Semigroups

(1) Background:

Consider: $\begin{cases} u'(t) = Au(t), & t \geq 0 \\ u(0) = u_0 \end{cases}$ (*) where

$u: [0, \infty) \longrightarrow X$. Banach Space. A is a linear operator on X. To let the equation make sense.

Def: $u: [0, \infty) \longrightarrow X$. n.v.s is differentiable if

$u(t) = u(t_0) + V_{t_0}(t - t_0) + o(|t - t_0|)$. $\forall t_0 \in \overline{\mathbb{R}^+}$.

Denote: $V(t) = u'(t)$.

Rmk: If $X = \mathbb{C}$. Then the unique solution of the equation is $u_0 e^{tA} = u$. ($A \in \mathbb{C}$).

Next, we want to prove the unique solution of (*) is $u(t) = e^{tA} u_0$.

① Define: e^{tA} for A is LO:

Note that: $e^{tz} = \sum_{k=0}^{\infty} \frac{(tz)^k}{k!}$. for $z \in \mathbb{C}$.

Def: $e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$. if $\sum \frac{t^k}{k!} \|A\|^k < \infty$.

Rmk: $\|e^{tA}\| \leq \sum \frac{t^k}{k!} \|A\|^k = e^{t\|A\|}$.

② Check: $e^{tA}u_0$ satisfies (*):

<u>Lemma.</u> If $BC = CB$. Then $e^{B+C} = e^B e^C = e^C e^B$.

<u>Pf:</u> $\left\| \sum_0^N \sum_0^N \frac{B^m C^n}{m!\,n!} - \sum_{m+n\le N} \frac{B^m C^n}{m!\,n!} \right\|$

$$\le \sum_{\substack{m+n\ge N \\ m\le N\ n\le N}} \frac{\|B\|^m \|C\|^n}{m!\,n!} = \sum_{m=0}^N \frac{\|B\|^m}{m!} \sum_{n=0}^N \frac{\|C\|^n}{n!} - \sum_0^N \frac{(\|B\|+\|C\|)^k}{k!} \to 0$$

$$u(t+h) - u(t) = (e^{hA} - I)\, e^{tA} u_0$$

$$= \sum_1^\infty \frac{h^k A^k}{k!} u(t)$$

$$= h \cdot Au(t) + o(|h|) \quad (h \to 0)$$

$\Rightarrow u'(t) = Au(t)$. satisfies (*).

③ Check $e^{tA}u_0$ is uniqueness:

If $u(t)$. $\tilde{u}(t)$ both satisfy (*). Set $V = u - \tilde{u}$

$$\therefore \begin{cases} AV = V'(t) \\ V(0) = 0. \end{cases} \xrightarrow{\text{Set } W(t) = e^{-tA} V(t)} \begin{cases} W'(t) = 0 \\ W(0) = 0 \end{cases}$$

For $\forall\, l \in X^*$. $\langle l, W(t) \rangle =: W_l(t)$

$\therefore \frac{d}{dt} \langle l, W(t) \rangle = \langle l, W'(t) \rangle = 0$. $W_l(0) = 0$.

i.e. $W_l(t) \equiv 0$. $\forall\, l \in X^*$. $\therefore W(t) \equiv 0$.

<u>Rmk:</u> Define $\int_{t_0}^t u(s) = \lim_{\|T\| \to 0} \sum u(\xi_i)(s_i - s_{i-1})$

for conti. function $u(s)$ in X.

(2) Unbounded Linear Operator:

Next. We consider linear operator A densly defined on Banach space X. $A: D(A) \subset X \to X$. Which is closed. ($G(A)$ is closed)

<u>Def</u>: $\lambda \in \rho(A)$ if $\lambda I - A: D(A) \to X$ is bijection. Denote: $R_\lambda(A) = (\lambda I - A)^{-1}$

<u>Rmk</u>: By Closed Graph Thm. $R_\lambda(A)$ is bdd

<u>Rmk</u>: Since $A \circ A$ doesn't make sense generally. ($R(A) \nsubseteq D(A)$). So we can't define e^{tA} as above!

<u>Def</u>: One-parameter semigroups of operators over $V^{\mathbb{C}}$. Banach Space is $(P_t)_{t \in \mathbb{R}_{\geq 0}} \subset \mathcal{L}(V)$. s.t.

i) $P_{t+s} = P_t \circ P_s$. $\forall t, s \in \mathbb{R}_{\geq 0}$ ii) $P_0 = I$. identity.

① C_0 semigroups:

<u>Def</u>. Semigroups (P_t) is strongly conti if: $\lim_{t \downarrow 0} \|P_t x - x\| = 0$ for any $x \in V$. Denote set of such semigroups by C_0

<u>Rmk</u>: It's equi with weakly conti.:

$$\lim_{t \downarrow 0} \langle x^*, P_t x \rangle = \langle x^*, x \rangle. \ \forall x \in V. \ x^* \in V^*.$$

Lemma. $(P_t) \subseteq C_0 \iff \exists\, \delta, c > 0,\ D \underset{\text{dense}}{\subseteq} V$ s.t.

i) $\sup\limits_{[0,\delta]} \|P_t\| \leq c$ ii) $\lim\limits_{t \downarrow 0} \|P_t x - x\| = 0,\ \forall x \in D.$

Pf: $(\Rightarrow)$. If $\forall \delta > 0$, $\sup\limits_{0 \leq t \leq \delta} \|P_t\| = \infty$. Then:

$\exists (t_n) \to 0,\ \|P_{t_n}\| \to \infty.$

By UBP: $\exists x \in V,\ \sup\limits_n \|P_{t_n} x\| = \infty.$

It's absurb: $\sup\limits_n \|P_{t_n} x\| \leq \|x\| + \sup \|P_{t_n} x - x\| < \infty$

$(\Leftarrow)$ It's routine.

