Gaussian Free Field $\mathbb{Z}^d$.

(1) Definitions and Properties:

① Denote: $\Omega = \mathbb{R}^{\mathbb{Z}^d}$. $\Omega_\Lambda = \mathbb{R}^\Lambda$. $(\Lambda \overset{\text{finite}}{\subset} \mathbb{Z}^d)$

For each $i \in \mathbb{Z}^d$. refer ω_i to spin at i.

To define the Hamiltonian of GFF. we have following natural assumptions :

i) Only spins at nearest-neighbours vertices of $\mathbb{Z}^d$ will interact.

ii) Interaction favours agreement of neighbour spins.

iii) Favour Localization of spin ω_i near zero, i.e. low - energy state.

Def: i) Hamiltonian of the GFF in $\Lambda \subset\subset \mathbb{Z}^d$ is :

$$\mathcal{H}_{\Lambda,\beta,m}(\omega) =: \frac{\beta}{4d} \sum_{\substack{i \sim j \\ \{i,j\} \in E(\bar{\Lambda})}} (\omega_i - \omega_j)^2 + \frac{m^2}{2} \sum_{\Lambda} \omega_i^2 .$$

where β is inverse temperature. $m \geq 0$ is mass.

$E(\bar{\Lambda}) = \{ e = \{x,y\} \in \mathbb{Z}^d \mid \exists x \text{ or } y \in \Lambda \}$

<u>Rmk</u>: Denote $|\nabla F(e)| = |F(x) - F(y)|$

Set $\Sigma_\Lambda(F) =: \sum_{e \in E(\bar{\Lambda})} |\nabla F(e)|^2$. which

is Dirichlet energy of F in Λ.

ii) $\mathcal{F} = \sigma(\{\text{cylinders of } B_\Lambda = \bigotimes_\Lambda B_{\mathbb{R}^1}\}_{\Lambda \subset\subset \mathbb{Z}^d})$

For $\Lambda \subset\subset \mathbb{Z}^d$. $\eta \in \Omega$. Gibbs dist. for GFF in Λ with b.c. η is a p.m.:

$\forall A \in \mathcal{F}$. $\mu^\eta_{\Lambda,\beta,m}(A) = \int e^{-\mathcal{H}_{\Lambda,\beta,m}(\omega_\Lambda \eta_{\Lambda^c})} I_A(\omega_\Lambda \eta_{\Lambda^c}) \prod_\Lambda d\omega_i \,/\, Z^\eta_{\Lambda,\beta,m}$. Where $Z^\eta_{\Lambda,\beta,m} =: \int e^{-\mathcal{H}_{\Lambda,\beta,m}(\omega_\Lambda \eta_{\Lambda^c})} d\prod \omega_i$

<u>Rmk</u>: i) $Z^\eta_{\Lambda,\beta,m} < \infty$. $\forall \eta \in \Omega$. $m \geq 0$. $\beta > 0$.

<u>Pf</u>: Cut off some edges in $E(\bar{\Lambda})$.

St. produce a tree: 

remove each leaves per time.

ii) For free b.c. &. $Z^{\varnothing}_{\Lambda,\beta,0} = \infty$.

<u>Pf</u>: The problem is that we can shift ω_i without increasing energy at i.

$\Rightarrow$ let some ω_i disappear in $\mathcal{H}$.

So $Z^{\varnothing}_{\Lambda,\beta,0} = (\int \square) \cdot \int_{\mathbb{R}} d\omega_i = \infty$

③ Distribution:

Set r.v. $\varphi_i : \Omega \to \mathbb{R}$. $\varphi_i(\omega) =: \omega_i$.

First, we will prove: $\varphi_{B(n)} \sim \mu^{n}_{B(n), m}$ is Gaussian vector. $\Lambda \subset\subset B(n) = [-n, n]^d \subset \mathbb{Z}^d$.

<u>Thm.</u> (Wick's Formula)

For φ_Λ is centered Gaussian vector.

i) $\mathbb{E}_\Lambda(\varphi_{i_1} \cdots \varphi_{i_{2n+1}}) = 0$. for $i_j \in \Lambda$.

ii) $\mathbb{E}_\Lambda(\varphi_{i_1} \cdots \varphi_{i_{2n}}) = \sum_{g} \prod_{\{l,l'\} \in g} \mathbb{E}_\Lambda(\varphi_{i_l} \varphi_{i_{l'}})$

Sum is over all pairings partition of $\{1, \cdots 2n\}$. i.e. $g = \Sigma\{l, l'\}$.

<u>Pf:</u> i) is odd function

ii) By $\mathbb{E}(e^{\sum_i^{2k} \lambda_i \varphi_i}) = e^{\frac{1}{2}\mathbb{E}((\sum_i^{2k} \lambda_i \varphi_i)^2)}$. (Taylor)

<u>Def.</u> $\varphi = (\varphi_i)_{\mathbb{Z}^d}$ is Gaussian field if $\forall \Lambda \subset\subset \mathbb{Z}^d$. $\varphi_\Lambda = (\varphi_i)_{i \in \Lambda}$ is Gaussian.

<u>Thm.</u> For $\varphi_{B(n)} \sim \mathcal{N}(\mu_{B(n)}, \Sigma_{B(n)})$. If $\forall i, j$. $\mu_i = \lim_{n \to \infty} (\mu_{B(n)})_i$. $\Sigma_{ij} = \lim_{n \to \infty} (\Sigma_{B(n)})_{ij}$ exists and finite. Then:

i) $\forall \Lambda \subset\subset \mathbb{Z}^d$. $\varphi_\Lambda = (\varphi_i)_\Lambda \xrightarrow{n\to\infty} N(\mu_\Lambda, \Sigma_\Lambda)$.

s.t. $\mu_\Lambda, \Sigma_\Lambda$ are restricted on μ, Σ.

ii) $\exists \tilde{\varphi}$ Gaussian field. $\forall \Lambda \subset\subset \mathbb{Z}^d$.

we have: $\tilde{\varphi}|_\Lambda \sim N(\mu_\Lambda, \Sigma_\Lambda)$.

Pf: i) Note it's limit of Gaussian dist.

ii) By Kolmogorov extension.

i) Discrete Green Id.:

Def: For $f = (f_i)_{\mathbb{Z}^d}$. For $e = \{i, j\} \in E(\mathbb{Z}^d)$.

$(\nabla f)_{ij} =: f_j - f_i$. $(\Delta f)_i =: \sum_{j\sim i} (\nabla f)_{ij}$

Lemma. (Discrete Green Id.)

