Random Spanning Trees.

(1) Uniform spanning trees:

Consider on finite connected graph D. with (V, E). where E can contain multiple edges.

<u>Def</u>: i) A spanning tree in D is a subset T of E st. (V, T) is a tree which doesn't contain a cycle and uses edges only once, connected.

ii) A uniform spanning tree (UST) in D is a random tree $\mathcal{T}$ that chooses uniformly among all spanning trees of D.

<u>Rmk</u>: Note that ST must use up the vertices of D.

<u>Def</u>: For $D \subset \mathbb{Z}^d$. finite subset

i) $\partial D = \{ x \in \mathbb{Z}^d \mid d(x, D) = 1 \}$, $\bar{D} = D \cup \partial D$.

ii) $\bar{E} = \bar{E}_{\bar{D}}$ set of edges at least one extremity in D.

iii) Next, we see all points in ∂D as one abstract point x_0.

set $\hat{D} = D \cup \{x_0\}$. with set of edges $\hat{E}$. def by replacing $\{x, y\}$ with $\{x, x_0\}$. if $y \in \partial D$. in E

iv) Denote this bijection map : $E \to \hat{E}$ by Ψ.

② **Loop-erased RWs:**

For $(z_0, e_1, z_1, e_2 \cdots e_m, z_m)$, s.t. $z_0 = z_m$.

e_i connects z_{i-1} and z_i. $e_i \in \hat{E}$. $z_i \in \hat{D}$.

It's called a loop with length m.

$\Rightarrow$ Denote it by $(z_0, z_1 \cdots z_m)$ simply.

<u>Def</u>: (Loop-erasure on path)

For path $z = (z_0, \cdots z_m) \in (\mathbb{Z}^d)^{m+1}$.

loop erasure of z is $L(z) = (L_0 \cdots L_\sigma)(z)$

defined iteratively : $L_0 = z_0$. $r_j = \max\{r \le m \mid z_r = L_j\}$. $L_{j+1} = z_{r_j+1}$. $\sigma = \min\{j \mid L_j = z_m\}$

<u>Rmk</u>: i) We erase loop of z chronologically

ii) The edge from L_j to L_{j+1} is the edge from z_{r_j} to z_{r_j+1} in z.

<u>e.g.</u> $\underline{1}\ \underline{2}\ \cancel{3}\ \cancel{4}\ \cancel{3}\ \cancel{2}\ \cancel{1}\ \cancel{2}\ \underline{1}$.

<u>Lemma.</u> For path $z = (z_0, z_1 \cdots z_\tau)$ of SRW.

τ is exit time of D. $z_0 \in D$. Then.

$\mathbb{P}(L(z) = (\eta_0, \cdots \eta_s)) =$

$(2d)^{-s}\, h_D(\eta_0, \eta_0)\, h_{D\setminus\{\eta_0\}}(\eta_1, \eta_1) \cdots h_{D\setminus\{\eta_0 \cdots \eta_{s-2}\}}(\eta_{s-1}, \eta_{s-1})$

Pf: $LHS = (2d)^{-s} \prod_{1}^{s} \sum_{k_i \geq 0} (2d)^{-k_i}$ #{ path $\eta_{i-1} \to \eta_{i-1}$

with k_i steps in $D/\{\eta_0 \cdots \eta_{i-2}\}$ }

$(2d)^{-s}$ is prob. of jump of each step.

$\sum \square$: means the prob. of erased loops

from $\eta_i \to \eta_i$ with finite length k_i.

② Wilson's Algorithm:

Next, we construct random tree in $D = \{x_i\}_1^n$.

i) (Z_n) is SRW starting at x_1. τ is the

first exit of D. Set $Z = (Z_k)_{k \leq \tau}$.

Consider Loop-erasure $L(Z) \triangleq X^{(1)}$

ii) Set $X^{(k)} = \begin{cases} X^{(k-1)}, & \text{if } x_k \in X^{(k-1)} \\ L(\tilde{Z}) \cup X^{(k-1)}, & \text{if } x_k \notin X^{(k-1)} \end{cases}$

where $\tilde{Z}$ is SRW starting at x_k, and stop

when it first exits $D / X^{(k-1)}$.

iteratively for each $k \in \{2, 3, \dots n\}$.

iii) We obtain $\mathcal{E}(X^{(n)}) = \mathcal{T}$ is random tree of $\hat{D}$.

Prop. Law of $\mathcal{T}$ is UST of $\hat{D}$.

Pf: 1°) For a outcome T for tree $X^{(n)}$.

First, relabel the points of D:

Denote path of SRW from x_1 to ∂D

by $\eta_1, \eta_2, \cdots \eta_{s-1}$.

Next choose the smallest index j: s.t.