Cor. $(P_t) \subseteq C_0 \Rightarrow \exists\, C, \gamma > 0$ s.t. $\|P_t\| \leq C e^{\gamma t},\ \forall t \geq 0$

Pf: suppose $t = n\delta + r$, $0 \leq r < \delta$.

$\|P_t\| \leq \|P_\delta\|^n \|P_r\| \leq C^{n+1} \leq C e^{t \cdot \frac{\ln C(c)}{\delta}}.$

prop. $(P_t) \subseteq C_0 \iff \xi_x : t \longmapsto P_t x$ is conti. $\forall x \in V.$

Pf: $\|P_{t_0}\| \leq C(t_0)$, $\|P_{t_0 - h}\| \leq C e^{\gamma t_0}$. Check: L.R

Lemma. $(P_t) \subseteq C_0$. Then for every $x \in U \subseteq V$.

ξ_x is differentiable on $\mathbb{R}_{\geq 0} \iff \xi_x$ is right-diff at $t=0$

Pf: $(\Leftarrow)$ right side is trivial. It equals $P_t \xi_x'(0)$ at t

Check: $\left\|\frac{1}{h}(P_{t-h} x - P_t x) - P_t \xi_x'(0)\right\| \leq$

$\left\|P_{t-h}\left(\frac{1}{h}(x - P_h x) - \xi_x'(0)\right)\right\| + \|P_{t-h}\xi_x'(0) - P_t \xi_x'(0)\|$

$\leq C e^{\gamma t} \left\|\frac{1}{h}(x - P_h x) - \xi_x'(0)\right\| + O(h) \to 0$

② **Infestimal Generators:**

Def: Infestimal generator $\mathcal{Q} : D(\mathcal{Q}) \subseteq V \to V$ of

$(P_t) \subseteq C_0$ is $\mathcal{Q}x = \mathcal{J}_x'(0) = \lim_{t \downarrow 0} \frac{P_t x - x}{t}$. where

$D(\mathcal{Q}) = \{ x \in V \mid \lim_{t \downarrow 0} (P_t x - x)/t$ exists in $V \}$.

Rmk: $D(\mathcal{Q}) \subseteq V$. l.s. And by: $\mathcal{J}_x$ is conti. $\forall x \in V$.

Def: $M_t x = \frac{1}{t} \int_0^t \mathcal{J}_x(s)\, ds = \frac{1}{t} \int_0^t P_s(x)\, ds$.

It's easy to check Frechet Derivate:

$$\frac{d}{dt} \int_0^t \mathcal{J}_x(s)\, ds = \mathcal{J}_x(t).$$

Thm. i) $D(\mathcal{Q})$ is dense in V. invariant under P_t.

ii) $\frac{d}{dt} P_t x = \mathcal{Q} P_t x = P_t \mathcal{Q} x$. $\forall x \in D(\mathcal{Q})$. $t \geq 0$.

Therefore. $\forall l \in D(\mathcal{Q}^*)$. $\forall x \in V$. $t \mapsto \langle l, P_t x \rangle$

is differentiable. $\frac{d}{dt} \langle l, P_t x \rangle = \langle \mathcal{Q}^* l, P_t x \rangle$.

iii) $\forall t \geq 0$. $x \in V$. $\int_0^t P_s x\, ds \in D(\mathcal{Q})$

iv) $P_t x - x = \mathcal{Q} \int_0^t P_s x\, ds$ for $\forall x \in V$

$= \int_0^t P_s \mathcal{Q} x\, ds$ $\forall x \in D(\mathcal{Q})$. $\forall t \geq 0$

Pf: i) $\forall x \in V$. Check: $\int_0^t P_s(x)\, ds \in D(\mathcal{Q})$.

i.e. $\lim_{h \to 0} \dfrac{\int_h^{t+h} P_s x\, ds - \int_0^t P_s(x)\, ds}{h} =$

$$\lim_{h \to 0} \frac{\int_t^{t+h} P_s(x)\, ds - \int_0^h P_s(x)\, ds}{h} = P_t(x) - x$$

by continuity of P_s. So we have iii)

$\Rightarrow \frac{1}{t} \int_0^t P_s(x)\, ds \xrightarrow{t \to 0} x$. By Rmk. above.

ii) is routine. with Lemma. Check right side.

iv) $\sup_{0\le s\le t} \| \frac{1}{h} (P_{s+h} x - P_s - h P_s Q x) \| \le \sup_{0\le s\le t} \| P_s \| \, O(h)$

$\lesssim e^{\gamma t} \, O(h) \to 0 \quad (h \to 0)$

$\therefore \| \frac{1}{h} (P_h - I) \int_0^t P_s x \, ds - \int_0^t P_s Q x \, ds \|$

$= \| \int_0^t \frac{1}{h} (P_{s+h} x - P_s x - h P_s Q x) \, ds \| \le t \, O(h)$

<u>Cor.</u> $\frac{d}{dt} P_t T_t x = Q P_t T_t x + P_t A T_t x$. where

Q, A are generators of C_0: P_t, T_t.