For $\Lambda \subset\subset \mathbb{Z}^d$. $f = (f_i)_{\mathbb{Z}^d}$. $g = (g_i)_{\mathbb{Z}^d}$. Then:

$$\sum_{\{i,j\}\in E(\bar{\Lambda})} (\nabla f)_{ij} (\nabla g)_{ij} = -\sum_{i\in\Lambda} g_i (\Delta f)_i + \sum_{\substack{i\in\Lambda \\ j\in\Lambda^c,\, j\sim i}} g_j (\nabla f)_{ji}$$

$$\sum_{i\in\Lambda} [f_i (\Delta g)_i - g_i (\Delta f)_i] = \sum_{\substack{i\in\Lambda \\ j\in\Lambda^c,\, i\sim j}} [f_j (\nabla g)_{ij} - g_j (\nabla f)_{ij}]$$

Pf: Only prove the former. Since the latter is difference of the former.

Next, assume $i \sim j$ in the following:

1') $\sum_{\{i,j\}\subset\Lambda} (\nabla f)_{ij}(\nabla g)_{ij} = -\sum_{\Lambda} g_i(\Delta f)_i + \sum_{\Lambda} g_i \sum_{j\in\Lambda^c}(f_j - f_i)$

2') $\sum_{\{i,j\}\in E(\bar\Lambda)} (\nabla f)_{ij}(\nabla g)_{ij} = \sum_{\{i,j\}\subset\Lambda} + \sum_{i\in\Lambda,\, j\in\Lambda^c}$ $\square$

<u>Def</u>: i) $(A_{ij})_{\mathbb{Z}^d}$. $A_{ij} = \begin{cases} -2d. & i=j \\ 1. & i\sim j \\ 0. & \text{otherwise} \end{cases}$ Denote

$\Delta_\Lambda = (A_{ij})_{i,j\in\Lambda}$.

<u>Rmk</u>: $(\Delta f)_i =: \sum_{j\in\mathbb{Z}^d} A_{ij} f_j$ for $f = (f_{ij})_{\mathbb{Z}^d}$.

ii) $f\cdot\Delta_\Lambda g =: \sum_{i,j\in\Lambda} \Delta_{ij} f_i g_j$

<u>Rmk</u>: By sym of Δ. $\Rightarrow f\cdot\Delta_\Lambda g = (\Delta_\Lambda f)\cdot g$.

ii) For massless case: to investigate Hamilton $\mathcal{H}_{\Lambda,0}$:

Assume $f = \eta$ outside Λ. and denote B_Λ as the boundary term (i.e. only depend on $\bar\Lambda^c$)

1') By Green Id.:

$$\mathcal{E}_\Lambda(f) = \sum_{\{i,j\}\in E(\bar\Lambda)} (f_i - f_j)^2 = \sum_{E(\bar\Lambda)} (\nabla f)_{ij}^2$$

$$\overset{\text{Green}}{=} -f\cdot\Delta_\Lambda f - 2\sum_{\substack{i\in\Lambda,\, j\in\Lambda^c \\ i\sim j}} f_i f_j + B_\Lambda.$$

2') Introduce $u = (u_i)_{\mathbb{Z}^d}$. Rewrite $-f\cdot\Delta_\Lambda f$:

$$-(f-u)\cdot\Delta_\Lambda(f-u) = f\cdot\Delta_\Lambda f - 2\sum_{i\in\Lambda} f_i(\Delta u)_i$$

$$+2\sum_{\substack{i\in\Lambda,\, j\in\Lambda^c \\ i\sim j}} f_i u_j + B_\Lambda.$$

$$\Rightarrow \ \mathcal{E}_\Lambda(f) = -(f-u)\cdot\Delta_\Lambda(f-u) - 2\sum_{i\in\Lambda} f_i(\Delta u)_i$$

$$+2\sum_{\substack{i\in\Lambda,\, j\in\Lambda^c \\ i\sim j}} f_i(u_j-f_j) + B_\Lambda.$$

To cancel the nonboundary terms except

$-(f-u)\cdot\Delta_\Lambda(f-u)$:

__Def__: u is harmonic in Λ if $(\Delta u)_i = 0,\ \forall i\in\Lambda$.

__Lemma__. There exists at most one solution for Dirichlet problem in Λ with b.c. η:

$$\begin{cases} (\Delta u)_i = 0, & \forall i\in\Lambda \\ u_i = \eta_i, & \forall i\in\Lambda^c \end{cases}$$

__Pf__: By MVT. and Max principle Thm.

__Rmk__: Choose u as the solution. Then:

$$\mathcal{E}_\Lambda(f) = -(f-u)\cdot\Delta_\Lambda(f-u) + B_\Lambda.$$

$$\Rightarrow \text{So: } \mathcal{H}_{\Lambda,\eta} = \frac{1}{2}(w-u)\left(\frac{1}{2d}\Delta_\Lambda\right)(w-u)$$

3') Random Walk representation:

We wonder if $\exists\ \Sigma_\Lambda$ positive-definite covariance matrix. s.t. $\Sigma_\Lambda^{-1} = \frac{1}{2d}\Delta_\Lambda$.

First, note that $\frac{1}{2d}\Delta_\Lambda = I_\Lambda - P_\Lambda$.

$P_\Lambda = (P_{ij})_{i,j\in\Lambda}$. $P_{ij} =: \begin{cases} 1/2d & . \ i\sim j \\ 0 & . \text{ otherwise.} \end{cases}$

Rmk: $(P_{ij})_{\mathbb{Z}^d}$ is trans. prob. for RSW. $(X_k)_{k\in\mathbb{Z}^+}$ on $\mathbb{Z}^d$.

Lemma. For $\Lambda \subset\subset \mathbb{Z}^d$. $\tau_{\Lambda^c} =: \inf\{n\geq 0 \mid X_n \in \Lambda^c\}$.

We have: $\mathbb{P}_i(\tau_{\Lambda^c} < \infty) = 1$. $\exists\ c = c(\Lambda) > 0$. s.t. $\forall i\in\Lambda$. $\mathbb{P}_i(\tau_{\Lambda^c} \geq n) \leq e^{-cn}$.

where $\mathbb{P}_i$ is law of X start at i

Pf: $R = \sup_{l\in\Lambda} \inf_{k\in\Lambda^c} \|k-l\|$.

$\Rightarrow$ During time interval R. $\exists$ prob. at least $(2d)^{-R}$ s.t. X exits Λ.

$\Rightarrow P_i(\tau_{\Lambda^c} > n) \leq (1-(2d)^{-R})^{[n/R]}$.

