$

~~A, B is domain of f, g. Then $\varphi = \{\phi_i\}_1^n \times \{\phi_i \cdot g_i\} : X$~~

$\longrightarrow (R \times R \cdots R) \times (R^{k_1} \times R^{k_2} \cdots \times R^{k_n})$

(5) Some fact about topological group.

1. If G is C_1. And satisfies $\left[\begin{array}{l}\text{Lindelöf or}\\ \text{separable.}\end{array}\right.$

$\Rightarrow G$ is C_2.

(Suppose $B_n \supsetneq B_{n+1} \cdots$)

Pf: i) $\{B_n\}$ is countable basis on e. D is the countable dense set. then $\{dB_n\}$ is the countable Basis!

ii) $\bigcup_{g\in G} gB_n$ cover G. then $\exists$ countable set C_n. st. $\bigcup_{g\in C_n} gB_n$ cover G. Then

$\bigcup_{g\in \bigcup C_i} gB_n$ is countable Basis. [By:

$\forall x$ and U_x. $\exists gB_n \cap U_x \ni c$. then $c^{-1}gB_n \cap c^{-1}U_x$ is the open nbd of e. $\exists B_k$. $B_k \subset (c^{-1}gB_n \cap c^{-1})$

$\therefore cB_k \subset U_x$. But x !] (Wrong!)

$\exists gB_n \ni x$. $\Rightarrow x \in gB_n \cap U_x \Rightarrow e \in x^{-1}gB_n \cap x^{-1}U_x$. $\exists B_k$. $e \in B_k \subset (\cdots)$ ✓.

2. Topological group is completely regular.

(6) Reasearch on perfect map.

Def: $f: X \longrightarrow Y$. f is close conti. surjection. And $f^{-1}(\{y\})$ is compact space 于 Y

Property: i) If $f^{-1}(\{y\}) \subset U$ (open set). then $\exists$ nbd of y: W. st $f^{-1}(W) \subset U$.

Pf: Prove: $f^{-1}(\{y\}) \subset U \Rightarrow f^{-1}(\{y\}) \cap (X-U) = \emptyset$

$\{y\} \cap f(X-U) = \{y\} \cap f(X-U) = \emptyset$. $\therefore \{y\} \in Y - f(X-U)$

Since $Y - f(X-U)$ is open $\therefore \exists W$. $\{y\} \subset W \subset Y - f(X-U)$

$\therefore f^{-1}(\{y\}) \subset f^{-1}(W) \subset U$.

If $p^{-1}(B) \subset U$. B is subspace of Y. then $\exists$ nbd of B: W. st. $p^{-1}(B) \subset p^{-1}(W) \subset U$

Pf: let $W = \bigcup_{b \in B} W_b$, W_b is nbd of b

such that $p^{-1}(\{b\}) \subset p^{-1}(W_b) \subset U$.

$\Rightarrow p^{-1}(W) = p^{-1}(\bigcup W_b) = \bigcup p^{-1}(W_b) \subset U$.

(Remark: it strengthens ii)!)

iii) If $\{x\}$ is close in X, then $\{y\}$ close in Y.

And $p^{-1}(\{y\})$ is close compact set! which mean p contains Hausdorff!

Application: (a) Cut off the "compact" part

If $p: X \to Y$, close conti surjection then X is normal $\Rightarrow Y$ is normal (by ii). iii))

(b) $p: X \to Y$. perfect map. If X regular then Y is regular.

Pf: $A \underset{close}{\subseteq} Y$. $y \notin A$. $p^{-1}(\{y\})$ cpt. $p^{-1}(A)$ close.

choose $y' \in p^{-1}(\{y\})$ $\exists$ $y' \in U_i \cap V_i \supset p^{-1}(A) = \emptyset$.

~~$\Rightarrow$ finite $\bigcup_i^n U_i$ cover $p^{-1}(\{y\})$. And $(\bigcup_i U_i) \cap (\bigcap_i V_i) = \emptyset$.~~

$\exists W$. $p^{-1}(\{y\}) \subset p^{-1}(W)$ $\bigcup_i^n U_i \cap (\bigcap_i^n V_i) = \emptyset$!

(c) X is locally cpt $\Rightarrow$ Y is locally compact!

pf: $y \in Y$. $p^{-1}(\{y\})$ close cpt. choose $y' \in p^{-1}(\{y\})$

$\exists$ cpt subspace $C_{y'}$ st. $y' \in U_{y'} \subset C_{y'}$

$\exists$ finite $\bigcup U_i$ cover $p^{-1}(\{y\})$. $\bigcup C_i$ is

the cpt subspace correspond it. And

$p(\bigcup C_i)$ is cpt subspace of Y. st.

$\exists W$. $y \subset W \subset p(\bigcup U_i) \subset p(\bigcup C_i)$

(d) X is C_2 $\Rightarrow$ Y is C_2.

pf: Assume $\mathcal{B}$ is the countable basis of X.

J is some finite index set. $J \subseteq \mathbb{Z}^+$

choose open set W st. $p^{-1}(W) \subset \bigcup_{j \in J} B_j$

$\Rightarrow$ let $U_J = \bigcup p^{-1}(W)$, then $p(U_J)$ open!

Claim: $\{p(U_J) \mid J \subseteq \mathbb{Z}^+$, finite set$\}$ is the countable basis of Y. (obviously!)

(e) Y is paracompact $\Rightarrow$ X is paracompact. (Don't need compact property?)

pf: Suppose $\mathcal{A}$ is open cover of X.

Then $\{Y - p(X - \bigcup_{\alpha \in J} A_\alpha) \mid J$ is finite$\}$ cover Y

Refine it $\Rightarrow$ $\{U_i\} = \mathcal{U}$. then $\{p^{-1}(U_i) \cap A_\alpha \mid A_\alpha \in \mathcal{A}\}$. ✓

(Remark: The way is wrong! If $f^{-1}(y)$ intersects infinite open set $A \in \mathcal{A}$. then y will be be cover!

Pf: $\mathcal{U}=\{U_i\}$ cover X. choose open set $B\subseteq Y$. st
$p^{-1}(B)\subseteq \bigcup_{j\in J} U_j$. J is finite set. Since if
$p^{-1}(\{y\})\subseteq \bigcup_{j\in J} U_j$. then $\exists B$ of y. $p^{-1}(\{y\})\subset p^{-1}(B)\subset \bigcup_{j\in J} U_j$.
Since $p^{-1}(\{y\})$ is cpt. $\therefore$ for arbitrary $y\in Y$. it holds.
(Then the proof above is true!) Then $\{B\}=\mathcal{B}$ is
open cover of Y. refinement $\mathcal{C}$. then $\{p^{-1}(C)\cap U_i \mid C\in\mathcal{C}\}$
is the locally finite open cover of X!

(7) Some examples:

1. S_Ω = limit point compact. normal.
$\bar{S}_\Omega$ = the same as S_Ω, but it's compact Hausdorff
$\Rightarrow$ But $S_\Omega\times\bar{S}_\Omega$ isn't normal. (While $\bar{S}_\Omega\times\bar{S}_\Omega$ is. by cpt Hausdorff!) Then wrong is Δ and $S_\Omega\times\{\Omega\}$. can't be seperated!

2. I_0^2: connected. but not path-connected. C_2
compact. But $I_0\times(0,1)$ isn't Lindelöf!

3. $\mathbb{R}_l$: C_1. separable. Lindelöf. but not C_2
totally disconnected. normal.
$\Rightarrow$ But $\mathbb{R}_l\times\mathbb{R}_l$ (Sorgenfrey Plane) isn't Lindelöf! Also not normal! The problem is Δ is discrete!

4. $\mathbb{R}_K$: Hausdorff (0 can't be seperated!)
C_2.

5. $\mathbb{R}^\omega$ in product topo: normal (By Tychonoff)
metrizable (uniform metric)

Stone - Čech compactification

Condition: If X has compactification Y. As a subspace of Y. X is completely regular. Verse vise, since X can be imbedded into $[0,1]^J$!

Lemma. If $X \xrightarrow{f} Z$. An imbedding into compact Hausdorff space. Then exist a compactification Y. st. $Y \xrightarrow{F} Z$, an imbedding. $F|_X = f$. And Y is unique which is determined up to iso-!

Pf: $X_0 = f(X)$. $\overline{X_0} = Y_0$. which is compact.

choose A. $A \cap X = \emptyset$. $Y = X \cup A$. st $(X, Y) \cong (X_0, Y_0)$

A should satisfy: $\exists$ bijection: $A \xrightarrow{k} Y_0 - X_0$

Then def: $H(x) = \begin{cases} h(x), & x \in X \\ k(x), & x \in A \end{cases}$. And U is open set iff $H(U)$ is open!

Uniqueness can be check easely: $H_1^{-1} \circ H_2 = I_{Y_1}$, $H_2^{-1} \circ H_1 = I_{Y_2}$

$\Rightarrow$ Now consider the continue maps which can extend on the compactification!