$x \in D(A) \subseteq D(Q)$.

<u>Cor.</u> The infestimal generator Q of (P_t)

$\in C_0$ determines a unique semigroup (P_t).

i.e. No two distinct semigroups have one

same generator.

<u>Pf.</u> If $\exists R_t$. st. $\begin{cases} \frac{d}{dt} P_t x = Q P_t x \\ \frac{dt}{dt} R_t x = Q R_t x \end{cases}$

for $\forall x \in D(Q)$.

Then: Consider $W(s) = P_{T-s} R_s$ on $[0,T]$

$\Rightarrow W(s) x = 0$. $\because \langle l, W(s) x \rangle = 0$. $\forall l \in V^*$

$\therefore W(s) x \equiv 0$. $\forall s \in [0,T]$. $\therefore W(0) x = W(T) x$.

<u>Lemma.</u> Q is generator of $(P_t) \in C_0$. $Q_h = \frac{P_h - I}{h}$.

Then: $\forall x \in V$. $P_t x = \lim_{h\to 0} e^{t Q_h} x$. uniformly with

$t \in K$. $\forall K$ cpt subset of $\mathbb{R}_{\ge 0}$.

Pf. $\|e^{t\mathcal{a}_h}\| \leq \|e^{-tI/h}\|\|e^{tP_h/h}\| \leq e^{-t/h}\sum_k t^k\|P_{kh}\|/h^k k!$

$$\lesssim e^{-t/h}\sum t^k e^{khy}/h^k k! \lesssim e^{tye^y}.$$

Fix s. set $z = t/n$. $t \in [0, s]$. We have:

$e^{t\mathcal{a}_h} - P_t = e^{nz\mathcal{a}_h} - P_{nz} = (e^{z\mathcal{a}_h} - P_z)\,\square$.

$\|\square\| = \|\sum_0^{n-1} e^{kz\mathcal{a}_h} P_{(n-k-1)z}\| \lesssim n\exp(sye^y + sy)$

Note: $x \in D(\mathcal{a})$. $\|(e^{z\mathcal{a}_h}x - P_z x)/z\| \leq \|\frac{(e^{z\mathcal{a}_h} - I)x}{z}\|$

$$+\|\frac{(P_z - I)x}{z}\| \xrightarrow{n\to\infty} \|\mathcal{a}_h x - \mathcal{a}x\| \xrightarrow{h\to 0} 0.$$

Rmk: Note for $(\tilde{\mathcal{a}}_t) = (e^{t\mathcal{a}_h})_{t\geq 0}$ satisfies:

$\|\tilde{\mathcal{a}}_t - \tilde{\mathcal{a}}_{t_0}\| \xrightarrow{t\to t_0} 0$. call it uniform conti SG.

$\Rightarrow \forall\ C_0$ can be approxi. by such SG.

prop. $\mathcal{a}$ is generator of SG $(P_t)_{t\geq 0}$. Then i). ii). iii) equi.

i) $D(\mathcal{a}) = V$. ii) $\lim_{t\downarrow 0}\|P_t - I\| = 0$. iii) $\mathcal{a} \in \mathcal{L}(V)$. $P_t = e^{t\mathcal{a}}$.

Rmk: Uniform Conti SGs $\Longleftrightarrow$ BLOs. unique corrsp.

③ Hille - Yosida Thm.

Lemma $(P_t) \subset C_0$. For $\lambda \in \mathbb{C}$. $\alpha > 0$. Set: $T_t = e^{\lambda t}P_{\alpha t}$. Then:

$\mathcal{a} \sim (P_t) \Rightarrow A = \lambda I + \alpha\mathcal{a} \sim (T_t)$. $D(A) = D(\mathcal{a})$.

$\sigma(A) = \alpha\sigma(\mathcal{a}) + \lambda$. $R_m(A) = \frac{1}{\alpha}R_{\frac{m-\lambda}{\alpha}}(\mathcal{a})$. $\forall m \in \sigma(A)$.

Pf: $\frac{d}{dt}T_t x = AT_t x \Rightarrow Ax = (\lambda I + \alpha\mathcal{a})x$

Thm. $(P_t) \subset C_0$. st. $\|P_t\| \leq ce^{\gamma t}$. Then:

i) If $Re\lambda > \gamma$. Then: $\lambda \in \rho(\mathcal{a})$. and $\forall x \in V$.

$$R_\lambda(\mathcal{a})x = \int_0^\infty e^{-\lambda s}P_s x\,ds$$

ii) If $\lambda \in \mathbb{C}$. s.t. $\int_0^\infty e^{-\lambda s} P_s x \, ds$ exists for $\forall x \in V$.

Then $\lambda \in \rho(\mathcal{Q})$. $R_\lambda(\mathcal{Q}) x = \int_0^\infty e^{-\lambda s} P_s x \, ds$. $\forall x \in V$.

iii) $\| R_\lambda(\mathcal{Q}) \| \le \dfrac{c}{Re(\lambda) - \gamma}$. for $\forall Re(\lambda) > \gamma$.

Pf: i) $Z_\lambda x = \int_0^\infty e^{-\lambda s} P_s x \, ds$ converges if $Re(\lambda) > \gamma$.

So it's well-def. $\forall x \in V$.