Lemma. $G_\Lambda = (I_\Lambda - P_\Lambda)^{-1}$ exists. symmetric. given by green func. in Λ of RSW: $i,j\in\Lambda$.

$G_\Lambda(i,j) =: \mathbb{E}_i\left(\sum_0^{\tau_{\Lambda^c}} I\{X_n = j\}\right)$.

Pf: Note that $I_\Lambda - P_\Lambda^{n+1} = (I_\Lambda - P_\Lambda)(I_\Lambda + \cdots + P_\Lambda^n)$

$$P_\Lambda^n(i,j) = P_i(X_n = j,\ \tau_{\Lambda^c} > n) \lesssim e^{-cn}.$$

Let $n \to \infty$. $\therefore G_\Lambda(i,j) = \sum P_\Lambda^n(i,j)$

Rmk: The key of $I_\Lambda - P_\Lambda$ is invertible

is from: P_Λ is substochastic.

i.e. $\sum_{j\in\Lambda} P_\Lambda(i,j) < 1$. $\forall i \in \partial_{in}\Lambda$.

Lemma The unique solution for Dirichlet problem in Λ with b.c η is $u = (u_i)_{\mathbb{Z}^d}$:

$$u_i =: \mathbb{E}_i(\eta_{X_{\tau_{\Lambda^c}}}).\quad \forall i \in \mathbb{Z}^d.$$

Pf: 1) $i \in \Lambda^c \Rightarrow \tau_{\Lambda^c} = 0$. $u_i = \eta_i$

2) $i \in \Lambda \Rightarrow$ First step analysis:

$$u_i = \sum_{j\sim i} P_i(X_1 = j)\, \mathbb{E}_i(\eta_{X_{\tau_{\Lambda^c}}} \mid X_1 = j)$$

$$\overset{(mp)}{=} \sum_{j\sim i} \frac{1}{2d} u_j.$$

Thm. Under $\mu_{\Lambda,0}^\eta$. $\varphi_\Lambda = (\varphi_i)_\Lambda$ is Gaussian with mean $\mu_\Lambda = (u_i)_\Lambda$ and covariance G_Λ def in above.

Rmk: We have alternative def on GFF:

Def: (By density)

Discrete GFF in Λ with Dirichlet b.c. (i.e. Zero b.c.) on $\partial\Lambda$ is the centered Gaussian vector $(\Gamma(x))_{x\in\Lambda}$. with density on $\mathbb{R}^n$ at (φ_x) is :

$$e^{-\frac{1}{2}\cdot \mathcal{E}_\Lambda(\varphi)/2d} \cdot {}^1/_Z .$$

Note that $\varphi \mapsto \mathcal{E}_\Lambda(\varphi)$ is positive definite bilinear form. So it's Gaussian dist. in this sense.

4) Laplace operator and Green Func :

Prop. (Resampling procedure)

$x\in\Lambda$. $\Gamma(x) \mid (\Gamma(y))_{y\in\Lambda/\{x\}} \sim$ density at $\varphi_x \mid (h_y)_{y\in\Lambda/x} \propto \exp(-\frac{1}{2}(\varphi_x - \bar{h}(x))^2$

i.e. it's normal dist. $N(\bar{h}(x), 1)$.

Pf: Check the density : $e^{-\frac{1}{2}\cdot\frac{1}{2d}\sum_{y\sim x}|\varphi_x - h(y)|^2}$

expand it : $\bar{h}(x) = \frac{1}{2d}\sum_{y\sim x} h(y)$.

Rmk: i) It depends only on values $h(y)$ at neighbours y of x.

ii) It means: $\forall x \in \Lambda$. $\Phi(x) - \bar{\Phi}(x)$

is std. Gaussian r.v. indept of $(\Phi(y))_{\Lambda/\{x\}}$

Cor. i) For $x \neq y$. $\Sigma_x(y) =: \mathbb{E}(\Phi(x)\Phi(y)) = \bar{\Sigma}_x(y)$

ii) $\Sigma_x(x) = \bar{\Sigma}_x(x) + 1$

Rmk: It means: $\Delta \Sigma_x(y)/2d = -\mathbb{1}\{y=x\}$.

Pf: i) $\mathbb{E}((\Phi(x) - \bar{\Phi}(x))\Phi(y)) \overset{\text{indept}}{=}$

$\mathbb{E}(\square)\,\mathbb{E}(\Phi(y)) = 0$

ii) $\mathbb{E}(\Phi(x)^2) = \mathbb{E}(\Phi(x)\bar{\Phi}(x)) +$

$\mathbb{E}((\Phi(x) - \bar{\Phi}(x))^2) + \mathbb{E}((\Phi(x)-\bar{\Phi}(x))\bar{\Phi}(x))$

$= \bar{\Sigma}_x(x) + 1 + 0$

Def: $\mathcal{F}_\Lambda = \{ h : \mathbb{Z}^d \to \mathbb{R}' \mid h|_{\Lambda^c} = 0 \}$

$\bar{\Delta} F(x) = \frac{1}{2d} \sum_{y \sim x} (F(y) - F(x))$.

$\bar{\Delta}_D = \bar{\Delta}$ in D. $\bar{\Delta}_D = 0$ outside D.

Rmk: $\bar{\Delta}_D : \mathcal{F}_D \to \mathcal{F}_D$ is injective LO.

$\Rightarrow \bar{\Delta}_D$ is bijective LO.

Note that $-\bar{\Delta}_\Lambda \cdot \Sigma = I \Rightarrow -\bar{\Delta}_\Lambda G_\Lambda = I$

i.e. G_Λ is inverse of $-\bar{\Delta}_D$

For GFF $(\varphi_i)_\Lambda$ with general b.c. η:

We have: a) For $d \le 2$. $\lim_{n\to\infty} G_{B(n)}(0,0) = \mathbb{E}_0(\sum_i I_{\{X_k=0\}})$

$= \infty$. So Var of φ_0 diverges.

by recurrence of RSW in $d \le 2$.

more precisely. $G_{B(n)}(0,0) \overset{n\to\infty}{\sim} \begin{cases} n. & d=1 \\ \log n. & d=2 \end{cases}$

For $d \ge 3$. $\lim_{n\to\infty} G_{B(n)}(0,0) = G(0,0) < \infty$.

b) For $d \ge 3$. $G = (G(x,y))_{x,y \in \mathbb{Z}^d}$.

It's translation invariant:

$G(x,y) = G(0,y-x) =: g(y-x)$.

