Sesquilinear Form.

(1) Definitions:

Next, we consider $\mathcal{H}^{\mathbb{C}}$. Hibert span on $\mathbb{C}$.

① **Def:** For $A \in \mathcal{L}(\mathcal{H})$. Numerical range is $W(A) =:$

$\{(Au, u) \mid \|u\| = 1, u \in \mathcal{H}\}$.

Rmk: It contains all values of diagonal represent in o.n.b. of A.

Thm. For $A \in \mathcal{L}(\mathcal{H})$. Then $\sigma(A) \subset \overline{W(A)}$

Rmk: Recall we have proved it for self-adjoint operator in $\mathcal{H}^{\mathbb{R}}$ case.

Pf: If $\lambda \in \sigma(A)$. Then one of follows happen:

1') $A - \lambda$ isn't injective

So $\exists u$. $Au = \lambda u$, $\|u\| = 1$. $(Au, u) = \lambda \in W(A)$.

2') $\overline{R(A-\lambda)} \neq \mathcal{H}$.

So $R(A-\lambda)^{\perp} = N(A^* - \bar{\lambda}) \neq \{0\}$. $\exists v'$. st.

$A^* v' = \bar{\lambda} v'$. $\|v'\| = 1$. $\therefore (v', A^* v') = \lambda = (Av', v')$

3') $A - \lambda$ is injective. But $R(A-\lambda)$ isn't closed.

So. It doesn't exist $c > 0$. st.

$\|(A-\lambda)x\| \geq c\|x\|$. $\Rightarrow \exists (u_k)$. $\|u_k\| = 1$. and

$\|(A-\lambda)u_k\| \leq \frac{1}{k} \to 0$. So: $|(Au_k, u_k) - \lambda(u_k, u_k)| \to 0$

① **Def:** Sesquilinear form (SLF) on $\mathcal{H}$ is function a with $D(a) \subset \mathcal{H}$. $a: D(a) \times D(a) \to \mathbb{C}$. St. a is antilinear. i.e.

$$\begin{cases} a(u, \lambda v_1 + \beta v_2) = \bar{\lambda}\, a(u, v_1) + \bar{\beta}\, a(u, v_2) \\ a(\lambda u_1 + \beta u_2, v) = \lambda\, a(u_1, v) + \beta\, a(u_2, v) \end{cases}$$

Rmk: If a is densely defined i.e. $D(a)$ is dense. and $\exists\, c(u)$. St. $|a(u,v)| \leq c(u) \|v\|$. $\forall v \in D(a)$

Then $v \mapsto a(u,v)$ can be extended to $\mathcal{H}$. for each fix u. Apply Riesz Represent Thm.

$\exists\, f_u \in \mathcal{H}$. $a(u,v) = (f_u, v)$. $\forall v \in \mathcal{H}$.

Def: A is associated operator of SLF a densely defined if $A: u \mapsto f_u$. $D(A) = \{u \in D(a) \mid \exists\, c(u).\ |a(u,v)| \leq c(u)\|v\|.\ \forall v\}$

Denote: $W(a) = \{a(u,u) \mid \|u\| = 1.\ u \in D(a)\}$. $a(u) \triangleq a(u,u)$.

Thm. For $a(u,v)$ densely defined SLF with associated operator A. If $\lambda \notin \overline{W(a)}$. Then $\exists\, c > 0$. St.

$\|(A-\lambda)u\| \geq c\|u\|$. $\forall u \in D(A)$.

Pf: $\exists\, \delta > 0$. St. $|\lambda - a(u)| \geq \delta > 0$. $\forall u \in D(a)$. $\|u\| = 1$.

$\Rightarrow |a(w) - \lambda \|w\|^2| \geq \delta \|w\|^2$. $\forall w \in D(a)$.

Set $\tilde{w} = (A-\lambda)u$. $\therefore (\tilde{w}, u) = a(u) - \lambda\|u\|^2$.

So: $\delta\|u\|^2 \leq |(\tilde{w}, u)| \leq \|\tilde{w}\|\|u\|$. $c = \delta$.

Rmk: It also implies: $A - \lambda$ is injective.

Moreover. A is CLO $\Rightarrow R(A-\lambda)$ is closed.

③ Hermitian SLF:

<u>Def</u>: SLF $a(u,v)$ is Hermitian if $a(u,v) = \overline{a(v,u)}$.