Note: $\lim\limits_{h \to 0} \dfrac{P_h - I}{h} Z_\lambda x = \lim\limits_{h \to 0} \int_0^\infty \dfrac{e^{-\lambda s} (P_{s+h} x - P_s x)}{h} ds$

$= \lim\limits_{h \to 0} \dfrac{e^{\lambda h} - 1}{h} \int_0^\infty e^{-\lambda s} P_s x \, ds - \dfrac{e^{\lambda h}}{h} \int_0^h e^{-\lambda s} P_s x \, ds$

$= \lambda Z_\lambda(x) - x$. i.e. $\mathcal{Q} Z_\lambda x = \lambda Z_\lambda x - x$

$\Rightarrow (\lambda I - \mathcal{Q}) Z_\lambda x = x$. Next, show $\lambda I - \mathcal{Q}$ is injection

By contradiction: $\exists x \in D(\mathcal{Q}) / \{0\}$. $\mathcal{Q} x = \lambda x$

By Uniqueness: $P_t x = e^{t\lambda} x$. contradict!

ii) Set $T_t = e^{-\lambda t} P_t$. Its generator $A = \mathcal{Q} - \lambda I$.

$\dfrac{T_h - I}{h} \int_0^\infty T_s x \, ds = -\dfrac{1}{h} \int_0^h T_s x \, ds \xrightarrow{h \to 0} -x$.

$\therefore \int_0^\infty T_s x \, ds \in D(A)$. $A \int_0^\infty T_s x \, ds = -x$. $\forall x \in V$

For injection: Note: $\lim\limits_{t \to \infty} \int_0^t T_s x \, ds = \int_0^\infty T_s x \, ds$.

By A is CLO. $x \in D(A) \Rightarrow \int_0^\infty T_s A x \, ds = A \int_0^\infty T_s x \, ds = -x$.

If $\exists \eta \in D(A) / \{0\}$. $A\eta = 0 \Rightarrow \eta = 0$. Contradict!

$\therefore 0 \in \rho(A)$. i.e. $\lambda \in \rho(\mathcal{Q})$. And equation holds.

iii) is direct from i).

<u>Lemma.</u> $\mathcal{A}$ is generator of $(P_t) \subset C_0$. Then $\mathcal{A}$ is CLO.

<u>Pf.</u> If $x_n \to x$, $\mathcal{A}x_n = \eta_n \to \eta$. $\exists \lambda \in \rho(\mathcal{A})$. $(Re\lambda > \gamma)$

$\therefore (\lambda I - \mathcal{A})x_n = \lambda x_n - \eta_n \to \lambda x - \eta$

On the other hand. $(\lambda I - \mathcal{A})^{-1}$ is conti. $\therefore \eta = \mathcal{A}x$.

<u>Lemma.</u> (Resolvent Equations)

If $a, b \in \rho(A)$. A is unbdd linear operator.

Then: $R_a(A) - R_b(A) = (b-a) R_b(A) R_a(A)$.

<u>Pf:</u>
$$\begin{cases} (aR_a(A) - AR_a(A))R_b(A) = R_b(A) \\ R_a(A)(bR_b(A) - AR_b(A)) = R_a(A) \end{cases}$$

subtract the two equations.

Check: $AR_a(A) = -I_V + aR_a(A)$

$R_a(A)A = -I_{D(A)} + aR_a(A)$.

<u>Rmk:</u> It means $R_a(A)$. $R_b(A)$ are commutative.

<u>Lemma.</u> $\mathcal{A}: D(\mathcal{A}) \to V$. densely defined CLO. If $\exists \gamma$. and $C > 0$. st. $[\gamma, \infty) \subset \rho(\mathcal{A})$. $\|\lambda R_\lambda(\mathcal{A})\| \le C$. for all $\lambda \ge \gamma$. Then:

i) $\forall x \in V$. $\lim_{\lambda \to \infty} \|\lambda R_\lambda(\mathcal{A})x - x\| = 0$

ii) $\forall x \in D(\mathcal{A})$. $\lim_{\lambda \to \infty} \|\lambda R_\lambda(\mathcal{A})\mathcal{A}x - \mathcal{A}x\| = 0$

Pf: i) $\| \lambda R_\lambda(a) x - x \| = \| R_\lambda(a) a x \|$

$\leq \frac{c}{|\lambda|} \| a x \| . \to 0$

It hold for $\forall x \in D(a)$. Let $\lambda > \gamma$.

For $z \in V$. Since $D(a)$ is dense. By approxi.

ii) By $a R_\lambda(a) x = R_\lambda(a) a x$. $\forall x \in D(a)$

Thm. (Hille-Yosida)

For unbdd linear operator L with $D(L) \subseteq V$.

Then L is generator of (S_t). with $\|S_t\| \leq$

$M e^{at}$. $M > 0$. $a \in \mathbb{R}^1$. $\forall t \geq 0$ $\Leftrightarrow$ L is densely def.

cLO. "$\forall \lambda \in \mathbb{C}$." $Re(\lambda) > a$. Then: $\lambda \in \rho(L)$. With:

$\| R_\lambda(L)^n \| \leq M / (Re(\lambda) - a)^n$: $M > 0$. $\forall n \in \mathbb{N}^+$.

Rmk: Replace $\lambda \in \mathbb{C}$ by $\lambda \in \mathbb{R}^1$. It still holds.

Pf: $(\Rightarrow)$. $R_\lambda^n(L) =: R_\lambda^n = \int_{\mathbb{R}^n_{\geq 0}} e^{-\lambda(t_1 + \cdots t_n)} S(t_1 + \cdots t_n) \, d\vec{t}$

$(\Leftarrow)$ The ideal: (Consider $\lambda \in \mathbb{R}^1$. Then we proved Rmk)

Set $L_\lambda = \lambda L R_\lambda(L)$. "Yosida Approxi".

prove: $L_\lambda \to L$ as $\lambda \to \infty$.