We have: $g(x) \sim (1+|x|)^{2-d}$. $d \ge 3$

(i.e. $(1+|x|)^{2-d} \lesssim g(x) \lesssim (1+|x|)^{2-d}$).

iii) Next, we investigate massive case:

consider $\mathcal{H}_{\Lambda,m}$ containing term $\frac{m^2}{2}\sum_{i\in\Lambda}\varphi_i^2$.

$\mathcal{H}_{\Lambda,m} = \frac{1}{2}(\varphi-u)\cdot(-\frac{1}{2d}\Delta_\Lambda + m^2)\cdot(\varphi-u) + \sum_\Lambda \varphi_i((-\frac{1}{2d}\Delta_\Lambda +$

$m^2)u)_i + \frac{1}{2d}\sum_{\substack{i\in\Lambda, j\in\Lambda^c \\ i\sim j}}\varphi_i(u_j-\varphi_j) + B_\Lambda$.

<u>Def</u>: $u = (u_i)_{\mathbb{Z}^d}$ is m-harmonic in Λ if:

$((-\frac{1}{2d}\Delta + m^2)u)_i = 0$. $\forall i \in \Lambda$.

We say u solves the massive Dirichlet problem in Λ with b.c. η. if:

$$\begin{cases} u \text{ is } m\text{-harmonic on } \Lambda. \\ u_j = \eta_i \text{ on } \Lambda^c. \end{cases}$$

Rmk: Note that $(m^2+1)u_i = \bar{u}_i$. So the maximal value won't show up inside Λ. $\Rightarrow$ satisfies MVT.

So: the solution is unique.

Next, we consider its Random Walk representation:

consider $\mathbb{Z}^d_* = \mathbb{Z}^d \cup \{*\}$. "$*$" is coffin state.

$$p_m(i,j) = \begin{cases} 1/2d \cdot 1/1+m^2. & i,j \in \mathbb{Z}^d.\ i \sim j. \\ m^2/1+m^2 & i \in \mathbb{Z}^d.\ j = * \\ 1 & i = j = * \\ 0 & \text{otherwise} \end{cases}$$ for $m \in (0,1)$.

$Z = (Z_k)_{k\geq 0}$ is killed RW on $\mathbb{Z}^d$ with prob. transition p_m. with p.m. $\mathbb{P}^m_i$ (start at i)

Denote: $\tau_* =: \inf\{n \geq 0 \mid Z_n = *\}$.

Rmk: $\mathbb{P}^m_i(\tau_* > n) = (1+m^2)^{-n}$.

$\mathbb{P}^m_i(Z_n = j) = \mathbb{P}^m_i(\tau_* > n)\, \mathbb{P}_i(X_n = j)$.

Thm For $\Lambda \subset\subset \mathbb{Z}^d$. $d \geq 1$. η is b.c. Set $\eta_* = 0$

Then, under $\mu^\eta_{\Lambda, m}$. $\varphi_\Lambda = (\varphi_i)_\Lambda$. is Gaussian

with mean $u^m_\Lambda = (u^m_i)_\Lambda$. and COV. $G_{m,\Lambda}$:

$$\begin{cases} u^m_i =: \mathbb{E}^m_i (\eta_{Z_{\tau_{\Lambda^c}}}). \\ G_{m,\Lambda}(i,j) =: \frac{1}{1+m^2} \mathbb{E}^m_i (\sum_0^{\tau_{\Lambda^c}} I\{ Z_n = j \}). \end{cases}$$

Rmk: Consider conti. killed RW $(Y_t)_{t \geq 0}$

with killed rate m^2. and jump

from x to nearest nbd at rate

$1/2d$. So its killing time $X \sim Exp(1+m^2)$.

for non-coffin state.

$$\Rightarrow G_{m,\Lambda}(i,j) = \mathbb{E}_i (\int_0^{\sigma_{\Lambda^c}} I\{Y_s = j\} ds).$$

$\sigma_{\Lambda^c} = \inf\{ t \geq 0 \mid Y_t \notin \Lambda \}$.

Pf: 1°) By one-step analysis:

u^m_i Solves massive Dirichlet Problem.

2°) Note $-\frac{1}{2d}\Delta_\Lambda + m^2 = (1+m^2) I_\Lambda - P_\Lambda$

$= (1+m^2)(I_\Lambda - P_{m,\Lambda})$.

By $P^k_{m,\Lambda}(i,j) \leq \mathbb{P}^k_m(i,j) \leq (1+m^2)^{-k}$

$\Rightarrow I_\Lambda - P_{m,\Lambda}$ is invertible.

St. $G_{m,\Lambda} = \frac{1}{1+m^2} (I_\Lambda + P_{m,\Lambda} + \cdots + P^n_{m,\Lambda} \cdots)$

(2) <u>Thermodynamic Limit:</u>

<u>Def</u>: i) $\mathcal{M}_0(\Omega_\Lambda)$ is set of p.m. on the measurable space $(\Omega_\Lambda, \mathcal{P}(\Omega_\Lambda))$.

ii) Specification $\pi = (\pi_\Lambda)_{\Lambda \subset\subset \mathbb{Z}^d}$ of GFF is defined by the equation:

$\pi_\Lambda^m(\cdot \mid \eta) = \mu^\eta_{m,\Lambda}(\cdot)$

iii) Set of Gibbs measure compatible with π is $\mathcal{G}(m) =: \{\mu \in \mathcal{M}_0(\Omega) \mid \mu \pi_\Lambda^m(A) =: \int \pi_\Lambda^m(A \mid \omega)\, \mu(d\omega) = \mu(A)$ $\forall \Lambda \subset\subset \mathbb{Z}^d,\ A \in \mathcal{P}(\Omega_\Lambda)\}$.

<u>Rmk</u>: Gibbs measure is naturally defined as some sense of stationary dist.

<u>Prop.</u> $\mu \in \mathcal{G}(m) \Leftrightarrow \forall \Lambda \subset\subset \mathbb{Z}^d,\ \forall A \in \mathcal{F}$.

We have: $\mu(A \mid \mathcal{F}_{\Lambda^c})(\omega) = \pi_\Lambda^m(A \mid \omega)$. μ-a.s.

① <u>Massless Case:</u>

i) <u>Low Dimension: $d = 1, 2$</u>

Thm. When $d = 1$ or 2. The massless GFF has no infinite volume measure: $\mathcal{G}(0) = \emptyset$.

Rmk: It's because $\lim_{n\to\infty} G_{B(n)}(i,j) = \infty$ follows from recurrence of SRW.