$x_j \notin \{\eta_i\}_1^{s-1}$. generate a SRW begin

at x_j. until hitting $\partial D \cup \{\eta_i\}_1^{s-1}$.

Denote the path by $\eta_s, \cdots \eta_{s'-1}$.

Iteratively until D is exhausted.

$\Rightarrow$ Obtain $\eta_1 \cdots \eta_n$. $\{\eta_i\} = \{x_i\}$.

2°) By Lemma above.

$$\mathbb{P}(\mathcal{T} = T) = (2d)^{-n} \prod_{j=1}^{n} h_{D/\{\eta_1 \cdots \eta_{j-1}\}}(\eta_j, \eta_j)$$

$$= (2d)^{-n} |h_D|$$

doesn't depend on order of $\{\eta_i\}_1^n$.

__Cor.__ Number of spanning trees in $\hat{D}$

$= (2d)^n |-\bar{\Delta}_D|$.

__Rmk:__ i) $\mathbb{P}(\mathcal{T}$ is constructed without any

erasing $| \mathcal{T} = T) = (2d)^{-n} / \mathbb{P}(\mathcal{T} = T)$

$= |-\bar{\Delta}_D| = 1/|h_D|$.

ii) Denote λ is the loop from x_1 to

x_1 that erased when performing the

algorithm. which dist. like $Z =$

$(z_0, \cdots z_\ell)$. $z_0 = x_1$. ℓ is the last time at x_1 before hitting x_0 firstly.

<u>prop.</u> $\mathbb{P}(\lambda = (z_0, \cdots z_r)) = (2d)^{-r} / h_D(x_1, x_1)$

<u>Pf:</u> $LHS = \mathbb{P}_{x_1}((X_0, X_1 \cdots X_r) = (z_1, z_1 \cdots z_r),\ \widetilde{H}_{\{x_1\}} \circ \theta_r > T_D \circ \theta_r)$

$\overset{mp}{=} \mathbb{P}_{x_1}((X_0 \cdots X_r) = (z_0, \cdots z_r))\ \mathbb{P}_{x_1}(\widetilde{H}_{\{x_1\}} > T_D)$

$= (2d)^{-r} / h_D(x_1, x_1)$.

$\mathbb{P}_{x_1}(\widetilde{H}_{\{x_1\}} > T_D) = 1/h_D(x_1, x_1)$ follows from the last exit id.

③ <u>Generalisations:</u>

i) <u>With killing:</u>

Consider in finite graph $\hat{D}_\partial = \hat{D} \cup \{\partial\}$.

endowed with killing measure $k = (k(x_i))_{x_i \in \widetilde{E}_\partial}$

<u>Denote</u>: $c_e = \begin{cases} k(x_i). & e = \{x_i, \partial\}. \\ 1/2d. & e \in \hat{E}. \end{cases}$

<u>Thm.</u> $\mathbb{P}(\mathcal{T} = T =: (\eta_1, \cdots \eta_n)) = \prod_{e \in T} c_e \cdot \det h_{D,k}$.

<u>Pf:</u> $LHS = \prod_{T} c_e \cdot \hat{\prod_{i}} h_{D/\{\eta_1 \cdots \eta_{k-1}\}, k}(\eta_k, \eta_k)$

ii) <u>General Graph:</u>

$D = (V, \mathcal{E})$ is finite connected graph. (d_i) is number of neighbour of x_i in D.

Thm. $\mathbb{P}(\mathcal{T} = T) = \prod_1^n 1/\lambda_i \cdot \det h_D$

iii) **Weighted Spanning Trees:**

Def: A c-weighted spanning tree in D is a random spanning tree $\mathcal{T}$. s.t.

$\mathbb{P}(\mathcal{T} = T) = \prod_{e \in T} c_e \Big/ \sum_T \prod_{e \in T} c_e$. Where $c = c(e)$ is transition prob. along e.

Thm Consider $c = c(e)$ is conductance func. on E. Then the random tree $\mathcal{T}$ constructed by Wilson Algorithm is a c-weighted spanning tree.

Besides, $P(\mathcal{T} = T) = \prod_{e \in T} c_e \cdot \det(h_{D,c})$

Pf: Note $P(\mathcal{T} = T) \propto \prod_{e \in T} c_e$.

Rmk: It's not UST.

④ **Conti. Case Wilson Algorithm:**

Consider replace SRW in Wilson Algorithm above by conti. time RW (X_t). with the jump rate $1/2d$ along each edge and holding time $\sim \operatorname{Exp}(1)$

(2) Occupation Field:

Def: i) Consider Wilson's Algorithm constructing UST of $\hat{D}$ in discrete time. For $x \in D$. Set $V(x)$ is cumulative time spent at x by all SRW in the algorithm.

ii) Similar as above: set $W(x)$ is occupation time spent at x in conti. case.

iii) $V = (V(x))_{x \in D}$. $W = (W(x))_{x \in D}$ are called occupation fields in Wilson's algorithm.