Besides $L_\lambda \in \mathcal{L}(V)$. correspond e^{tL_λ}.

where $e^{tL_\lambda} \longrightarrow S_t$ as $\lambda \to \infty$.

s.t. S_t satisfies the conditions.

1) $L R_\lambda$ is uniform bdd for large λ:

for $x \in D(L)$. $\| L R_\lambda x \| = \| R_\lambda L x \|$

$= \| (\lambda R_\lambda - 1) x \| \leq (M \lambda (\lambda - a)^{-1} + 1) \| x \|$

So extend LR_λ from $D(L)$ to V.

2°) $LR_\lambda x \xrightarrow{\lambda\to\infty} 0$:

Check $x \in D(L)$: It's from Lemma above

3°) $L_\lambda x \to Lx$. $\forall x \in D(L)$

$\| L_\lambda x - Lx \| = \| (\lambda R_\lambda - I) Lx \| = \| L R_\lambda Lx \|$. By 2°).

4°) Define $S_\lambda(t) = e^{L_\lambda t}$. C_0-semigroup for $L_\lambda \in \mathcal{L}(V)$.

(Note: for each $\lambda \neq a$. L_λ is bdd)

5°) $\| S_\lambda(t) \| \leq M e^{\lambda a t/(\lambda - a)}$.

Note $L_\lambda = -\lambda + \lambda^2 R_\lambda$ $\therefore S_\lambda(t) = e^{-\lambda t} \cdot e^{\lambda^2 t R_\lambda}$

By expansion: $\| S_\lambda(t) \| \leq e^{-\lambda t} \sum \frac{t^n \lambda^{2n}}{n!} \| R_\lambda^n \|$. By condition.

So we obtain: $\overline{\lim_{\lambda\to\infty}} \| S_\lambda(t) \| \leq M e^{at}$.

6°) $\lim_\lambda S_\lambda(t) x$ exists. $\forall t \geq 0$. $x \in V$.

Let λ, μ large enough s.t. $\max [\| S_\lambda(t) \|, \| S_\mu(t) \|] \leq 2M e^{at}$

$\| \frac{\partial}{\partial s} S_\lambda(t-s) S_\mu(s) x \| = \| S_\lambda(t-s) S_\mu(s) (L_\mu - L_\lambda) x \|$

$\leq 4M^2 e^{2at} \| (L_\mu - L_\lambda) x \|$.

(Note: $L_\lambda = -\lambda + \lambda^2 R_\lambda$. So there commutative.

Since: $\| S_\lambda(t) x - S_\mu(t) x \| = \| \int_0^t \partial_s S_\lambda(t-s) S_\mu(s) x \, ds \|$

$\leq 4M^2 t e^{2at} \| (L_\mu - L_\lambda) x \| \to 0$

7°) Define $S(t) x = \lim_{\lambda\to\infty} S_\lambda(t) x$.

$\therefore \| S_t x \| \leq e^{at} \cdot M$. $S_t \circ S_r = S_{t+r}$ follows from $S_\lambda(t)$.

By Letting $\lambda \to \infty$. It's truly semigroup.

And for $\forall$ fix $x \in D(L)$. bdd interval of t.

$S_\lambda(t)x \longrightarrow S(t)x$. uniform with t.

With $\|S(t)\| \le Me^{at}$. By Lemma. in ① $\Rightarrow S_t \in C_0$.

8') generator of $S(t)$: $\hat{L}$ coincides with L.

$$\frac{S_\lambda(t)x - x}{t} = \frac{1}{t}\int_0^t S_\lambda(s) L_\lambda x \, ds . \quad x \in D(L)$$

set $\lambda \to \infty$. Then $t \to 0$. $\therefore \hat{L}x = Lx \quad \therefore D(L) \subseteq D(\hat{L})$

With $Re(\lambda) > a$. $\lambda - L$ and $\lambda - \hat{L}$ are bijection $\Rightarrow L = \hat{L}$.

Rmk: $R_\lambda(L)$ is bdd isn't result of $\sigma(L)$ falls in half-space.

e.g. $V = \bigoplus_{n \ge 1} \mathbb{C}^2$. equipped with Euclidean norm

$L = \bigoplus_{n \ge 1} L_n$. where $L_n : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$. given by

$L_n = \begin{pmatrix} in & n \\ 0 & in \end{pmatrix}$. $\sigma(L_n) = \{in\}$.

$$\frac{n}{|\lambda - in|^2} \le \|R_\lambda^{(n)}\| =: \|(\lambda I_2 - L_n)^{-1}\| \le \frac{n}{|\lambda - in|^2} + \frac{\sqrt{2}}{|\lambda - in|}$$

$\|R_\lambda(L)\| =: \sup_{n \ge 1} \|R_\lambda^{(n)}\|$.