Pf: If $\exists\, \mu \in \mathcal{G}(0)$. Then $\mu = \mu\, \pi^{0}_{B(n)}$

$$\mu(\varphi_0 \in [a,b]) = \int \mu^{n}_{B(n),\eta}(\varphi_0 \in [a,b])\, \mu(d\eta)$$

But $\mu^{n}_{B(n),\eta}(\varphi_0 \in [a,b]) \le C / \sqrt{G_{B(n)}(0,0)}$

$\Rightarrow \mu(\varphi_0 \in [a,b]) \xrightarrow{n\to\infty} 0$. Contradict!

Rmk: To reduce the fluctuation of each spins φ_i (i.e. $\mathrm{Cov}(\varphi_i)$)

Set $\tilde{\varphi} = (\tilde{\varphi}_i)_{\mathbb{Z}^d}$. $\tilde{\varphi}_i = \varphi_i - \varphi_0$.

has a well-def thermodynamic limit.

ii) High Dimension: $d \ge 3$

Note that $G(i,j) =: \lim_{n\to\infty} G_{B(n)}(i,j)$ exists now.

Def: η is harmonic on $\mathbb{Z}^d$ if $(\Delta\eta)_i = 0$. $\forall i$.

Rmk: There're infinite $\mathbb{Z}^d$-harmonic f. eg:

$u : S = \{(i_1 \cdots i_d) \in \mathbb{Z}^d,\ i_d \in \{0,1\}\} \to \mathbb{R}'$

can extend to $\mathbb{Z}^d$. s.t. $\Delta u = 0$.

Thm. For $d \geq 3$, we have $|\mathcal{G}(0)| = \infty$.

Moreover, given $\forall\ \eta$ harmonic on $\mathbb{Z}^d$.

Then $\exists$ Gaussian Gibbs measure μ_0^η.

with mean η and cov. $G(i,j)$.

Rmk: Precisely, $ex\,\mathcal{G}(0) = \{\mu_0^\eta \mid \eta \text{ is harmonic}\}$

i.e. μ_0^η's above are extremal elements of $\mathcal{G}(0)$

Pf: Fix η harmonic. Construct $\varphi = (\varphi_i)_{\mathbb{Z}^d}$

GF with mean (η_i) and cov

$G(i,j) = \lim\limits_{n\to\infty} G_{B(n)}(i,j)$. by limiting.

Set μ_0^η, $\mathbb{E}_0^\eta$ are law and expectation

of $(\varphi_i)_{i\in\mathbb{Z}^d}$. prove: $\mu_0^\eta \in \mathcal{G}(0)$.

Lemma. For $a_i(w) = \mathbb{E}_0^\eta(\varphi_i \mid \mathcal{F}_{\Lambda^c})(w)$, $i \in$

$\Lambda \subset\subset \mathbb{Z}^d$. Then: $a_i(w) = \mathbb{E}_i(w_{X_{\tau_{\Lambda^c}}})$

μ - a.s.

Rmk: Note that $a_i(w) = \sum_{k\in\partial_{ex}\Lambda}$

$w_k\, \mathbb{P}_i(X_{\tau_{\Lambda^c}} = k)$. i.e. finite

linear combination of (w_i).

$\Rightarrow (a_i)_{\mathbb{Z}^d}$ is Gaussian Field.

$\underline{\text{Pf}}$: Check: $\mathbb{E}_0^n(\varphi_j(\varphi_i - \mathbb{E}_i(\varphi_{X_{\tau_{\Lambda^c}}}))) = 0$. $j \in \Lambda^c$

It follows from:

$$h(i,j) = \sum_{k \in \partial \Lambda} \mathbb{P}_i(X_{\tau_{\Lambda^c}} = k)\, h(k,j)$$

$$= \mathbb{E}_i(h(X_{\tau_{\Lambda^c}}, j)).\ i \in \Lambda.\ j \in \Lambda^c.$$

$\underline{\text{Cor.}}$ Under μ_0^n. $(\varphi_i - a_i)_\Lambda$ is indept of $\mathcal{F}_{\Lambda^c}$.

$\underline{\text{Pf}}$: By uncorrelated of Normal dist.

Return to the pf:

By criteria. show: $\mu_0^n(A \mid \mathcal{F}_{\Lambda^c})(\omega) = \mu_{0,n}^{\omega}(A)$

for $\forall A \in \mathcal{F}_\Lambda$. $\Lambda \subset\subset \mathbb{Z}^d$. μ_0^n - a.s.

Note that $\mathbb{E}(e^{it_\Lambda \cdot \varphi_\Lambda} \mid \mathcal{F}_{\Lambda^c}) = e^{it_\Lambda a_\Lambda} \cdot \mathbb{E}_0^n(e^{it_\Lambda(\varphi_\Lambda - a_\Lambda)})$

Next, check $\mathrm{cov}_0^n(\varphi_i - a_i, \varphi_j - a_j) = h_\Lambda(i,j)$.

$\underline{\text{Prop}}$: If $d \geq 3$. $m = 0$. Then $\forall \mu_0^n \in \mathrm{ex}\,\mathcal{G}(0)$. Satisfies:

$$\mathrm{cov}_0^n(\varphi_i, \varphi_j) = \frac{r(d)}{\|j-i\|_2^{d-2}}(1 + o(1))\quad r(d) > 0.\ \text{as } \|i-j\|_2 \to \infty.$$

② $\underline{\text{Massive Case}}$:

$\forall d \geq 1$. Note $h_m(i,j) \triangleq \lim_{n \to \infty} h_{m,B(n)}(i,j)$

$$= \frac{1}{1+m^2} \sum_{n\geq 0} \mathbb{P}_i^m (Z_n = j) .$$ converges. if $m>0$.

<u>Thm.</u> $\forall\, d \geq 1$. $|\mathcal{G}(m)| = \infty$ for $m>0$.

i.e. given any m-harmonic function η on $\mathbb{Z}^d$.

There exists Gaussian Gibbs measure μ_m^η with

mean η and covariance $G_m(i,j)$.

<u>Rmk:</u> $\mathrm{ex}\, \mathcal{G}(m) = \{ \mu_m^\eta : \eta$ is m-harmonic $\}$.

i) <u>Decay of $G_m(i,j)$:</u>

<u>Def:</u> $\xi_m(i) = \lim_{l\to\infty} -\frac{1}{l} \log G_m(0, li)$.