<u>Thm.</u> a is SLF. Then follows equi.:

i) $a(u,v)$ is Hermitian.

ii) $a(u) \in \mathbb{R}^1$. $\forall u \in D(a)$

iii) $Re(a(u,v)) = Re(a(v,u))$. $\forall u,v \in D(a)$

<u>Pf</u>: i) $\Rightarrow$ ii) trivial. iii) $\Rightarrow$ i) consider $a(iu,v)$.

ii) $\Rightarrow$ iii) consider: $a(iu+v) \in \mathbb{R}^1$

<u>prop.</u> $a(u,v)$. $b(u,v)$ are Hermitian SLFs.

If $|a(u,v)| \leq M|b(u,v)|$. $\forall u,v \in D(a) \cap D(b)$. Then:

$|a(u,v)|^2 \leq M^2\, b(u,u)\, b(v,v)$. $\forall u,v \in D(a) \cap D(b)$.

<u>Pf</u>: WLOG. suppose $M=1$. (or set $\tilde{b} = Mb$).

Fix $u,v \in D(a)$. Let $w = e^{i\theta}u$. st. $a(w,v) \in \mathbb{R}^1$.

Set: $p(t) = b(w,w)t^2 + 2a(w,v)t + b(v,v)$.

Note: $4a(wt,v) = a(tw+v) - a(tw-v)$

$\therefore |2a(wt,v)| \leq \frac{1}{2}b(tw+v) + \frac{1}{2}b(tw-v)$

$= b(w,w)t^2 + b(v,v)$

$\Rightarrow \Delta_p \leq 0$. Obtain the conclusion!

<u>Cor.</u> a. b are SLFs. b is Hermitian. If

$|a(u,u)| \leq M|b(u,u)|$ Then: $|a(u,v)| \leq$

$4M^2\, b(u,u)\, b(v,v)$. $\forall u,v \in D(a)$.

Pf. To Hermitianize a: Set $\begin{cases} a_1(u,v) = \frac{1}{2}(a(u,v) + \overline{a(v,u)}) \\ a_2(u,v) = \frac{1}{2i}(a(u,v) - \overline{a(v,u)}) \end{cases}$

$\Rightarrow a = a_1 + i a_2$

Note: a_1, a_2 are Hermitian. Apply Prop.

Cor. If $b(u,v)$ is Hermitian SLF, $b(u) \geq 0$, $\forall u \in D(b)$.

Then: $|b(u,v)|^2 \leq b(u) b(v)$, $b^{\frac{1}{2}}(u+v) \leq b^{\frac{1}{2}}(u) + b^{\frac{1}{2}}(v)$.

Pf. $|b(u)| = b(u)$. The last follows from the former.

④ Numerical Range:

Thm. For a is SLF, $W(a)$ is convex in $\mathbb{C}$.

Pf: $\forall u, v \in D(a)$, $\|u\| = \|v\| = 1$.

1°) $a(u) = a(v) \Rightarrow \theta a(v) + (1-\theta) a(u) \in W(a)$, trivial.

2°) $a(u) \neq a(v)$. We fix $\theta \in (0,1)$.

prove: $\exists w \in D(a)$, $\|w\| = 1$, st. $a(w) = \theta a(v) + (1-\theta) a(u)$.

The ideal is using intermediate value of conti. $f \in \mathbb{R}^1$. Since w should be linear combination of u, v.

Set $\gamma \in \mathbb{C}$, $|\gamma| = 1$, st.

$\begin{cases} \gamma a(u) = X + iM. \\ \gamma a(v) = Y + iM. \end{cases}$ Then find w: $\begin{cases} \gamma a(w) = Z + iM. \\ Z = (1-\theta) X + \theta Y. \end{cases}$

Set $h(t) = \gamma a\left(\frac{t e^{i\psi} u + (1-t) v}{\|t e^{i\psi} u + (1-t) v\|}\right) - iM$. $\begin{cases} h(0) = X \in \mathbb{R}^1 \\ h(1) = Y \in \mathbb{R}^1. \end{cases}$

$\|t e^{i\psi} u + (1-t) v\| \neq 0$, $t \in [0,1]$. (ψ is undetermined)

Otherwise: $\begin{cases} \|t e^{i\psi} u\| = \|(1-t) v\| \\ a(t e^{i\psi} u) = a((t-1) v) \end{cases} \Rightarrow \begin{cases} t = \frac{1}{2} \\ a(u) = a(v) \end{cases}$