Theorem. X is completely regular, $\exists$ compactification Y. st. for every bounded continue map $f: X \to R$, then f can extend to $g: Y \to R$. Y and g is unique!

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Pf: $\{f_\alpha\}_{\alpha\in A}$ is the family of bounded continuous maps from X to $\mathbb{R}$. $I_\alpha = [\inf f_\alpha(X), \sup f_\alpha(X)]$.

Def: $h: X \longrightarrow \prod_{\alpha\in A} I_\alpha$, $h = (f_\alpha)_{\alpha\in A}$. h is an imbedding since $\{f_\alpha\}_{\alpha\in A}$ seperate point and close sets.

By Lemma. $\exists\ H: Y \longrightarrow \prod I_\alpha$. then for $\forall f$, $\Rightarrow f\in\{f_\alpha\}_{\alpha\in A}$.

Suppose $f = f_\beta \Rightarrow f_\beta = \pi_\beta(H)$ unique!)

2) H is unique.

Suppose $f_\beta: X \to I_\beta \Rightarrow g_\beta: Y \longrightarrow I_\beta$.

Then $H = (g_\beta)_{\beta\in A}$. check conti. imbed.

3) Y is unique.

$$\begin{tikzcd} X \arrow[r, "i_1"] \arrow[dr, "j_1"'] & Y_1 \arrow[d, "f_1"] \\ & Y_2 \end{tikzcd} , \qquad \begin{tikzcd} X \arrow[r, "i_2"] \arrow[dr, "j_2"'] & Y_2 \arrow[d, "f_2"] \\ & Y_1 \end{tikzcd}$$

Suppose Y_1, Y_2 are different compactifications.

$\Rightarrow f_1\circ f_2|_X = id$. then by uniqueness of f_1, f_2

and $f_1\circ f_2 = id$ extend $X \to Y_1$. $\therefore f_1\circ f_2 = id$!

Remark: The compactiesfication has universal property.

which means: $\forall$ compactification Y' of X. $\exists ! g: Y \longrightarrow Y'$

, conti. close surjection and $g|_X = id$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "i"'] & Y' \arrow[d, "g"] \\ & Y \end{tikzcd}$$

g is unique. by Y is Hausdorff.

And by Y' compact $\Rightarrow g$ is close. $g|_X = id$ is ob!

Since $g(Y')$ is cpt space containing X.

$\therefore g(Y') = Y$. by the uniqueness of Y

Actually g is projection!

Some conclusions:

1. The condition of a metrizable space which can be compact.

Pf: Note that cpt metric $\Rightarrow$ separable

$\Rightarrow$ inherits to subspace. $\therefore$ If X is separable metric.

Choose A is dense countable set in X.

Def: $f_\alpha = \min_{x_\alpha \in A}\{d(x, x_\alpha), 1\}$. then $f : (f_\alpha)_{x_\alpha \in A}$.

$\therefore X \xrightarrow[\text{imbed}]{} [0,1]^N$ is a compactification.

2. X is completely regular. X is connected

$\Leftrightarrow$ $\beta(X)$ is connected.

pf: ($\Rightarrow$) $\beta(X)$ is closure of X

($\Leftarrow$) If $X = A \cup B$, which a seperation.

Then $\exists$ $f : A \cup B \to [0,1]$. $f(A) = 0$. $f(B) = 1$

By clopen of A, B $\Rightarrow$ $\bar{f} : \beta(X) \to [0,1]$.

$\therefore$ $\beta(X)$ is unconnected. contradict!

Paracompact and Metrizable

(1) Some about locally finite-thing:

If $\mathcal{A}$ is locally finite $\Rightarrow$ $\begin{cases} \text{i)} \ \overline{\bigcup_{A \in \mathcal{A}} A} = \bigcup_{A \in \mathcal{A}} \bar{A} \\ \text{ii)} \ \{\bar{A} \mid A \in \mathcal{A}\} \text{ is locally finite} \\ \text{iii)} \ \mathcal{B} \subseteq \mathcal{A}. \text{ locally finite} \end{cases}$

(2) Paracompact.

Lemma If X is regular. $\mathcal{U}$ is an arbitrary open cover.

i) X has countably locally finite refinement $\Leftrightarrow$ iv) locally finite open refinement.

$\Updownarrow$ $\Updownarrow$

ii) X has locally finite refinement $\Leftrightarrow$ iii) locally finite close refinement.

Pf: i) $\Rightarrow$ iv) Suppose $\mathcal{B} = \bigcup \mathcal{B}_n$ is a CLFR of $\mathcal{U}$.

Use the "retract way". $V_n = \bigcup_{B \in \mathcal{B}_n} B$. then $S_n(u) = u - \bigcup_{i<n} V_i$. for $u \in \mathcal{B}_n$.

Then $S_n(u) \subseteq u$. Denote $\mathcal{C}_n = \{S_n(u) \mid u \in \mathcal{B}_n\}$.

Then $\bigcup \mathcal{C}_n$ is L-F-R which is needed!

ii) $\Rightarrow$ iii)

By regularity, $\mathcal{U}$ admits a refinement $\mathcal{A}$,

s.t. $A \in \mathcal{A} \Rightarrow \bar{A} \subset u$. use ii) on $\mathcal{A} \Rightarrow \mathcal{C}$.

Then $\mathcal{D} = \{\bar{C} \mid C \in \mathcal{C}\}$ is what we need!

iii) $\Rightarrow$ iv)

"Extend Way". use iii) on $\mathcal{U} \Rightarrow \mathcal{B}$.

Then for $\forall x \in X$. $\exists u_x$ of x. s.t u_x intersect $\mathcal{B}$

only finite B. use iii) on $\{u_x\} \Rightarrow \mathcal{C}$

Def: $\mathcal{C}(B) = \{C \mid C \subseteq X - B.\ C \in \mathcal{C}\}$.

$E(B) = X - \bigcup_{C \in \mathcal{C}(B)} C$. $\Rightarrow B \subseteq E(B)$. $F(B) = \{u \mid B \subseteq u.\ u \in \mathcal{U}\}$.

Then $\{E(B) \cap F'_{(F' \in F(B))}\}$ is what we need!

Theorem: i) Regular Lindelöf is paracompact

ii) Stone Theorem: Metrizable Space is paracompact

iii) The product of a compact space X and paracompact space Y. $X\times Y$ is paracompact.

iv) X is regular. If $X = \bigcup_{i}^{n} X_i$, X_i is close paracompact.

or $X = \bigcup_{i}^{n} X_i$. X_i is close paracompact, And $\bigcup \text{int} X_i = X$

$\Rightarrow X$ is paracompact.

Pf: i) obvious! ii) $\mathcal{A}$ is an open cover. Now use "retract way"

By Well-order Axiom, give $\mathcal{A}$ a well-order!

$S_n(U) = \{ x \in U \mid B(x, \frac{1}{n}) \subset U \}$. $T_n(U) = S_n(U) - \bigcup_{V<U} V$

Then $x \in T_n(U)$, $y \in T_n(V) \Rightarrow d(x,y) > \frac{1}{n}$

$\Rightarrow$ Let $T_n(U)$ be open set: $E_n(U) = \bigcup_{x\in T_n(U)} B(x, \frac{1}{3n})$

$\Rightarrow d(x,y) > \frac{1}{3n}$. If x, y fall differently!

Def: $\mathcal{E}_n = \{ E_n(U) \mid U \in \mathcal{A} \}$. $\bigcup \mathcal{E}_n$ is CLF.

Actually. We construct countably locally discrete.

Since $\forall x \in X$. $B(x, \frac{1}{6n})$ intersects one element in $\mathcal{E}_n$. $\forall n$.

iii) $\{U_\alpha\}_{\alpha\in A} = \mathcal{U}$ is open cover of $X\times Y$. then $\exists$ finite open sets $\{U_{ik} \mid 1\le k\le n\}$ cover $X\times\{y\}$. By pipe lemma.

$\exists\ X\times W \subseteq \bigcup_{1}^{n} U_{ik}$, cover $X\times\{y\}$. Then $\mathcal{W} = \{W\}$

cover $Y \Rightarrow$ LFC $\mathcal{W}' = \{W'\}$ cover Y.

Then $\{(X\times W')\cap U_\alpha \mid W' \in \mathcal{W}', U_\alpha \in \mathcal{U}\}$ is LFR!

iv): 1) $\mathcal{A}$ is open cover of X. $V_i = \{X_i \cap A \mid A \in \mathcal{A}\}$ cover X_i

$\Rightarrow$ LFR $= W_i = \{X_i \cap W \mid W$ is open in $X\}$.

$\Rightarrow W_i' = \{(X - \bigcup_{j \neq i} X_j) \cap W \mid W \in W_i\}$. $\bigcup_i W_i$ is LFC!

2) $W_i' = \{\text{int} X_i \cap W \mid W \underset{\text{open}}{\subseteq} X, W \in W_i\}$.