Note: $\begin{cases} \sigma(L) = \{in\}_{n \ge 1} \subseteq \mathbb{R}_+^2 \text{ fall in half-space} \\ \|R_{a+in}\| \ge n/a^2 . \text{ doesn't satisfies bdd.} \end{cases}$

Cor. From ($\Leftarrow$) Part. $S_t = \lim_\lambda S_\lambda(t)$. satisfies:

$[S_t . R_\lambda(L)] = 0$

Pf: $e^{tL_m} R_\lambda = \Sigma \frac{t^n L_m^n}{n!} R_\lambda = R_\lambda \Sigma \frac{t^n L_m^n}{n!}$

$= R_\lambda e^{tL_m}$. Let $m \to \infty$

Diagram:

$$\begin{tikzcd}
 & (P_t)_{t\geq 0} \subseteq C_0 \arrow[dl, shift right, "Ax=\lim_{t\to 0}\frac{P_tx-x}{t}"'] \arrow[dr, "R_\lambda(A)=\int_0^\infty e^{-\lambda t}P_t\,dt.\ \mathrm{Re}(\lambda)>\gamma"] & \\
(A.D(A)) \arrow[ur, shift right, "\text{Yosida}"'] \arrow[rr, shift left, "R_\lambda(A)=(\lambda I-A)^{-1}"] & & R_\lambda(A).\ \lambda\in\rho(A) \arrow[ll, shift left, "A=\lambda I-R_\lambda(A)."]
\end{tikzcd}$$

(4) Self-adjoint generators:

Def: i) A is self adjoint operator with $D(A)\subseteq H$. Hilbert space if $A=A^*$. A is symmetric if $A\subset A^*$

ii) $B: H\to H$. densely def on $D(B).\subseteq H$. Hilbert space. B is negative definite if $(Bx,x)\leq 0$. $\forall x\in D(B)$. Denote it by $B\leq 0$. Similarly for $B\geq 0$.

Rmk: i) A is self adjoint. Then: $A\leq 0 \iff \sigma(A)\subseteq(-\infty,0]$.

ii) A is injective self adjoint $\Rightarrow R(A)$ is dense. A^{-1} is self-adjoint. $(A: D(A)\xrightarrow{\sim} R(A))$

Pf. i) follows from: $\sigma(A)\subseteq[m,M]$. $\begin{cases} M=\sup(Au,u) \\ m=\inf(Au,u)\end{cases}$

ii) $N(A)=\{0\}=R(A)^\perp \Rightarrow R(A)$ is dense.

$A^{-1}: R(A)\to D(A)$. which is well-def on $R(A)$.

Then follows by def: $D(A^*)=\{\eta\in H: (Ax,\eta)$ is conti$\}$

Thm. (Spectral Decomposition)

i) L is self-adjoint operator on separable Hilbert space H. Then: $\exists$ finite measure space (M,μ). $k: H\cong L^2(M,\mu)$. unitary. $f_L: M\to\mathbb{R}^1$. measurable. associated with ~

multiplication F_L. $F_L(f)(x) = f_L(x) f(x)$. s.t.

$L = K^{-1} F_L K$. $K(D(L)) = \{ g \mid f_L(x) g(x) \in L^2(M,\mu) \}$

ii) $(A, D(A))$ is self adjoint on separable Hilbert space

H with cpt resolvent. Then $\exists (\sigma_n) \subseteq \mathbb{R}^1$. and

unitary operator: $U: H \to l^2$. s.t. $A = U^{-1} A_\sigma U$.

where $A_\sigma : l^2 \to l^2$. $A_\sigma (x_n) = (\sigma_n x_n)$ with

$D(A_\sigma) = \{ x \in l^2 \mid (\sigma_n x_n) \in l^2 \}$.

Moreover. $\exists \{ \phi_n \}$ orthonormal basis which is eigen-vectors of A. $Ax = \sum \sigma_n (x, \phi_n) \phi_n$.

<u>Rmk</u>: i) It's likewise that for matrix A. $A = P^{-1} B P$.

ii) From i): It allows us to define: $C_L(f) =$

$f(L) = K^{-1} (f \circ F_L) K$ for $f: \mathbb{R}^1 \to \mathbb{R}^1$. which

is measurable. Note that: $f(L)$ is self adjoint.

$fg(L) = K^{-1} (fg \circ F_L) K = K^{-1} (f \circ F_L) K K^{-1} (g \circ F_L) K$

$= f(L) g(L) = g(L) f(L) \Rightarrow$ homomor.

<u>Thm</u>: (Resolvent Calculus)

A is self adjoint on Hilbert space H. Then $\exists$ unique projection-value measure E on $B_{\mathbb{R}^1}$. s.t. $\operatorname{supp}(E) \subseteq \sigma(A)$. $E(\sigma(A)) = I$. $A = \int_{\sigma(A)} \lambda \, dE(\lambda)$.

Moreover. denote $E_{xy} = (E(\cdot) x, y)$. for $x, y \in H$.

$E_{x,y}$ is Borel measure. $D(A) = \{ x \in H \mid \int_{\mathbb{R}^1} \lambda^2 \, dE_{xx} < \infty \}$

$(Ax, y) = \int_{\mathbb{R}^1} \lambda \, dE_{xy}(\lambda)$. $\forall x \in D(A)$.

Rmk: For $\forall f : \mathbb{R}^1 \to \mathbb{C}$. $\mathcal{B}_{\mathbb{R}}$-measurable. Def:

$$f(A) = \int_{\sigma(A)} f(\lambda)\, dE(\lambda). \quad D(f(A)) = \{ x \in H \mid$$

$$\int_{\mathbb{R}^1} |f(\lambda)|^2 \, dE_{xx}(\lambda) < \infty \}.$$ Then: We have:

$f(A)$ is self-adjoint $\Leftrightarrow$ f is real-valued.