Rmk: For $(\xi_{i,j})_{j \leq n, i \geq 1} \overset{i.i.d}{\sim} \exp(1)$. indept of V. we have:

$$(\widetilde{W}(x_j))_{j=1}^n =: (\sum_1^{V(x_j)} \xi_{i,j})_{j=1}^n \overset{d}{\sim} W = (W_i)_0$$

prop. i) $V(x_1) \sim \text{Geo}(1/h_D(x_1, x_1))$.

ii) $W(x_1) \sim \exp(1/h_D(x_1, x_1))$

Pf: i) $\mathbb{E}_{x_1}(V(x_1)) = \mathbb{E}_{x_1}(\sum_{k=0} I\{X_k = x_1\}) = h_D(x_1, x_1)$

$\mathbb{P}_{x_1}(V(x_1) > m+n \mid V(x_1) > m) =$

$\mathbb{P}_{x_1}(V(x_1) \circ \theta_{H_{x_1}^m} > n \mid V(x_1) > m) \overset{m.p.}{=}$

$\mathbb{P}_{x_1}(V(x_1) > n)$. $\Rightarrow$ Memoryless prop.

ii) Similar as i)

prop. (Laplace Transf.)

For $k: D \to \mathbb{R}^{\geq 0}$. Then, we have

$$\mathbb{E}(e^{-\sum_i^n k(x_i) W(x_i)}) \overset{1)}{=} \mathbb{E}(\prod_i^n (1+k(x_i))^{-V(x_i)})$$

$$\overset{2)}{=} \det(-\bar{\Delta}_D) / \det(-\bar{\Delta}_D + I_k)$$

where $I_k = \mathrm{diag}\{k(x_1), \cdots k(x_n)\}$.

Rmk: Note that W and V are indept of the constructed spanning trees T since the order of (x_i) doesn't influence W, V.

Pf: 1°) $LHS = \mathbb{E}(\mathbb{E}(e^{-\sum_{i=1}^n \sum_{j=1}^{V(x_i)} k(x_i) \xi_{ij}} | V))$

$= \mathbb{E}(\prod_i^n \mathbb{E}(e^{-k(x_i)\xi_{ij}})^{V(x_i)})$

$= RHS$

2°) Work on conti-time case:

Coupling T with killing version T'

T' is a spanning tree constructed from Wilson's Algorithm on conti. time RW with killing measure $k = (k(x_i))_{i \leq n}$.

Denote $\widetilde{E}_\partial \triangleq \{\{x_i, \partial\}\}_{1 \leq i \leq n}$.

Note $T' \overset{d}{=} T \iff T' \cap \widetilde{E}_\partial = \emptyset$.

$\mathbb{P}(\mathcal{J}' \cap \widetilde{E}_0 = x \mid \mathcal{J} = T) =$

$\mathbb{P}(\forall x_k,\ 1 \le k \le n$. RW won't be killed in the interval $W(x_k) \mid \mathcal{J} = T)$

$$\overset{\text{indpt}}{=} \prod_1^n \mathbb{P}(\text{No kill in } W(x_k) \mid \mathcal{J} = T)$$

$$\overset{\text{Poi.}}{=} \prod_1^n e^{-k(x_i) W(x_i)}$$

$$\Rightarrow \mathbb{P}(\mathcal{J}' \cap \widetilde{E}_0 = x) = \mathbb{E}(e^{-\sum_1^n k(x_i) W(x_i)})$$

$$\text{LHS} = \sum_{T \in \hat{D}} \mathbb{P}(\mathcal{J}' \cap \widetilde{E}_0 = x,\ \mathcal{J}' = T)$$

$$= \sum_{T \in \tilde{D}} \mathbb{P}(\mathcal{J}' = T).$$

$$= \sum_{\hat{D}} (2d)^{-n} \cdot |\hat{G}_{D,k}| = |G_{D,k}| / |G_D|$$

<u>Cor.</u> Suppose (φ^1), (φ^2) are indpt GFF with Dirichlet boundary in D

$$\Rightarrow (\varphi^1)^2/2 + (\varphi^2)^2/2 \sim W$$

<u>Pf</u>: Directly by Laplace Transf.

<u>Rmk</u>: If set $\widetilde{V}(x_i) = V(x_i) - 1 \ge 0$.

Then: $\mathbb{E}(\prod_1^n (1 + k(x_i))^{-\widetilde{V}(x_i)}) =$

$$\prod_1^n (1 + k(x_i)) \cdot \frac{|I - \bar{\Delta}_D|}{|I - \bar{\Delta}_D + \mathcal{I}k|}$$

$$= |I - \bar{\Delta}_D| / |I - \bar{\Delta}_{D,k}|.$$