Property of paracompact:

i) The (close) subspace of paracompact space is paracompact.

ii) Paracompact Hausdorff is normal

pf: i) $\mathcal{A}$ cover Y. $\{A \cap Y \mid A \in \mathcal{A}\} \cup \{X - Y\}$

is cover of X, too $\Rightarrow$ refinement $\cap Y$ is LFR!

ii) $x \in X$, $A \underset{\text{close}}{\subseteq} X$. $(\bigcup_{a \in A} U_a) \cup \{X - A\}$. $\overline{U_a} \cap \{x\} = \emptyset$

cover $X \Rightarrow$ refinement $\mathcal{B}$. choose $B \in \mathcal{B}$.

s.t. $B \cap A \neq \emptyset$. then make an union!

$\boxed{\text{Criterion of paracompact space:}}$

i) X is Hausdorff. then X is paracompact $\Leftrightarrow$ $\forall$ open cover $\mathcal{U}$ of X admits a patition of unity subordinate to it.

ii) X is locally compact Hausdorff. then X is paracompact $\Leftrightarrow$ X is union of disjoint σ-cpt open sets. (σ-cpt: countable union of cpt sets)

pf: i) ($\Leftarrow$) It's trival

($\Rightarrow$) replace $\mathcal{U}$ with locally finite refinement.

Denote $\mathcal{U}$, too!

Def: $\mathcal{U}$-admissible collection:

A family of func. $\{\phi_u \mid u\in J\}$, s.t.

i) $J\subseteq \mathcal{U}$ ii) $X \xrightarrow{\phi_u} [0,1]$, $\sup\phi_u \subseteq u$, $\forall u\in J$.

iii) $\bigcup_{u\in J} \phi_u^{-1}(0,1]$ together with $\bigcup_{w\in \mathcal{U}/J} w$ cover X

1') Suppose $A=\{$all $\mathcal{U}$-addmissible collection$\}$ with "$\leq$" $(\subseteq)$

Then A admits a maximal chain $\mathcal{C}\subseteq A$.

Let $C_0=\bigcup_{c\in\mathcal{C}} c=\{g_u \mid u\in J_0\}$. we hope $J_0=\mathcal{U}$

2') Claim 1. $C_0\in A$

only prove C_0 satisfies iii). Suppose $\exists x\notin g_u^{-1}(0,1]$ for every u and $x\notin u'$, $u'\in\mathcal{U}/J_0$. But by $\mathcal{U}$ is Locally finite refinement, $\exists\{u_i\}_1^n$ cover $\{x\}$.

Then $\{u_i\}_1^n\subseteq J_0$, since $\mathcal{C}$ is maximal chain.

$\therefore \exists C\in C_0$, $C=\{g_u \mid u\in J_1\}$, s.t. $\{u_i\}_1^n\subseteq J_1$

$\therefore x\in\left(\bigcup_{u\in J_1} g_u^{-1}(0,1]\right)\cup\left(\bigcup_{w\in\mathcal{U}/J_1} w\right)=X$. but $\{u_i\}_1^n$ is the only intersections of $\{x\}$. $\therefore x\in\bigcup_{u\in J_1} g_u^{-1}(0,1]$, contradict!

3') Claim 2. $J_0=\mathcal{U}$

If $\exists u_0\in\mathcal{U}/J_0$. Let $Y=\bigcup_{u\in J_0} g_u^{-1}(0,1]\cup\bigcup_{\substack{v\in\mathcal{U}/J_0\\ w\neq u_0}} w$

$\therefore X=u_0\cup Y$. Let $Z=X/Y$, close.

And $Z\subset u_0$. By normality, $\exists V$, $Z\subseteq V\subseteq\bar{V}\subseteq u_0$

∴ Use Urysohn construction: $\exists f: X \to [0,1]$

s.t. $f(Z) = \{1\}$, $f(X - U_0) = \{0\}$. ∴ $Z \subseteq f^{-1}(0,1]$. $f^{-1}(0,1] \cup Y = X$.

Denote $f = g_{u_0}$, then $J_0 \cup \{u_0\}$ is larger family! contradiction!

$\Rightarrow X = \bigcup\limits_{u \in U} g_u^{-1}(0,1]$. Let $e_u = g_u / \sum\limits_{u \in U} g_u$

iii) ($\Leftarrow$) $X = \bigcup\limits_{\iota \in \Lambda} U_\iota$, U_ι is σ-cpt. Note that if $\forall \iota \in \Lambda$

U_ι is paracompact, then X is paracompact.

∴ Suppose $X = U_\iota$, then $X = \bigcup\limits_{1}^{\infty} C_k$, C_k is cpt close.

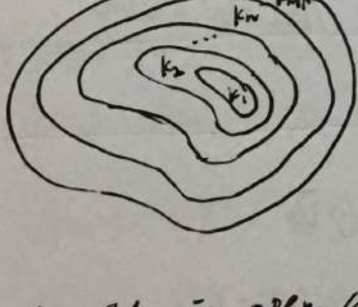
Let $K_1 = C_1$. for $x \in C_1$, $\exists V_{1,1}$ s.t. $x \in V_{11} \subset C$ (compact subspace) $\subset U_i$??

$\Rightarrow \overline{V}_{11}$ is compact, then by compactness of $C_1 \Rightarrow \bigcup\limits_{1}^{n} V_{1,i}$ cover C_1

Let $K_2 = C_2 \cup (\bigcup\limits_{1}^{n} \overline{V}_{1,i})$, which is compact. (then ... K_n cpt!)

$C_2 \subseteq K_2$ and $K_1 \subseteq \operatorname{int} K_2$, repeat the process (cover K_i)

$\Rightarrow C_n \subseteq K_n \subseteq \operatorname{int} K_{n+1}$. then $\underbrace{K_n / \operatorname{int} K_{n-1}}_{\substack{\shortparallel \\ Z_n, \text{ close} \\ \text{cpt in } K_n}} \subseteq \underbrace{\operatorname{int} K_{n+1} / K_{n-2}}_{\substack{\shortparallel \\ W_n, \text{ open}}}$



i.e. we constructed many cpt bands, each of which contained in an open set. And these are countable. cover X

If $\mathcal{U}$ is open cover of X, $\mathcal{O}_n = \{U \cap Z_n \mid U \subset W_n, U \in \mathcal{U}\}$. or intersects with W_n!

then By compactness of Z_n. $\exists$ finite $U \cap Z_n$ cover Z_n.

denote $\{U_{n,i}\}_1^m$. then $\bigcup \{U_{n,i}\}_1^m$ is locally finite.

Because $\forall x \in X$, x will fall into a Z_n which is seperated by W_n!

~~Cor. C_2, locally compact Hausdorff~~ is paracompact.

Remark: Actually, the func from partition of unity can be seen as the func. def locally. Since $f(X-U)=\{0\}$, which can be ignored. Then we can splice these func. to construct a new func.

(2) Metrizable:

Lemma: If X is regular, having a countably locally finite Basis $\mathcal{B}$, then X is normal perfectly

pf: 1) $\forall W \underset{open}{\subseteq} X$, $\exists$ family of open sets $\{U_n\}$

st. $W = \bigcup U_n = \bigcup \overline{U_n}$

prove: $\mathcal{B} = \bigcup \mathcal{B}_n$. choose $B \in \mathcal{B}_n$, st. $\overline{B} \subset W$.

Denote $C_n = \{B \mid B \in \mathcal{B}_n, \overline{B} \subset W\}$.

$\therefore U_n = \bigcup\limits_{B \in C_n} B$. $\overline{U_n} = \bigcup\limits_{B \in C_n} \overline{B}$, prove: $\bigcup \overline{U_n} = W = \bigcup U_n$

$\Rightarrow$ Then $\forall$ close set in X is G_δ

2) $C, D \underset{close}{\subseteq} X$. $X - C = \bigcup \overline{U_n}$, $X - D = \bigcup \overline{V_n}$.

each $\overline{U_n}$, $\overline{V_n}$ intersects C, D is null. [resp.]

We can easily construct!

Theorem. i) [Nagata-Smirnov]: X can be metrizable

$\Leftrightarrow$ X is regular and has countably locally finite basis.

ii) [Smirnov]: X is metrizable $\Leftrightarrow$ X is paracompact Hausdorff and locally metrizable.

Pf: i) $(\Leftarrow)$ $\mathcal{B} = \cup \mathcal{B}_n$. then for n and $B \in \mathcal{B}_n$

construct $f_{n,B}: X \longrightarrow [0, \frac{1}{n}]$, $x \in B$, $f(x) > 0$

$x \notin B$, $f(x) = 0$. Let $J = \mathbb{Z}^+ \times \mathcal{B}$, Def:

$F: X \longrightarrow [0,1]^J$, $F(x) = (f_{n,B}(x))_{(n,B) \in J}$

which is imbedding. Since it separates any x and U.

Now we will imbed X into $(\mathbb{R}^J, \bar{\rho})$ $\Leftrightarrow$ Prove: F conti.