$f(A) g(A) = g(A) f(A) = fg(A)$. on $D(fg(A))$

Thm. Self-adjoint negative definite operator A on Hilbert space is generator of contraction Semigroups S_t. which is self-adjoint as well.

Rmk: Converse is true: generator A of contraction semigroup S_t which is self-adjoint. is negative definite and self-adjoint.

Pf: Note that $(0, \infty) \subseteq \rho(A)$. for $\forall \lambda > 0$:

$$((\lambda I - A)\eta, \eta) \geq \lambda(\eta, \eta). \quad \text{Then:}$$

$$\lambda \|(\lambda I - A)^{-1} x\|^2 = \lambda((\lambda I - A)^{-1} x, (\lambda I - A)^{-1} x)$$

$$\leq (x, (\lambda I - A)^{-1} x)$$

$$\leq \|x\| \, \|(\lambda I - A)^{-1} x\|.$$

$\therefore \|R_\lambda(A)\| \leq 1/\lambda$. Apply Hille-Yosida Thm.

Note: $A(\varepsilon, t) = e^{tA(I - \varepsilon A)^{-1}} \xrightarrow{\varepsilon \to 0} S_t$. (Yosida. approxi.)

It's self-adjoint. by expansion. (Or Use: $\lambda A R_\lambda$)

$$\Rightarrow (S_t x, \eta) = \lim (A(\varepsilon, t) x, \eta) = \lim (x, A(\varepsilon, t)\eta) = (x, S_t \eta)$$

for $\forall x \in H$.

Conversely. $\|S_t\| \le 1 \Rightarrow (0,\infty) \subseteq \rho(A)$.

$R_\lambda(A) = \int_0^\infty e^{-\lambda t} S_t \, dt$. self-adjoint. injective.

So that $\lambda I - A$ is self-adjoint.

<u>Cor.</u> $S_t = e^{tA}$. in sence of resolvent calculus.

<u>Pf:</u> $\int_{\mathbb{R}^1} |e^{t\lambda}|^2 \, dE_{x,x}(\lambda) = \int_{-\infty}^0 |e^{t\lambda}|^2 \, dE_{xx}$

$\le (E(\sigma(A))x, x) = \|x\|^2. \ \forall x \in H$.

$\Rightarrow e^{tA} \in \mathcal{L}(H)$.

$\frac{\partial}{\partial t} e^{tA} = \int_{\mathbb{R}^1} \lambda e^{t\lambda} \, dE_x = A e^{tA}$.

<u>Rmk:</u> Analogously. define square root

of $-A$: $\sqrt{-A} = \int_{-\infty}^0 \sqrt{-\lambda} \, dE(\lambda)$.

where A is self-adjoint. $A \le 0$.

$\int_{-\infty}^0 |\sqrt{-\lambda}|^2 \, dE_{xx} = \int_{-1}^0 -\lambda \, dE_{xx} + \int_{-\infty}^{-1} -\lambda \, dE_{xx}$

$\le \|x\|^2 + \int_{-\infty}^0 |\lambda|^2 \, dE_{xx} < \infty$

$\Rightarrow \forall x \in D(A)$. Then : $x \in D(\sqrt{-A})$

Besides : $\sqrt{-A}\sqrt{-A}\, x = \int_{\mathbb{R}^1} |\sqrt{-\lambda}|^2 \, dE(\lambda) x$

$= -Ax. \ \forall x \in D(A)$.

<u>prop.</u> L is self-adjoint. $L \le 0$. Then e^{tL} maps H to

$D(I-L)^\tau$. $\forall \tau \in \mathbb{R}^1$. $t \ge 0$. $\exists C_\tau$. st.

$\|(I-L)^\alpha e^{tL}\| \le C_\alpha(1+t^{-\alpha})$. holds

Pf. $(I-L)^\alpha$ is well-def in sense of Calculus. $\alpha \notin \mathbb{Z}$.

with $I-L \in \mathcal{L}(H)$. so $\alpha \in \mathbb{Z}$.

$$\|(I-L)^\alpha e^{tL}\| = \left\| \int_{\sigma(L)} (1-\lambda)^\alpha e^{t\lambda}\, dE(\lambda) \right\|$$

$$\le \sup_{\lambda \le 0} (1-\lambda)^\alpha e^{t\lambda}$$

$$\lesssim_\alpha \sup_{\lambda\le 0} (1+(-\lambda)^\alpha)\, e^{t\lambda} \le C_\alpha(1+t^{-\alpha})$$

⑤ Adjoint Semigroups:

prop. $S_t \in C_0$. on B. Then: $S^*(t) \in C_0$ on $B^+ = \overline{D(L^*)}$ in B^*. With generator $L^+ = L^*|_{D(L^+)}$

$D(L^+) = \{x \in D(L^*) \mid L^* x \in B^+\}$.

Rmk: generally. $S(t) \in C_0 \not\Rightarrow S^*(t) \in C_0$.

e.g. $B = C([0,1], \mathbb{R}^1)$. $S(t)$ is heat semi-group. with Neumann boundary condition.

However. we can restrict $S^*(t)$ on a smaller space.

Pf. 1°) S_t is bdd on B^+. $\forall t \ge 0$.

Note: $\|S_t\| = \|S_t^*\|$. And:

$S_t^*: D(L^*) \longrightarrow D(L^*)$. extend to B^+.

($\langle l, S_t x\rangle$ is differentiable. $\forall l \in D(L^*)$)

2') $S_t^* x \to x$. $\forall x \in D(L^*) \underset{\text{dense}}{\subseteq} B^+$.