And $h(t)$ is real valued by choosing an appropriate φ. Note:

$$\gamma a(te^{i\varphi}u + (1-t)v) - i\|te^{i\varphi}u + (1-t)v\|^2\mu.$$

$$= \ell + t(1-t)\left[\gamma e^{i\varphi}a(u,v) + \gamma e^{-i\varphi}a(v,u) - i\mu(e^{i\varphi}(u,v) + e^{-i\varphi}(v,u)\right]$$

$\Leftrightarrow$ Choose φ. st. $\operatorname{Im}(e^{i\varphi}(\gamma a(u,v) - i\mu(u,v)) + e^{-i\varphi}(\square)) = 0$

So. $\exists t_0$. set $w = \dfrac{t_0 e^{i\varphi}u + (1-t_0)v}{\|t_0 e^{i\varphi}u + (1-t_0)v\|}$

(1) Semipositive definite SLF:

<u>Def:</u> $b(u,v)$ is semipositive definite if $b(u) \geq 0$. $\forall u \in D(b)$.

<u>Lemma.</u> $S = \{u \in D(b) \mid b(u) = 0\}$ is linear space.

<u>Pf:</u> By $b^{\frac{1}{2}}(u+v) \leq b^{\frac{1}{2}}(u) + b^{\frac{1}{2}}(v)$.

<u>Rmk:</u> We can construct a inner product space from such $b \geq 0$: Set $\langle\langle u, v\rangle\rangle = b(u,v)$. on space $D(b)/\overline{S}$.

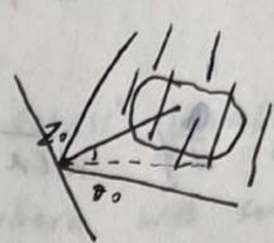
(2) Closed SLF:

<u>Def:</u> SLF $a(u,v)$ is closed if $(u_n) \subset D(a) \to u$ in H. $a(u_n - u_m) \to 0$. then: $u \in D(a)$. $a(u_n - u) \to 0$.

Next. We consider densely defined. closed SLF a associated with operator A.

Lemma W is closed convex set. If $W \neq \mathbb{C}$, half plane, strip or a line. Then: $\exists z_0, \theta_0, \theta$

$\forall z \in W$, $|\arg(z-z_0) - \theta_0| \le \theta < \frac{\pi}{2}$.



Pf. By Hahn-Banach Thm:

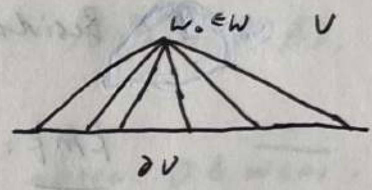
$\exists f$ separate W and $t \notin W$, $f \in \mathbb{R}^*$.

$f(t) < \alpha < f(z)$, $\forall z \in W$: $\inf\limits_{W} f(z) = \beta$ exist.

$\Rightarrow V = \{f \ge \beta\}$ is half plane contain W.

and ∂V contains a point $P \in W$.

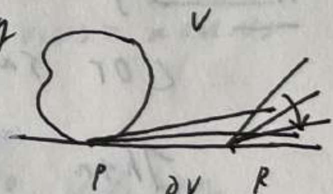
1°) $\partial V \not\subseteq W$.



otherwise: by convexity, $\exists$ strip lies in W. Contradict!

2°) $\exists R \in \partial V \cap W^c$. set $z_0 = R$.

If θ_0 doesn't exist. Then for any line starts at R, $\exists$ point between ∂V and it. i.e. $\exists z_n \in W \to \partial V$.



Since W is closed, $\overline{Pz_n} \to \partial V$ (tend to fall in) which implies $R \in W$.

Cor. If $\overline{W(a)}$ contained in the domain above.

Then, $\exists |\gamma| = 1$, $k > 0$, $k_0 \in \mathbb{R}'$, s.t. for $u \in D(a)$.

$|a(u)| \le k\,[\,\mathrm{Re}(\gamma a(u)) + k_0 \|u\|^2\,]$.

Pf: By condition: $|\mathrm{Im}(e^{-i\theta_0}(z-z_0))| \le \tan\theta\, \mathrm{Re}(e^{-i\theta_0}(z-z_0))$

set $\gamma = e^{-i\theta_0}$, $z = a(u)$. Then:

$|\mathrm{Im}\,\gamma a(u)| \le |\mathrm{Im}\,\gamma z_0| + \tan\theta\, \mathrm{Re}(e^{-i\theta_0}(z-z_0))$

denote $k_0 = \tan\theta \operatorname{Re} z_0 + |\operatorname{Im} \gamma z_0| / \tan\theta$.