$\Leftrightarrow$ Fix x_0, $\forall \varepsilon$, $\exists W$ nbd of x, st $\rho(F(x_0), F(x)) < \varepsilon$.

1) Fix n. for x_0, $\exists U_n$, st, $U_n \cap B \neq \emptyset$, $B \in \mathcal{B}_n$, only finite B.

then choose V_n, st $V_n \subseteq B_n$, $|f_{n,B}(x) - f_{n,B}(x_0)| < \frac{\varepsilon}{2}$, by conti and finite of B!

for such B. then choose N, st $\frac{1}{N} < \frac{\varepsilon}{2}$

2) Let $W = \bigcap_1^N V_i$. then we are done

$(\Rightarrow)$ Let $\mathcal{A}_m = \{B_{\bar{\rho}}(x, \frac{1}{m}) \mid x \in X\}$. by paracompact of metric space

$\exists$ locally finite refinement $\mathcal{B}_m$. then $\bigcup \mathcal{B}_m = \mathcal{B}$. CLF Bases!

ii) $(\Rightarrow)$ It's trival

$(\Leftarrow)$ Prove: X has countably locally finite basis

choose $\mathcal{C} = \{C \mid C$ is a metrizable nbd of x, $x \in X\}$

cover X. $\Rightarrow$ locally finite refinement $\mathcal{B}$.

Let $\mathcal{D}_m = \{B_{\rho}(x, \frac{1}{m}) \mid x \in B, B \in \mathcal{B}\}$. $\mathcal{D} = \cup \mathcal{D}_m$ is what we need!

ρ metric in B!

Some conclusion: T_1 space X doesn't have locally finite bases. except it's discrete.

pf: If $\mathcal{B}$ is locally finite basis of X.

Then it's point-finite, too. $B_x = \{B \mid B \subseteq \mathcal{B},\ x \in B\}$ is finite. Since it's T_1 space.

$\therefore \forall y \in X - \{x\}$, $\exists W$, s.t. $y \notin W \ni x$, then $\exists B$, s.t. $x \in B \subset W$. then $\bigcap\limits_{B \in B_x}^{\text{Finite}} B = \{x\}$. clopen!

Complete metrizable Space and function Space.

(1) Some complete metric spaces:

$$\begin{cases} \mathbb{R}^k : \text{give } \ell \text{ or } d \\ \mathbb{R}^\omega : \text{give } D(x,y) = \sup\{\frac{\bar{d}(x_i, y_i)}{i}\} \end{cases}$$

<u>Theorem</u>: 1. If (Y, d) is complete. then $(Y^J, \bar{e})$ is complete.

pf: Note. $\forall \alpha \in J$. $d(f_n(\alpha), f_m(\alpha)) \le \bar{e}(f_n, f_m) < \frac{\varepsilon}{2}$

2, X is topo space. (Y, d) is metric space. Then $C(X,Y)$, $\mathcal{B}(X,Y)$ are close in $(Y^X, \bar{e})$.

pf: If $\{f_n\} \subseteq C(X,Y)$, $\{f_n\} \to f$. then f is conti.

$\therefore f \in C(X,Y)$

If $\{g_n\} \subseteq \mathcal{B}(X,Y)$, $\exists N$, $n > N$. s.t. $\bar{d}(g_n, g) < 1$

(if $\{g_n\} \to g$.) $r(g_N(X)) = N \Rightarrow r(g(X)) < N+1$

$\therefore g \in \mathcal{B}(X,Y)$

$\boxed{3}$. $\exists\, \varphi = (X, d) \longrightarrow$ complete metric space.

And φ is isometry, $\overline{\varphi(X)}$ is unique completion of X. which is determined by homeo-!

pf: Embed (X, d) into $(\mathcal{B}(X, \mathbb{R}), e)$

since $\mathbb{R}$ is complete $\therefore$ $\mathcal{B}(X, \mathbb{R})$ is complete.

→ remark: If X is cpt. or $\mathcal{B}(X, \mathbb{R})$, then we can replace $\bar{e}$ by e since it's bounded!

1°) Def: $\phi_a(x) = d(x, a) - d(x, b)$, conti. $X \to \mathbb{R}$

And $|d(x,a) - d(x,b)| \le d(a,b)$. $\therefore \phi_a$ is bounded!

2°) Def: $(X, d) \xrightarrow{\varphi} (\mathcal{B}(X, \mathbb{R}), e)$

$\varphi(a) = \phi_a$. then prove: φ is isometry.

$e(\phi_a, \phi_b) = \sup\{ d(\phi_a(x), \phi_b(x)) \}_{x \in X}$

$= \sup\{ |d(x,a) - d(x,b)| \}_{x \in X} \le d(a,b)$

But if $x = a$. then $e(\phi_a, \phi_b) = d(a,b)$!

3°) Uniqueness: If: $\begin{array}{ccc} (X,d) & \xrightarrow{\varphi} & (\overline{\varphi(X)}, D) \\ & \searrow_{\varphi'} & \downarrow f \\ & & (\overline{\varphi'(X)}, D') \end{array}$ then Prove: $f = \varphi'\varphi^{-1}$ is iso-

$\{a_n\}$ is cauchy seq in $\overline{\varphi(X)}$. let $\varphi^{-1}(a_n) = \tilde{a}_n$.

$\varphi'(\tilde{a}_n) = a_n'$. then $D'(a_n', a_m') = D'(\varphi'(\tilde{a}_n), \varphi'(\tilde{a}_m))$

$= d(\tilde{a}_n, \tilde{a}_m) = d(\varphi^{-1}(a_n), \varphi^{-1}(a_m)) = D(a_n, a_m)$

$\therefore \{a_n'\}$ is also cauchy seq. If $\{a_n\} \to a$. $\{a_n'\} \to a'$

Then def: $f: (\overline{\varphi(X)}, D) \to (\overline{\varphi'(X)}, D')$. by $f(a) = a'$

If $\begin{matrix}\{b_n\} \searrow \\ \{a_n\} \nearrow\end{matrix} a$. check f is well def!

By $D(a_n, b_n) \le D(a_n, a) + D(b_n, b) < \varepsilon$

$\Rightarrow D(a_n, b_n) = D'(a_n', b_n') < \varepsilon$.

Some conclusions:

1. for some ε. 若 $\forall x \in X$. $B_d(x,\varepsilon)$ has cpt closure.

Then (X,d) is complete.

Pf: Consider $\{x_k\}_N^\infty$. when $n,m > N$. $d(x_n, x_m) < \varepsilon$

$\therefore$ It all falls into some cpt-closure of ε-ball!

since it's complete!

2. (X,d_X), (Y,d_Y) complete metric space. $A \subseteq X$

If $A \xrightarrow{f} Y$. uniform conti. then f can uniquely extend to $\bar{A} : \xrightarrow{g} Y$. g is uniform conti.

Pf: Def $g(x) = \begin{cases} f(x). & x \in A \\ \lim\limits_{n\to\infty} f(x_n). & x \notin A.\ x \in \bar{A} \end{cases}$ $\{x_n\} \to x$.

check 若 g is well-def and uniform conti.

3. Peano curve fills the space:

$I = [0,1]$. then $\exists \varphi : I \to I^2$. epi-

Cor. $\exists f : I \to I^n$. $\forall n$. epi-

Pf: By Induction on n. $n=1$ trivial.

若 $\exists g : I \to I^{n-1}$. then $I \xrightarrow{\varphi} I \times I \xrightarrow{id \times g} I^n$

$(id \times g) \circ \varphi = f$. conti.

(2) Compactness in Metric Space.

✡ (X,d) is compact $\iff$ It's complete and totally bounded.

Pf: ($\Rightarrow$) Trivial

($\Leftarrow$) Prove (X,d) is sequential compact. Suppose $\{x_i\}$ in (X,d)

$\bigcup_1^n B_d(x_i,1)$ cover X $\Rightarrow \exists$ $B_d(x_k,1)$ contains infinite x_i.

Collect these index of points. Denote J_1.

Also. $\bigcup_1^{n'} B_d(x_i,\frac{1}{2})$ cover X $\Rightarrow \exists B_d(x_k,\frac{1}{2})$ contain infinite points whose index is in J_1. Denote these index of point : J_2

$\cdots\cdots$ Use $\bigcup_1^{n''} B_d(x_i,\frac{1}{n})$ cover X, and we have $J_1 \supset J_2 \supset \cdots \supset J_k$

$n_1 \in J_1$, choose $n_2 > n_1 \in J_2$ $\cdots$ $n_k > n_{k-1} \in J_k$. $\{x_{n_k}\}$ is Cauchy seq!

Some conclusions:

1. (X_n,d_n) is totally bounded for every n. Then $(\prod X_n, D)$ is totally bounded.