<u>prop.</u> (Exponential Decay)

$\forall\, d \geq 1$. $i \in \mathbb{Z}^d$. Then $\xi_m(i)$ exists and

$G_m(0,i) \leq G_m(0,0)\, e^{-\xi_m(i)}$. Moreover.

$$\log(1+m^2) \leq \frac{\xi_m(i)}{\|i\|_1} \leq \log(2d) + \log(1+m^2)$$

<u>Pf:</u> $\forall j \in \mathbb{Z}^d$. $\tau_j = \min\{ n \geq 0 \mid Z_n = j \}$.

$$G_m(0, li) = \mathbb{P}_0^m (\tau_{li} < \tau_*)\, G_m(li, li)$$

$$= \mathbb{P}_0^m (\tau_{li} < \tau_*)\, G_m(0,0).$$

$$\Rightarrow \xi_m(i) = \lim_{l\to\infty} -\frac{1}{l} \log \mathbb{P}_0^m (\tau_{li} < \tau_*)$$

Since $G_m(0,0)$ is finite. Next. check Fekete :

1) $\mathbb{P}_0^m(Z_{(l_1+l_2)i} < Z_*) \geq \mathbb{P}_0^m(Z_{l_1 i} < Z_\square < Z_*)$

$= \mathbb{P}_0^m(Z_{l_1 i} < Z_*)\,\mathbb{P}_0^m(Z_{l_2 i} < Z_*)$.

by MP at $Z_{l_1 i}$ and translation invariant

: $l_1 i \leftrightarrow (l_1+l_2)i \iff 0 \leftrightarrow l_2 i$.

2) $(1+m^2)\, h_m(i,j) = \sum_{n\geq 1} \mathbb{P}_i^m(Z_n = j)$

$$\leq \sum_{n\geq |j-i|_1} (1+m^2)^{-n} \leq \frac{1+m^2}{m^2}(1+m^2)^{-|j-i|_1}$$

$(1+m^2)\, h_m(i,j) = \mathbb{P}_i^m(Z_j < Z_*)\,(1+m^2)\, h_m(0,0)$

$$\geq \mathbb{P}_i^m(Z_j < Z_*)$$

$$\geq (2d(1+m^2))^{-|j-i|_1}$$

where we chose nearest path in last $\geq$.

<u>Thm.</u> $\forall d \geq 1$. If for b.c. η satisfies the condition:

$$\overline{\lim_{k\to\infty}} \max_{|i|_1 = k} \frac{\log|\eta_i|}{k} < \log(1+m^2). \text{ Then: } \mu_m^\eta = \mu_m^0.$$

for every $\mu_m^\eta \in \text{ex}\,\mathcal{G}(m)$.

<u>Rmk:</u> If b.c. grow not too fast. Then:

the correspond GF is unique.

In other word, the only m-harmon

func. with subexponential growth is 0

Pf: Set $\varepsilon > 0$. s.t. $e^{\varepsilon}/(1+m^2) < 1$.

By condition: $|\eta_i| \le e^{\varepsilon n}$. $\forall n$ large. $i \in \partial_{ex} B(n)$.

$\Rightarrow |\mathbb{E}_i^m(\eta_{X_{\tau_{B(n)^c}}})| \le e^{\varepsilon n} \mathbb{P}_i^m(\tau_{B(n)^c} > d_1(i, B(n)^c))$

$\le e^{\varepsilon n}(1+m^2)^{-n+|i|} \xrightarrow{n\to\infty} 0$

ii) Limit as $m \downarrow 0$:

prop. (Divergence of Var in $d = 1, 2$)

For (φ_i) is massive GFF on $\mathbb{Z}^d$. Then:

$$\mathrm{Var}_m(\varphi_0) \overset{m\downarrow 0}{\sim} \begin{cases} 1/\sqrt{2}m & \text{if } d = 1 \\ \frac{2}{\pi}\log|m| & \text{if } d = 2 \end{cases}$$

Pf: Set $\lambda > 0$. s.t. $e^{\lambda} = 1+m^2$.

$\Rightarrow \mathrm{Var}_m(\varphi_0) = \sum_{n\ge 0} e^{-\lambda(n+1)} \mathbb{P}_0(X_n = 0)$

Apply local limit Thm. to estimate $\mathbb{P}_0(X_{2k} = 0)$.

Thm. For $d = 1$. $m > 0$. $\forall i, j \in \mathbb{Z}^1$. We have: exists

$\xi_m, A_m > 0$ s.t. $\lim\limits_{m\downarrow 0} \frac{\xi_m}{m} = \sqrt{2}$. $G_m(i,j) = A_m e^{-\xi_m|j-i|}$.

Pf: Set $e^{\lambda} = 1+m^2$. Note: $I_{\{X_n = j\}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik(X_n - j)} dk$

$\Rightarrow G_m(0,j) = \frac{1}{1+m^2} \mathbb{E}_0(\sum e^{-\lambda n} I_{\{X_n = j\}})$

$= \frac{1}{2\pi(1+m^2)} \int_{-\pi}^{\pi} e^{-ikj} \sum_{n\ge 0} (e^{-\lambda}\phi(k))^n dk$

$$= \frac{1}{2\pi(1+m^2)} \int_{-\pi}^{\pi} \frac{e^{-ikj}}{1-e^{-\lambda}\phi(k)} dk$$

where $\mathbb{E}(e^{ikX_n}) = \phi(k)^n$.

$\Rightarrow$ Apply Residue Formula on $\frac{e^{-izj}}{1-e^{-\lambda}\cos z}$.

<u>Thm.</u> (Decay of $\xi_m(i)/\|i\|_2$ in all dim)

There exists $m_0 > 0$, and const. $\alpha \in (0, \delta]$.

s.t. $\forall\, 0 < m < m_0$, $i \in \mathbb{Z}^d$. We have:

$\alpha m \|i\|_2 \leq \xi_m(i) \leq \delta m \|i\|_2$.

<u>Pf:</u> idea: • Time of survival $\sim 1/m^2$:

and $\mathbb{P}_0^m(Z_t > r/m^2) = (1+m^2)^{-\frac{r}{m^2}} \leq e^{-\frac{r}{2}}$

for $\forall r > 0$, s.t. $r/m^2 \in \mathbb{Z}^+$, m large.

• In a period with time $\leq r/m^2$,

the distance of walker walk $\sim r/m$:

$$\mathbb{P}_0^m(\|Z_{r/m^2}\| \geq r/m) \leq \frac{\mathbb{E}_0(\|X_{r/m^2}\|^2)}{r^2/m^2}$$

$$= \frac{r/m^2}{r^2/m^2} = 1/r.$$

1) Fix $r \geq 8$, s.t. $r/m^2 \in \mathbb{Z}^+$, $m/r < 1$.

Set $T_k = \inf\{t > T_{k-1} \mid \|Z_t - Z_{T_{k-1}}\|_2 \geq r/m\}$, $T_0 = 0$.