$\Rightarrow |\operatorname{Im} \gamma a(\frac{u}{\|u\|})| \le \tan\theta (\operatorname{Re} \gamma a(\frac{u}{\|u\|}) + k_0)$

Thm. If $\overline{W(A)}$ is contained in the domain above.

Then $\exists\, b(u,v)$ Hermitian SLF. $D(b) = D(a)$. s.t.

$\exists\, c > 0$. $\frac{1}{c}|a(u)| \le b(u) \le |a(u)| + c\|u\|^2$. $\forall u \in D(a)$.

Pf: Set $b_1(u,v) = \frac{1}{2}[\gamma a(u,v) + \overline{\gamma a(v,u)}]$

$b = b_1 + k_0 (u,v)$. is Hermitian SLF.

Besides. $b(u) = \operatorname{Re}(\gamma a(u)) + k_0\|u\|^2$. set $c = k$.

Rmk: b is closed SLF. easy to check.

Thm. If $\overline{W(A)}$ is contained in the domain above.

(or say $\overline{W(A)}$ isn't $\mathbb{C}$. half-plane. strip. line)

Then A is closed. $\sigma(A) \subset \overline{W(a)} = \overline{W(A)}$.

Pf: 1°) For $\begin{cases} (u_k) \to u \\ Au_k \to f \end{cases}$. Note $|a(u_k - u_j)| \to 0$

$\Rightarrow u \in D(a)$. and $|a(u_k - u)| \to 0$ by a closed.

So $a(u_k, v) = (Au_k, v) \to (f, v) = a(u, v)$

by $|a(u_k - u, v)| \le 4c^2 |b(v)|^{\frac{1}{2}} |b(u_k - u)|^{\frac{1}{2}} \to 0$.

$\therefore f = Au$. and $u \in D(A)$. $\Rightarrow A$ is CLO.

2°) If $\lambda \notin \overline{W(A)}$.

By Thm above: $\|u\| \le c\|(A - \lambda)u\|$. $\forall u \in D(A)$.

Since A is closed. So: $R(A-\lambda)$ is closed.

Next. prove: $R(A-\lambda) = \mathcal{H}$.

$\forall f \in \mathcal{H}$. Set: $F: v \in \mathcal{H} \mapsto (v, f)$.

Note: $\lambda \notin \overline{W(a)} \Rightarrow |a(u) - \lambda \|u\|^2| \geq \delta \|u\|^2$. $\forall u \in D(a)$.

$\Rightarrow |F(v)| \leq \|v\| \|f\| \leq C |a_\lambda(u)|^{\frac{1}{2}}$. where we set

$a_\lambda(u, v) = a(u, v) - \lambda(u, v)$.

Similiarly $\exists b_\lambda$. Hermitian SLF. $|a_\lambda(u)| \leq C |b_\lambda(u)|$.

besides $b_\lambda \geq 0$. define inner product by b_λ.

Use Riesz Represent$^{(*)}$ (Lemma below). $\exists u \in D(a_\lambda)$. st.

$(f, v) = a_\lambda(u, v)$. $\Rightarrow (A-\lambda) u = f$. So $\lambda \in \rho(A)$.

<u>Lemma</u>: $\overline{W(a)}$ contained in domain above. $0 \notin \overline{W(a)}$.

If $\forall$ LF. f on $D(a)$. st. $|f(v)| \leq C |a(v)|^{\frac{1}{2}}$

Then. $\exists w, u \in D(a)$. st. $f(v) = a(v, w) = \overline{a(u, v)}$

<u>Def</u>: LO. A is closable if $(x_k) \subset D(A) \to 0$. $Ax_k \to \eta$.

$\Rightarrow \eta = 0$. (So A is closed $\Rightarrow$ closable)

<u>Thm</u>. A is densely defined LO on $\mathcal{H}$. st. $W(A) \neq \mathbb{C}$.

Then A has a closed extension.

<u>Thm</u>. LO has a closed extension $\Leftrightarrow$ it's closable.

<u>Pf</u>: $(\Leftarrow)$ Define $\hat{A}$: $D(\hat{A}) = \{x \in X \mid \exists x_n \in D(A) \to x$.

$\exists \eta \in Y$. $Ax_n \to \eta\}$. $\hat{A}x = \eta$. well-def by closable.

<u>Rmk</u>: $\hat{A}$ is the smallest closed extension of A.