Pf: Recall that ε-ball in $(\prod X_n, D)$ is $\prod(X_n,d_n)\times \prod_1^N B_{d_i}(x_i,\varepsilon)$

for every (X_n,d_n), $\forall \varepsilon$, $\exists$ finite $\bigcup_{i=1}^{k} B_{d_n}(x_{ni},\varepsilon)$ cover X_n.

Then collect x_{ni}. $\Rightarrow J_n = \{x_{ni}\}_1^k$. $J = \{(\vec{x}) \mid x_i \in J_i\}$.

J is finite. If $D(x,y)<\varepsilon$. $\Rightarrow d_i(x_i,y_i)<2\varepsilon$.

$\therefore \bigcup_{x\in J} B_D(x,\varepsilon)$ cover X !

2. If $\mathcal{F}$ is subset of $C(X,Y)$. (Y,d) metric space satisfies one of the conditions

i) $\mathcal{F}$ is finite

ii) $\mathcal{F}$ is uniform conti $\Rightarrow \mathcal{F}$ is equiconti.

iii) $\forall x \in X$, $\exists U$ of x. st

$\mathcal{F}'$ is uniform bounded.

$\{f(x) \mid f \in \mathcal{F}\}$.

(3) point or cpt convergence:

① Topo of pointwise convergence.

for $x \in X$, $U \underset{open}{\subseteq} Y$. $S(x, U) = \{f \mid f \in Y^X, f(x) \in U\}$.

is the subbasis!

② Topo of compact convergence.

for compact subspace C and ε, $f \in Y^X$

$B_C(f, \varepsilon) = \{g \mid e(f(x), g(x)) < \varepsilon, \forall x \in C\}$.

is its basis!

③ Compact-open topo

for cpt subspace C and $U \underset{open}{\subseteq} Y$.

$S(C, U) = \{f \mid f(C) \subseteq U, f \in C(X, Y)\}$.

is its basis!

$\Rightarrow$ Topos in Y^X:

uniform $\supseteq$ Topo of cpt convergence $\supseteq$ Topo of pointwise convergence

when X is cpt. coincide trivial!

when X is discrete, coincide. Since cpt set is finite in X.

Remark: compact-open topo coincides with topo of cpt convergence. So the metric in Y does nothing with it!

pf: $\forall f \in S(C, U)$. $f(C)$ is cpt. $\therefore$ if $y \in f(C)$

$\exists$ nbd of y: $U_y = B_d(y, \varepsilon_y) \subseteq U$. ~~$\cup B_d(y, \varepsilon_y)$ cover $f(C)$.~~

since $\bigcup B_d(y,\varepsilon_y)$ is finite. Denote $\varepsilon = \min\{\varepsilon_y\}$. $\Rightarrow f(C) \subset U(f(C),\varepsilon) \subset U$

Then $B_C(f,\varepsilon) \subset S_C(C,U)$

for $f \in B_C(f,\varepsilon)$. By continue of f. choose V_x. st.

$f(V_x) \subset B(f(x),\frac{\varepsilon}{4})$ $\therefore f(\overline{V_x}) \subset B(f(x),\frac{\varepsilon}{3})$, cover C by V_x.

$\Rightarrow$ let $C_x = \overline{V_x} \cap C \Rightarrow f \in \bigcap_1^n S_C(C_x, B(f(x),\frac{\varepsilon}{3})) \subset B_C(f,\varepsilon)$

Property of topo of compact convergence:

Def: compactly generated: If for every compact subspace C.

If $A \cap C$ is open. then A is open

$\Rightarrow$ Proposition: If X is locally cpt or C_1. then X is cpt generated.

Pf: 1°) $\forall x \in A$. $\exists C$ cpt subspace. contains U of x.

$\Rightarrow$ by $A \cap C$ open in C. $\therefore A \cap C \cap U$ open in U

$\therefore A \cap U$ open in X $\therefore x \in A \cap U \subset A$

2) $x \in \overline{B}$. then $\exists \{x_n\} \subseteq B$. $\{x_n\} \to x$.

$C = \{x_n\} \cup \{x\}$. cpt subspace of $X \overset{\text{if}}{\Rightarrow} B \cap C$ close.

since $\forall x_n$ is contained in $B \cap C$ $\therefore x \in B \cap C$.

$\therefore x \in B$. $\therefore \overline{B} = B$. B is close!

Lemma. If X is compact generated. If $\forall$ cpt subspace C. $f|_C$ is conti $\Rightarrow f$ is conti.

Pf: $f^{-1}(V) \cap C = (f|_C)^{-1}(V)$, V open

$\Rightarrow$ Theorem: $C(X,Y)$ is close set in Y^X under topo of cpt convergence, if X is cpt generated.

Pf: If f is limit point of $C(X,Y)$, for $\forall$ $B_C(f,\frac{1}{n})$, choose f_n, then $f_n \rightrightarrows f$.

f is continue in every C! $f \in C(X,Y)$

Remark: Note that $f_n: X \to Y$, converge to f under cpt convergent topo $\Leftrightarrow$ $f_n|_C \rightrightarrows f|_C \Rightarrow f$ is conti!

Property of cpt-open topo

1. Under cpt-open topo, if Y is Hausdorff or regular, Then $C(X,Y)$ is Hausdorff or regular.

Pf: 1) $f(x) \in V \cap U \ni g(x) = \emptyset$, then $S(\{x\},V) \cap S(\{x\},U) = \emptyset$.

2) If $f \in S(C,U)$, $f(C)$ is cpt. close.

$\forall f(x) \in f(C)$ $\exists$ nbd U_x, s.t $f(x) \in U_x \subset \bar{U}_x \subset f(C)$

then $f(C)$ can be covered by finite U_x. And $\cup \bar{U}_x \subset U$

$f \in S(C, \cup U_x) \subset \overline{S(C,\cup U_x)} \subset S(C,U)$

Remark: If $\bar{U} \subset V$, then $\overline{S(C,U)} \subset S(C,V)$

since $f \in \overline{S(C,U)} \Rightarrow f(C) \subseteq \bar{U} \subset V$

2. If Y is locally cpt Hausdorff, under cpt-open topo

$\Rightarrow \varphi: C(X,Y) \times C(Y,Z) \longrightarrow C(X,Z)$, conti

$\varphi((g,f)) = f \circ g$.

pf: If $S(C,W)$ open in $C(X,Z)$. 若 $f\circ g \in S(C,W)$

$\Rightarrow f^{-1}(W)$ open. $g(C)$ cpt. choose $x \in g(C)$.

Since Y is regular. $\Rightarrow \exists U_x$. $x \in U_x \subset \bar{U}_x \subset U$. and $\bar{U}_x$ cpt!

$\therefore \bigcup_{i}^{n} U_{x_i}$ cover $g(C)$, $\overline{\bigcup_{i}^{n} U_{x_i}} = \bigcup_{i}^{n} \bar{U}_{x_i} \subset U$

Let $V = \bigcup_{i}^{n} U_{x_i}$. $\Rightarrow (g,f) \in S(C,V) \times S(\bar{V},W) \subset \varphi^{-1}(S(C,W))$

3. evaluation map: $e: X \times C(X,Y) \longrightarrow Y$.

by $e(x,f) = f(x)$. If X is locally cpt Hausdorff

$C(X,Y)$ is under cpt-open topo. then e is conti.

pf: V is open set in Y. 若 $f(x) \in V$. $x \in f^{-1}(V)$

$\Rightarrow$ By regularity: $\exists U_x$. s.t. $x \in U_x \subset \bar{U}_x \subset f^{-1}(V)$. $\bar{U}_x$ cpt.

then $U_x \times S(\bar{U}_x, V) \subset e^{-1}(V)$

Remark: If "f" "x" are both changing. then "e" is still continue. "e" is uniform conti above f and x.

$\Rightarrow$ <u>Def</u>: $F: Z \longrightarrow C(X,Y)$. by $f: X \times Z \rightarrow Y$

$F(z)(x) = f(x,z)$. F is induced by f.

Give: $C(X,Y)$ open-cpt topo. If f is conti. then

F is conti. Moreover. If X is locally cpt Hausdorff.

then verse vise.

pf: $(\Leftarrow)$ X is locally cpt Hausdorff $\Rightarrow e$ is conti.

$\therefore X \times Z \xrightarrow{id \times F} X \times C(X,Y) \xrightarrow{e} Y$ conti.

($\Rightarrow$) If $F(z_0) \in S(C, U) \Rightarrow \forall x \in C.\ F(z_0)(x) \in U.$

$\therefore f(x, z_0) \in U.\ \therefore f(C \times z_0) \subset U.\ C \times z_0 \subset f^{-1}(U)$

By pipe lemma. $\exists W.\ C \times z_0 \subset C \times W \subset f^{-1}(U)$

$\therefore W \subset F^{-1}(S(C, U))$ then F is conti!

Remark: Homotopic: f and g are homotopic

if $h: X \times [0,1] \longrightarrow Y$. for $\forall x$. $h(x,0) = g$.

$h(x,1) = f$. h conti. say h is homotopy between g and f.