$\Rightarrow T_M \leq Z_j$, $M = [\|j\|_2 / r/m]$, the least times of $\{\|Z_\cdot - z_\cdot\|_2 \geq r/m\}$ happens for $0 \to j$.

By MP: $\mathbb{P}_0^m(Z_j < Z_*) \leq \mathbb{P}_0^m(T_1 < Z_*)^m$

(Replace in $G_m(0,j) = \mathbb{P}_0^m(Z_j < Z_*)\, G_m(0,0)$.

Then we obtain lower bound estimation)

$\Rightarrow$ Estimate $\mathbb{P}_0^m(T_1 < Z_*) \leq \mathbb{P}_0^m(T_1 \leq r/m^2) + \mathbb{P}_0^m(Z_* > \frac{r}{m^2})$

2) $\mathbb{P}_0^m(Z_j < Z_*) \geq \mathbb{P}_0^m(X_{[\|j\|_2/m]} = j)\ \mathbb{P}_0^m(Z_* > \|j\|_2/m)$

with Local Limit Thm.

(3) <u>Variation on Markov property:</u>

① <u>With nonzero b.c.:</u>

Next, we use the density def of GFF: $(I(x))$

in D with b.c. $\eta = f$ on ∂D.

<u>Rmk:</u> If $f \equiv c$. Then $(I(x) - c)_D$ is GFF with

Dirichlet condition.

<u>prop.</u> (Version I)

For $O \subseteq D$. Then $(I(x))_{x\in O} \mid (I(x))_{x\notin O}$

$= (f)_{x\notin O} \sim$ GFF in O with b.c. $\eta = f|_{\partial O}$.

<u>Pf:</u> Directly check the density.

Def: For $F_1, F_2 : \mathbb{Z}^d \to \mathbb{R}^1$ with finite support.

set $$\langle F_1, F_2 \rangle = -\sum_{\mathbb{Z}^d} F_1(x)\, \bar{\Delta} F_2(x)$$

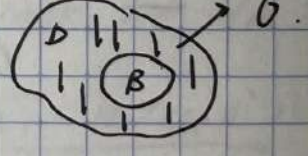
$$= \frac{1}{2} \cdot \frac{1}{2d} \sum_{x} \sum_{y \sim x} (F_1(y) - F_1(x))(F_2(y) - F_2(x))$$

Rmk: Note if $B \subset D$. $\mathrm{supp}(F_1) \subset B$. $\Delta F_2 = 0$ in B

then $\langle F_1 + F_2, F_1 + F_2 \rangle = \langle F_1, F_1 \rangle + \langle F_2, F_2 \rangle$

Prop. For $(I(x))_D$ with Dirichlet b.c. If F solves Dirichlet problem in D with $F = f$ on ∂D. Then $(I(x) + F(x))_D$ is a GFF with b.c. $\eta = f$ on ∂D.

Pf: Investigate the value of density at $(y_x + F(x))_D$. by Rmk above.



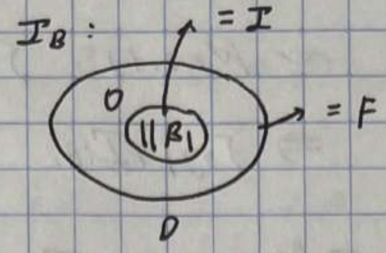
Def: For $B \subset\subset D$. $O = D / B$

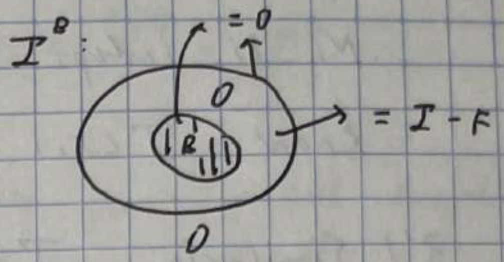
i) $(I_B(x))_{x \in D}$ is process. s.t.

$I_B = I$ in B. $I_B = F$ in O. where F solves Dirichlet problem in O with b.c. $\eta = I$ on ∂O.

ii) $(I^O(x))_{x \in D}$ is process. s.t. $I^O = I - I_B$

Rmk:





Prop. (Version II)

I^B is a $GFF^{Diri.}$ on O. which is indept of I_B.

Pf: 1°) I^B is $GFF^{Diri.}$ follows from Prop. above.

2°) $(I^B)_D \mid (I_B)_D = (I^B)_D \mid (I_B)_B \cup (I^B)_B$

$= (I^B)_O \cup (I^B)_B$

by Version I.

Rmk: $G_D(x,y) = \mathbb{E}(I(x) I(y))$

$= \mathbb{E}(I^B(x) I^B(y)) + \mathbb{E}(I_B(x) I_B(y))$

$= G_O(x,y) + \mathbb{E}(I_B(x) I_B(y))$. $\forall x, y \in D$.

③ Algorithm:

suppose $(I)_D$ is GFF with Dirichlet b.c. def by density.

Denote $D = \{x_1 \cdots x_n\}$. $B_j = \{x_1 \cdots x_j\}$. $O_j = D / B_j$.

1°) Note $I(x_1) \sim N_1 \cdot \sqrt{G_D(x_1, x_1)}$. $N_1 \sim N(0,1)$.

and I^{B_1} is GFF in O_1 indept of $I(x_1)$.

2°) Obtain I_{B_1} from $I(x_1) \Rightarrow$ Note $I(x_2) = I^{B_1}(x_2)$

$+ I_{B_1}(x_2)$. $I^{B_1}(x_2) \sim N_2 \cdot \sqrt{G_{O_1}(x_2, x_2)}$. where

N_1 is indept of $N_2 \sim N(0,1)$. So we know $I(x_1)$. $I(x_2)$ $\Rightarrow$ Obtain I_{B_2}

3') Similarly. $I(x_3) = I^{B_2}(x_3) + I_{B_2}(x_3)$.

iterate it continuously.

$\Rightarrow$ We can write $I(x) = \sum_1^n N_j \cdot \sqrt{h_{D_{j-1}}(x_j, x_j)} \cdot V_j(x)$.

where $N_j \overset{i.i.d}{\sim} N(0,1)$. for $\forall x \in D$.

To determine $V_j(\cdot)$. Calculate $\mathbb{E}(I(x) I(y))$.

Rmk: Different exploration order of D will induce different decomposition of I.

③ Local Sets:

Def: In the same prob. space as GFF $(I(x))_D$

$R \subset D$ is random set. We say the coupling (R, I) is local if $\forall$ fix $B \subset D$. I^B in D is indept of $\sigma((I_B, R = B))$

Rmk: If A is indept of I. Then (A, I) is clearly local.