(4) Ascoli Theorem:

Lemma. 1. $\mathcal{F} \subseteq (C(X,Y), \bar{e})$. $\mathcal{F}$ is totally bounded then $\mathcal{F}$ is equiconti.

pf: $\{B(f_i, \frac{\varepsilon}{3})\}_1^n$ cover $\mathcal{F}$. then $\forall x_0 \in X$

for $\{f_i\}_1^n$. $\exists U$. st $x \in U$. $d(f_i(x), f_i(x_0)) < \frac{\varepsilon}{3}$. $\forall i$.

Then $\forall x \in U$. $\forall f \in \mathcal{F}$. f fall in $B(f_k, \frac{\varepsilon}{3})$

$\therefore\ d(f(x), f_k(x)) < \frac{\varepsilon}{3}$. $d(f(x_0), f_k(x_0)) < \frac{\varepsilon}{3}$.

And $x \in U$, we have $d(f_k(x), f_k(x_0)) < \frac{\varepsilon}{3}$ $\therefore d(f(x), f(x_0)) < \varepsilon$!

Lemma. 2. $\mathcal{F} \subseteq (C(X,Y), \bar{e}/e)$. $\mathcal{F}$ is equiconti. If X, Y are cpt, then $\mathcal{F}$ is totally bounded

Pf: $\forall a \in X$. $\exists U_a$ st $\forall x \in U_a$. $d(f(x), f(a)) < \frac{\varepsilon}{3}$. for $\forall f \in \mathcal{F}$.

$\{U_{a_i}\}_1^n$ cover X by cpt. $\delta = \frac{\varepsilon}{3}$. use finite open set V_k whose diameter $< \delta$. then $\{V_k\}_1^m$ cover Y.

~~Let J contains all func. $\alpha: \{a_i\}_1^n \Rightarrow \{k\}_1^m$~~

$\therefore J$ is finite. Then collect all the func. f_α

Such that. 计 $f_\alpha \in \mathcal{F}$. $f_\alpha(a_i) \in V_{\alpha(a_i)}$. $\forall i$

$\{f_\alpha\}$ is finite. Then claim $\{B_{\bar{e}}(f_\alpha, \varepsilon)\}$ cover $\mathcal{F}$

prove: $\forall f \in \mathcal{F}$. for $f(a_i)$, choose $V_{\alpha(a_i)}$ cover $f(a_i)$. → choose i after U_k is determined!

Then $\Leftrightarrow$ prove $f \in B_{\bar{e}}(f_\alpha, \varepsilon)$

$\Leftrightarrow$ $\bar{e}(f, f_\alpha) < \varepsilon$. $\Leftrightarrow \forall x \in X$. $\bar{d}(f(x), f_\alpha(x)) < \varepsilon$.

since x fall in some U_k. $\Rightarrow \bar{d}(f(x), f(a_k)) < \frac{\varepsilon}{3}$

By $f, f_\alpha(a_k) \in V_{\alpha(a_k)}$ $\therefore \bar{d}(f(a_k), f_\alpha(a_k)) < \frac{\varepsilon}{3}$

And $f_\alpha \in \{f_\alpha\}$ $\therefore \bar{d}(f_\alpha(a_k), f_\alpha(x)) < \frac{\varepsilon}{3}$

$\Rightarrow$ Classic version of

Ascoli Theorem: X is cpt. $(\mathbb{R}^n, d)$ metric space

$d = d$ or e. 计 $\mathcal{F} \subseteq \mathcal{C}(X, \mathbb{R}^n), \bar{e})$. Then:

$\mathcal{F}$ has cpt closure $\Leftrightarrow$ $\mathcal{F}$ is equiconti and point-bounded.

Pf: Denote $\mathcal{G} = \bar{\mathcal{F}}$

($\Rightarrow$) If $\mathcal{G}$ is cpt. $\Rightarrow \mathcal{G}$ is bounded so $\mathcal{F}$ bounded

And $\mathcal{G}$ is totally bounded $\Rightarrow \mathcal{F} \subseteq \mathcal{G}$. so $\mathcal{F}$ equiconti by Lemma 1.

($\Leftarrow$) Claim 1. $\mathcal{G}$ is equiconti. and point-bounded

If $g \in \mathcal{G}$. then $B_{\bar{e}}(g, \frac{\varepsilon}{3}) \cap \mathcal{F} \neq \emptyset$. $\exists f \in \mathcal{F}$.

$f \in B_{\bar{e}}(g, \frac{\varepsilon}{3})$. $\therefore \bar{e}(g, f) < \frac{\varepsilon}{3}$. for $\forall x \in X$.

$\bar{d}(f(x), f(a)) < \frac{\varepsilon}{3}$. $\forall f$.

But for $\forall a \in X$. $\exists U_a$. $x \in U_a$.

$\Rightarrow \bar{d}(g(x), g(a)) < \varepsilon$ is obvious! And g is

point-bounded by $\forall g \neq g$. $g \in \mathcal{G}$ $\exists f \in \mathcal{F}$. s.t.

$\bar{e}(f', g') < \frac{\varepsilon}{3}$. But $\because f' \in \mathcal{F}$ point-bounded

$\therefore \forall x \in X$. $\bar{d}(f(x), f'(x)) < M \Rightarrow \bar{d}(g(x), g'(x)) < M+1$

Claim. 2. Embed $\mathcal{G}(X)$ into cpt subspace Y in $(\mathbb{R}^n, d)$

$\forall a \in X$. $\exists U_a$. $\forall x \in U_a$. $\bar{d}(f(x), f(a)) < \varepsilon$. $\forall f \in \mathcal{G}$.

Then cover X by $\{U_{a_i}\}_1^n \Rightarrow \{g(a_i) \mid g \in \mathcal{G}, 1 \le i \le n\}$

is bounded $\therefore \{g(a_i) \mid g \in \mathcal{G}, 1 \le i \le n\} \subseteq U(0, N)$

$\Rightarrow \mathcal{G}(X) \subseteq \overline{U(0, N+1)}$. $\overline{U(0, N+1)}$ is cpt

Then $\mathcal{G}(X)$ is complete. $\therefore \mathcal{G}$ is complete, since $\mathcal{G} \subseteq C(X, \mathbb{R}^n), \rho)$

And by Lemma. 2. $\mathcal{G}$ is totally bounded $\Rightarrow \mathcal{G}$ is cpt!

Generalization of Ascoli Theorem:

(*) $\Leftarrow$ point-bdd

Give $C(X, Y)$ topo of cpt convergence.
$\mathcal{F} \subseteq C(X, Y)$. If: $\mathcal{F}$ is equicontinuous
and $\mathcal{F}_a = \{f(a) \mid f \in \mathcal{F}\}$ has cpt closure. (*)
Then $\mathcal{F}$ has cpt-closure. Moreover, if X is cpt Hausdorff, then verse vice!

($\Rightarrow$)

pf: Denote $\mathcal{G} = \overline{\mathcal{F}}$. closure in Y^X with product topo. $\Leftrightarrow$ topo of point conv.

$\therefore C(X, Y) \nsubseteq Y^X$ because there have different topo sense

Claim 1. $\mathcal{G}$ is cpt subspace of Y^X.

Denote $C_a = \overline{\mathcal{F}_a} \Rightarrow \prod C_a$ cpt. close by Tychonoff.

And $\mathcal{F} \subset \prod C_a \Rightarrow \mathcal{G} = \overline{\mathcal{F}} \subset \overline{\prod C_a} = \prod C_a$.

~~$\therefore \mathcal{G}$ is cpt subspace. Since it's close~~

Claim 2. $\mathcal{G}$ is equiconti. and $\forall g \in \mathcal{G}$ is conti.

$\forall a \in X$. $\exists U_a$. st $\forall f \in \mathcal{F}$. $\bar{d}(f(a), f(x)) < \frac{\varepsilon}{3}$.

prove: 对 $x \in U_a$. then $\bar{d}(g(x), g(a)) < \varepsilon$. $\forall g \in \mathcal{G}$

$\exists$ open set $V_x = S(x, B(g(x), \frac{\varepsilon}{3})) \cap S(a, B(g(a), \frac{\varepsilon}{3}))$, $V_x \cap \mathcal{F} \neq \varnothing$ in Y^X

$\therefore \exists f \in V_x \cap \mathcal{F}$. $\Rightarrow$ $\bar{d}(g(x), g(a)) < \varepsilon$

Claim 3. topo of point convergent in $\mathcal{G}$ is same as topo of cpt covergence in $\mathcal{G}$ → Important Lemma!

Then $\mathcal{G}$ is cpt subspace in $C(X, Y)$

$\Leftrightarrow$ Prove: $\exists$ open set of topo of point convergence $\subseteq B_C(f, \varepsilon) \cap \mathcal{G}$. $f \in \mathcal{G}$.

Conver C by $\{U_{a_i}\}_1^n$. a_i is the equiconti-point.

$\Rightarrow f \in \bigcap_1^n S(a_i, B(f(a_i), \frac{\varepsilon}{3})) \cap \mathcal{G} \subset B_C(f, \varepsilon) \cap \mathcal{G}$

($\Leftarrow$) Denote $\mathcal{H}$ is a cpt subspace contains $\mathcal{F}$.

Then prove: $\mathcal{H}_a$ is cpt ($x \in X$). $\mathcal{H}$ is equiconti.

1°) $C(X, Y) \xrightarrow{j} \{a\} \times C(X, Y) \xrightarrow{e} Y$. conti.

$\mathcal{H}_a = e j(\mathcal{H})$ $\therefore \mathcal{H}_a$ is cpt.

2°) $\forall a \in X$. $\exists$ cpt subspace A. $a \in A$.

$\mathcal{H}$ is equiconti at a in $C(X, Y)$ $\Leftrightarrow$ $\mathcal{H}$ is equiconti at a in $C(A, Y)$

prove: $r: C(X, Y) \longrightarrow C(A, Y)$. conti.

$\forall f|_A \in B_C(f|_A, \varepsilon)$ in $C(A, Y)$. $\exists B_C(f, \varepsilon)$ in $C(X, Y)$

$f \in B_C(f, \varepsilon) \subseteq r^{-1}(B_C(f|_A, \varepsilon))$. $\therefore r(\mathcal{H})$ is cpt

$\therefore \mathcal{H}$ is totally bonded in $C(A, Y)$, then equiconti!

since uniform topo concide with topo of cpt convergence in cpt subspace A

Remark: In summary.

i) If $\mathcal{F} \subseteq C(X,Y)$, give $C(X,Y)$ topo of point convergence, if $\mathcal{F} = \{f_n\} \rightrightarrows f$, $\mathcal{F}$ is equicontinuous. then $\mathcal{F} \rightrightarrows f$ under topo of cpt convergence.

Pf: Consider cpt subspace C.

ii) generalize i) If $\mathcal{F} \subseteq C(X,Y)$, $\mathcal{F}$ is equiconti. then topo of point convergence in $\mathcal{F}$ coincides with topo of cpt convergence in $\mathcal{F}$. (use i)!)

iii) Restrict map: $C(X,Y) \longrightarrow C(A,Y)$ conti.

Arzela-Ascoli:

$\mathcal{F} \subseteq C(X, \mathbb{R}^n)$, X is seperable, If

i) $\mathcal{F}$ is equiconti pointwise ii) $\mathcal{F}a$ is bounded, $a \in X$.

$\Rightarrow$ Then $\forall$ seq $\{f_n\}$ in $\mathcal{F}$ has subseq $\rightrightarrows f$, under topo of cpt convergence.

Pf: By classical version of Ascoli, it's trivial

Now we use another way.

Denote $A = \{a_i\}_1^\infty$ countable dense set in X.

We claim $\exists \{f_n\}$ converges pointwise on A.

Note that $\{f_n(a_1)\}$ bounded $\therefore \exists \{f_{n_i^{(1)}}(a_1)\}$ converges.

$\{f_{n_i^{(1)}}(a_2)\}$ bounded $\therefore \exists$ subseq $\{f_{n_i^{(2)}}(a_2)\}$ converges...

$\Rightarrow \exists \{f_{n_i^{(m)}}(a^m)\}$ converges. And $\{n\} \supseteq \{n_i^{(1)}\} \supseteq \{n_i^{(2)}\} \cdots \{n_i^{(m)}\}$

$$\begin{array}{cccccc} & a_1 & a_2 & & & a_k \\ f_1: & \underline{f_{n_1^{(1)}}} & f_{n_1^{(2)}} & - & - & f_{n_1^{(k)}} \\ f_2: & f_{n_2^{(1)}} & \underline{f_{n_2^{(2)}}} & - & - & f_{n_2^{(k)}} \\ & \vdots & \vdots & & & \vdots \\ f_k: & f_{n_k^{(1)}} & f_{n_k^{(2)}} & - & - & \underline{f_{n_k^{(k)}}} \end{array}$$

Choose the diagonal index. Since if $m>k$. Then $\{n_m^{(m)}\}$ will fall into $\{n_i^{(k)}\}$. $\therefore \{f_{n_m^{(m)}}(a_k)\}$ converges. for every k.

Claim 2, $\forall \varepsilon > 0$. $\forall x \in X$. $\exists U_x$ of x and N_x.
s.t. $k, l > N_x$. then $|f_k(x') - f_l(x')| < \varepsilon$. $\forall x' \in U_x$.

(Here, we denote $\{n_m^{(m)}\}$ by $\{m\}$.)

prove: By equicontis of $\{f_n\}$. $\forall \varepsilon . \forall x \in X . \exists U_x \ni x$.
s.t. $\forall x' \in U_x$. $\forall f_n$. $|f_n(x') - f_n(x)| < \frac{\varepsilon}{6}$ then
since A is dense $\therefore \exists a \in A \cap U_x$. then

$$|f_k(x') - f_l(x')| \le |f_k(x') - f_k(x)| + |f_k(x) - f_k(a)| \} \ x', a \in U_x$$
$$+ \underbrace{|f_k(a) - f_l(a)|}_{\text{converge}} + |f_l(a) - f_l(x)| + |f_l(x) - f_l(x')| < \varepsilon.$$

Then $\forall K \underset{cpt}{\subseteq} X$. K can be covered by such U_x. finite.
Pick $N = \max\{N_i\}$. we're done!

Approximate a particular func.
by continuous functions.

Def: (1) $F \subseteq C(X, \mathbb{R})$, closed under lattice operations
if $\forall f_1, f_2 \in F$, $\min\{f_1, f_2\} \in F$. $\max\{f_1, f_2\} \in F$.

(2) $\mathcal{F} \subseteq C(X,\mathbb{R})$. $g \in C(X,\mathbb{R})$, we say $\mathcal{F}$ approximates g uniformly on set S. if $\forall \varepsilon$. $\forall x \in S \subseteq X$. $\exists f \in \mathcal{F}$. s.t. $|f(x)-g(x)|<\varepsilon$.

Now following. we suppose X is cpt. And with metric d.

Lemma. 1. $\mathcal{A}$ is closed subalgebra of $\{f: X \to \mathbb{R} \mid f \text{ is bounded}\}$. Then $\mathcal{A}$ is closed under lattice operations.

Pf: Note that $\max/\min\{f,g\} = \frac{f+g \pm |f-g|}{2}$.

Then $\Leftrightarrow$ prove: $\forall h \in \mathcal{A}$. $|h| \in \mathcal{A}$.

Without loss of generality, suppose $|h(x)| \le 1$

Consider $f(t) = |t|$, use $\{(t^2+\frac{1}{n^2})^{\frac{1}{2}} \mid n \in \mathbb{Z}\}$ $\xrightarrow{\text{approximate}}$ $|t|$. Also. use Taylor $\xrightarrow{\text{approximate}}$ $(t^2+\frac{1}{n^2})^{\frac{1}{2}}$

Then $|t| \in \mathcal{A}$. if $t \in \mathcal{A}$. let $t = h(x)$. done!

Lemma. 2 $\mathcal{F} \subseteq C(X,\mathbb{R})$. close under lattice operations. Then $\overline{\mathcal{F}} = \{g \in C(X,\mathbb{R}) \mid \mathcal{F}$ approximates g uniformly on $\{p,q\}$. $\forall p,q \in X\}$.

Pf: 1°) $\overline{\mathcal{F}} \subseteq \{\cdots\}$ obviously.

Fix $(p,q) \in X \times X$,

2) If $g \in \{\cdots\}$. $\forall \varepsilon$, $\exists f_{p,q} \in \mathcal{F}$. s.t $\begin{cases} |f_{p,q}(p) - g(p)| < \varepsilon \\ |f_{p,q}(q) - g(q)| < \varepsilon. \end{cases}$

Denote $\begin{cases} U_{p,q} = \{x \mid f_{p,q}(x) < g(x) + \varepsilon\} \\ V_{p,q} = \{x \mid g(x) - \varepsilon < f_{p,q}(x)\}. \end{cases}$

which are open nbds of p, q. Then

Fix q. $\bigcup_{p \in X} U_{p,q}$ covers $X \Rightarrow \exists \bigcup_{i}^{n} U_{p_i,q} = X$

Let $f_q = \min_{1 \le i \le n} \{f_{p_i,q}\}$, then $\forall x \in X$.

$f_q \le f_{p_i,q} \le g(x) + \varepsilon$. and for $x = q$. $f_q(q) > g(q) - \varepsilon$

Let $V_q = \{x \mid g(x) - \varepsilon < f_q(x)\}$. open nbd of q.

Then $\bigcup_{i}^{m} V_{q_i}$ cover X. $f = \max_{1 \le i \le m} \{f_{q_i}\} \Rightarrow |f(x) - g(x)| < \varepsilon$, $\forall x \in X$

$\therefore \forall \varepsilon$, $\exists f \in \mathcal{F}$, s.t. $\forall x \in X$. $|f(x) - g(x)| < \varepsilon$, $\therefore g \in \overline{\mathcal{F}}$!