Prop. If $\forall$ fix $B \subset D$. $\{A = B\} \in \sigma(I_B)$. Then A is local

Lemma. For $B \subset B' \subset D$. $\Rightarrow (\mathcal{I}^B)^{B'/B} = \mathcal{I}^{B'}$

prop. If $(A_1, \mathcal{I})$, $(A_2, \mathcal{I})$ are two local coupling. st. $A_1 | \mathcal{I}$ indept of $A_2 | \mathcal{I}$. Then: $(A_1 \cup A_2, \mathcal{I})$ is a local coupling.

Pf: 1°) For $U, V \in \sigma(\mathbb{R}^D)$, $B = B_1 \cup B_2$.

prove: $\mathbb{P}(\mathcal{I}^B \in U, \mathcal{I}_B \in V, A_1 = B_1, A_2 = B_2)$

$= \mathbb{P}(\mathcal{I}^B \in U)\, \mathbb{P}(\mathcal{I}_B \in V, A_1 = B_1, A_2 = B_2)$

by condition on $\mathcal{I}$.

Besides, $\mathbb{P}(A_i = B_i | \mathcal{I}) = \mathbb{P}(A_i = B_i | \mathcal{I}_B)$.

2°) Fix B. Sum over B_1, B_2 st. $B = B_1 \cup B_2$.

Rmk: Not all local sets can be generated by such way (or in any algorithm)

(4) **Determinant of Laplacian:**

Next, we consider $D = \{x_1, \cdots x_n\}$ and calculate determinant of $(-\bar{\Delta}_D(x_i, x_j))_{n \times n}$

Rmk: i) $-(2d)\bar{\Delta}_D$ is integer-valued. So: $\det(-2d \cdot \bar{\Delta}_D) \in \mathbb{Z}$.

ii) Note $-\bar{\Delta}_D G_D = I \Rightarrow |G_D| = 1/\det(-\bar{\Delta}_D)$

prop. $\sqrt{|1-\bar{\Delta}_D|}/(2\pi)^{\frac{n}{2}}$ = density of GFF at $\vec{0}$.

Pf: Recall density of GFF is:

$$\frac{\sqrt{|1-\bar{\Delta}_D|}}{(2\pi)^{\frac{n}{2}}} \exp\left(-\frac{\mathcal{E}_D(y)}{2\times 2d}\right) dy_1 \cdots dy_n.$$

prop. $\det(G_D) = \prod_{j=1}^{n} G_{D/\{x_1, \cdots x_{j-1}\}}(x_j, x_j)$

Pf: By induction:

Note: $\mathbb{P}(\forall i \le n,\ |\Gamma(x_i)| \le \varepsilon/\sqrt{2\pi}) \overset{\varepsilon\to 0}{\sim}$

$\varepsilon^{-n}/|G_D|.$

$\mathbb{P}(\forall i \in \{2, \cdots n\}.\ |\Gamma(x_i)| \le \varepsilon/\sqrt{2\pi} \mid \Gamma(x_1) = 0)$

$\overset{\varepsilon\to 0}{\sim} \varepsilon^{-n}/|G_{D/\{x_1\}}|.$

follows from $\Gamma(x_i) = \Gamma^{\{x_1\}}(x_i).\ \forall i \ne 1$

which indept of $\Gamma(x_1) = \Gamma_{\{x_1\}}(x_1)$.

$\Rightarrow |G_D| = |G_{D/\{x_1\}}| \cdot G_D(x_1, x_1).$

prop. (Laplace Transform)

For Γ is GFF on D with Dirichlet b.c.

Then $\forall k_i(x) \ge 0.\ 1 \le i \le n.$ We have:

$$\mathbb{E}\left(e^{-\frac{1}{2}\sum_1^n k_i(x)\Gamma^2(x_i)}\right) = \sqrt{|1-\bar{\Delta}_D|/|1-\bar{\Delta}_D + \operatorname{diag}\{k_i\}|}$$

Pf: Write LHS in density of GFF.

(5) General GFF Model:

i) Def: (Massive GFF)

Massive GFF in D with Dirichlet b.c and mass func. k is centered Gaussian r.v. $(\Gamma(x))_D$ with density at (y_x):

$$\sqrt{|-\bar{\Delta}_D + \mathrm{diag}\{k(x_i)\}|} \Big/ (2\pi)^{\frac{n}{2}} \cdot \exp\Big(-\frac{1}{2}\cdot\Big(\frac{\mathcal{E}_D(y)}{2\lambda} + \sum_D k(x) y_x^2\Big)\Big)$$

Rmk: i) Covariance Σ is inverse of

$$-\bar{\Delta}_D + \mathrm{diag}\{k(x_i)\}_{x_i \in D}$$

ii) resampling property: $\forall x \in D$. Then:

$$\Gamma(x) \mid (\Gamma(y))_{y \neq x} \sim N\Big(\frac{\bar{\Gamma}(x)}{1+k(x)}, \ 1/(1+k(x))\Big)$$

iii) $k(x)$ can be seen as killing rate at each point x.

ii) Def: (GFF on electric networks)

Consider $G = (V, E)$. V is countable.

$e = \{x, y\} \in E \mapsto C_{x,y} = C_{y,x} \in [0, \infty)$ the conductance of each edge.

set $\lambda_x = \sum_{y \sim x} C_{y,x}$. $P(x,y) = C_{x,y}/\lambda_x$.

For $D \subset V$. GFF in D with Dirichlet b.c. for (V, c) is $(I(x))_D$ s.t. its density at $(\varphi_x)_D \propto \exp(-\frac{1}{2}\sum_{e\in E} (c_e \cdot (\nabla\varphi(e))^2))$

<u>Rmk:</u> i) Consider Y is CTMC. when at x its jumping rate is λx with trans. prob. to y : $c_{x,y}$.

Then $G_{D,c}(x,y) = \mathbb{E}_x(\int_0^{\tau_D} I\{Y_s = y\} ds)$

ii) massive GFF is a particular of it. with $\widetilde{V} = V \cup \{\partial\}$. $c_{x,\partial} = k(x)$. $c_{x,y} = 1/2d$.

iii) Define: $\Delta_{D,c} : f(x) \longmapsto \sum_{y\sim x} c_{x,y}(f(y) - f(x))$ for $f: D \to \mathbb{R}'$.

Then for covariance of $(I(x))$. Σ.

$-\Delta_{D,c}\Sigma = I_D$.

iv) $I(x) | (I(y))_{y\neq x} \sim N(\frac{\sum_{y\sim x} c_{xy} I(y)}{\lambda x}, 1/\lambda x)$