Stone-Weierstrass Approximation Theorem: $A \subseteq C(X, \mathbb{R})$, close subalgebra.

If A seperates points in X. (i.e. $\forall p, q \in X$. $p \ne q$. then $\exists f \in A$. st $f(p) \ne f(q)$) Then

(1) $\bar{A} = C(X, \mathbb{R})$

(2) $\exists x_0 \in X$. $\bar{A} = \{f \in C(X, \mathbb{R}) \mid f(x_0) = 0\}$

Pf: Case 1. $\forall x_0 \in X$. $\exists f \in A$, st $f(x_0) \ne 0$.

then $\forall (p, q) \in X \times X$. $p \ne q$. $\exists g \in A$.

st. $0 \ne g(p) \ne g(q) \ne 0$. By linear combination of g, g^2, we can take any assigned value

$\longrightarrow$ choose h, st. $h(p) \ne h(q)$ if $h(p) = 0$ or $h(q) = 0$ choose p or q. st $\exists f$. $f(p)$ or $f(q) \ne 0$.

$\therefore$ ~~$g = h + f$~~. which satisfies this condition!

at p.q. By $\begin{cases} c_1 g(p) + c_2 g^2(p) = \square \\ c_1 g(q) + c_2 g^2(q) = \Delta \end{cases}$

$\therefore$ By g, g^2. there can approximate every f. conti.

at p.q. Then by Lemma. 2. $\bar{A} = C(X, R)$

Case. 2. $\exists x_0$. st $\forall f \in A$. $f(x_0) = 0$.

$A_0 = R + A = \{c + f \mid c \in R, f \in A\}$. satisfies Case 1.

By Case 1. $\bar{A}_0 = C(X, R)$ $\therefore \bar{A} = \{f \in C(X, R) \mid f(x_0) = 0\}$

since $A \subset A_0$. $\bar{A} \subset \bar{A}_0 = C(X, R)$

Baire Space

Def. Empty Interior: If $\forall$ open set in $X \not\subset A$. except $\emptyset$. then A is empty interior

$\Leftrightarrow$ $\forall$ open set intersects $X - A \neq \emptyset$. $\Leftrightarrow$ $X - A$ is dense.

Def. Baire Space: X is baire space. if $\forall \{A_n\}$. which is countable family of close sets having empty interior. then $\bigcup A_n$ also has empty interior.

$\Leftrightarrow$ Replace open set by close set. then if A_n is dense. $\forall n$. $\bigcap A_n$ is also dense.

Lemma: Open subspace of Baire Space is Baire space.

pf: $\{A_n\}$ countable close sets in subspace Y.

$\bar{A}_n$ is closure of A_n 在 X. $\bar{A}_n \cap Y = A_n$

If $\bar{A}_n$ contains some open set B. then

$B \subseteq A_n$, too $\therefore B \cap Y \subseteq A_n \cap Y$. contradict!

$\therefore \{\bar{A}_n\}$ has empty interior in X. If $\exists$ open set $U \subseteq Y$.

$U \subseteq \bigcup A_n \Rightarrow U \subseteq \bigcup \bar{A}_n, U \underset{open}{\subseteq} X$. contradict!

Baire category

Theorem: cpt Hausdorff Space or Complete metric space is Baire Space.

pf: for $\{A_n\}$ in X. $\forall U \underset{open}{\subseteq} X$. prove: $\exists x \in U$. $x \notin \bigcup A_n$.

$\because A_1$ has empty interior $\therefore \exists a_1$. $a_1 \in U$, $a_1 \notin A_1$

for U. $\bar{U}_2 \cap A_1 = \emptyset$.

$\Rightarrow$ By regularity of X, $\exists U_2$, $a_1 \in U_2 \subset \bar{U}_2 \subset U$.

$\exists a_2$, $a_2 \in U_1$, $a_2 \notin A_2$, $\therefore \exists U_3$. $a_2 \in U_3 \subset \bar{U}_3 \subset U_2$, $\bar{U}_3 \cap A_2 = \emptyset$

... $\bar{U}_n \cap A_{n-1} = \emptyset$. $\bar{U}_n \subseteq U_{n-1}$

Then we have $\bar{U}_1 \supseteq \bar{U}_2 \supseteq \cdots \supseteq \bar{U}_n \supseteq \cdots \Rightarrow$ nested seq.

1) If X is cpt Hausdorff. Then by finite intersection property of $\{\bar{U}_n\}$. $\exists u \in \bigcap \bar{U}_n$. $u \cap A_n = \emptyset$. $\forall n$.

2) If X is complete metric. Since $\operatorname{diam} U_n \to 0$. (Let $\operatorname{diam} U_n < \frac{1}{n}$)

$\therefore$ for every $\bar{U}_n$. choose x_n, then $\{x_n\}$ cauchy seq.

$\Rightarrow \{x_n\} \to x$. $x \in \bar{U}_n$, $\forall n$. $\therefore x \in \bigcap \bar{U}_n$.

Remark: 1. Locally cpt Hausdorff is Baire Space.

Since its one-point compactification

is Baire Space. And it's open subspace

of a Baire Space!

2. If Y is G_δ set under X. which

is cpt Hausdorff or complete metric.

Then Y is Baire Space! e.g.

$\mathbb{R}/\mathbb{Q} = \bigcap_{n=1}^{\infty} \mathbb{R}/\{r_n\}$, $\{r_n\} = \mathbb{Q}$. with order ⇒ Baire!

Some conclusions:

1. X is Baire Space. if $X = \cup B_n$.

Then $\exists$ at least one k. $\bar{B}_k$ has interior

Pf: $X = \cup B_n \subset \cup \bar{B}_n = X$. if $\{\bar{B}_n\}$ is

family of close empty interior sets.

Then X is an open set contained in $\overline{\cup B_n}$!

2. If X is locally Baire. then X is Baire

Pf: $\{A_n\}$ in X. If $\exists\, U \underset{open}{\subseteq} X$. $U \subseteq \cup A_n$

Then $\exists\, x \in U$, $\exists\, U_x$ of x. Baire.

$x \in U_x \subseteq U$. And $A_n \cap U_x$ close in U_x

$U_x \subseteq (\bigcup A_n) \cap U_x = \bigcup (A_n \cap U_x)$, However.

$\text{int}_{U_x} (A_n \cap U_x) = U_x \cap \text{int} (A_n \cap U_x)$

$= \text{int} (A_n \cap U_x) \subseteq \text{int} A_n = \emptyset$. Contradiction!

3. X is Baire Space. $f_n \to f$ under point convergence. $f_n : X \to Y$. conti. then $f : X \to Y$. conti on a dense set.

pf: Def: $A_N(\varepsilon) = \{x \mid d(f_n(x), f_m(x)) \le \varepsilon, \forall n, m > N\}$.

Then it's close. And $A_N(\varepsilon) \subseteq A_{N+1}(\varepsilon)$

$\bigcup_{N=1}^{\infty} A_N(\varepsilon) = X$. because $\forall x_0 \in X$. $\forall \varepsilon$. $\exists N'$.

$d(f_n(x_0), f_m(x_0)) \le \varepsilon$. if $n, m \ge N'$

Then. $U(\varepsilon) = \bigcup_{N=1}^{\infty} \text{int} A_N(\varepsilon)$, $C = \bigcap_{k=1}^{\infty} U(\frac{1}{k})$

Prove: $U(\varepsilon)$ is dense. $\forall x_0 \in C$. $f(x)$ conti at x_0.

Pf: $\forall U \underset{\text{open}}{\subseteq} X$. since $\bigcup (\underline{U \cap A_N(\varepsilon)}) = U$. close in U.

$\therefore \exists U \cap A_N(\varepsilon)$ has nonempty interior.

$\therefore \exists N$. st. $U \cap \text{int} A_N(\varepsilon) \ne \emptyset$.

pf: $x_0 \in C = \bigcap U(\frac{1}{k})$. $\exists N$. st $\frac{1}{N} < \frac{\varepsilon}{3}$.

and $x_0 \in \text{int} A_k(\frac{1}{N})$. $\exists U_{x_0}$ of x.

$U_{x_0} \subseteq \text{int} A_k(\frac{1}{N})$ and $d(f_k(x_0), f_k(x)) < \frac{\varepsilon}{3}$.

for $\forall x \in U_{x_0}$.

Then claim: $x \in U_{x_0}$. then $d(f(x), f(x_0)) < \varepsilon$.

$\exists k' > k$. $d(f_{k'}(x_0), f(x_0)) < \frac{\varepsilon}{3}$. And by $x \in \text{int} A_k(\frac{1}{N})$

$\therefore d(f_{k'}(x), f_{k''}(x)) < \frac{\varepsilon}{3}$. let $k'' \to +\infty$, done!