It follows directly: $S_t^* x - x = \int_0^t S_s^* L^* x \, ds$

3') $D(L^+)$ is given by: R_λ^+ of S_t^* on B^+

is $R_\lambda^* |_{B^+}$.

<u>prop</u>. $\forall l \in B^*$. $\exists l_n \in B^+$. $l_n \xrightarrow{*} l$. as $n \to \infty$.

<u>Pf</u>: $l_n = n R_n^* l \Rightarrow l_n \in D(L^*) \subseteq B^+$.

By $\| n R_n x - x \| \xrightarrow{n \to \infty} 0$. $\therefore$ $\langle l_n, x \rangle \xrightarrow{n \to \infty} \langle l, x \rangle$

<u>Rmk</u>: It means B^+ is large enough to be dense

in B^* in weak*-topo sense.

(3) Analytic Semigroups:

<u>Def</u>: Semigroups S_t on B is analytic if $\exists \theta > 0$.

s.t. $t \longmapsto S_t$ has analytic extension on $\{\lambda \in \mathbb{C} \mid$

$|\arg \lambda| < \theta\}$. $S(e^{i\varphi} t) \subseteq C_0$, $\forall \varphi$. $|\varphi| < \theta$.

<u>Rmk</u>: Denote $S_\varphi(t) = S(e^{i\varphi} t)$. $\Rightarrow \| S_\varphi(t) \| \le M(\varphi) e^{a(\varphi) t}$.

since $S_\varphi(t) \subseteq C_0$.

<u>prop</u>. $\forall \theta' < \theta$. Then there $\exists M, a$. s.t. $\| S_\varphi(t) \| \le M e^{at}$.

$\forall t \geq 0$. $|\varphi| \le \theta'$.

<u>Pf</u>: $t e^{i\varphi} = t_1 e^{i\theta'} + t_2 e^{-i\theta'}$, where $0 \le t_1, t_2 \lesssim t$.

prop. $\forall \varphi$. $|\varphi| < \theta$. generate L_φ of S_φ is $e^{i\varphi}L$

where L is generator of S.

Pf. $R_\lambda x = \int_0^\infty e^{-\lambda t} S_t x \, dt$. for $\lambda > \mathrm{Re}(\gamma)$.

By $e^{-\lambda t} S_t$ is analytic in $\{|\arg t| < \theta\}$.

Set $t = e^{i\varphi} t$. $R_\lambda x = e^{i\varphi} \int_0^\infty e^{-\lambda e^{i\varphi} t} S(e^{i\varphi} t) x \, dt$

$\therefore R_\lambda = e^{i\varphi} R^{\varphi}_{\lambda e^{i\varphi}} \Rightarrow L_\varphi = e^{i\varphi} L$.

Thm. (Hille - Yosida)

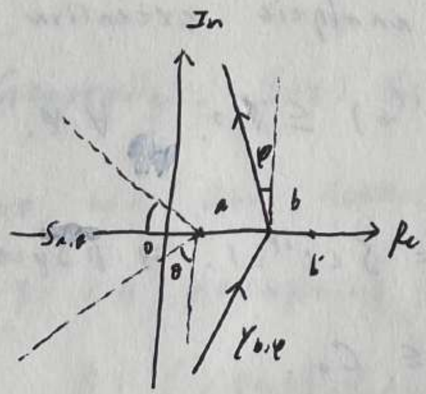
L is generator of analytic semigroup $S(t)$ on B

$\Longleftrightarrow \exists \theta \in (0, \frac{\pi}{2})$. $a \geq 0$. st. $\sigma(L) \subseteq S_{\theta,a} = \{\lambda \in \mathbb{C} \mid \arg(a-\lambda) \in [-\frac{\pi}{2}+\theta, \frac{\pi}{2}-\theta]\}$. and. $\exists M > 0$. st. $\|R_\lambda\| \leq \frac{M}{d(\lambda, S_{\theta,a})}$

for $\lambda \notin S_{\theta,a}$. Besides. L is densely def CLO.

Pf. ($\Rightarrow$) Apply Yosida Thm on $\{(S_\gamma(t))_{t \geq 0}\}_{|\gamma| < \theta}$

($\Leftarrow$)



Set $\varphi \in (0, \theta)$. $b > a$.

For t. $|\arg t| < \varphi$

Define $S(t)$ by:

$$S(t) = \frac{1}{2\pi i} \int_{\gamma_{\varphi,b}} e^{tz} R_z(L) \, dz$$

It's well-def. since $\|R_z\| \leq M$. on $\gamma_{\varphi,b}$

$(x, t) \longmapsto S(t)x$ is jointly conti by $S(t)$

is uniformly convergent on cpt set $\subseteq \{|\arg t| < \theta\}$.

Next. check $S(t)$ satisfies semigroup property.

Note : Choice of b is arbitrary since for $b' > a$.

$e^{zt} R_z$ is analytic among $\gamma_{\varphi,b}$, $\gamma_{\varphi,b'}$.

$\therefore S(s) S(t) = \int_{\gamma_{\varphi,b}} \int_{\gamma_{\varphi,b'}} e^{tz+sz'} R_z R_{z'} \, dz \, dz'$. where $b' > b$

with $R_z R_{z'} = \dfrac{R_{z'} - R_z}{z' - z}$. Then by Residue Formula

Then : $S(t) \subseteq C_0$. its correspond generator is $\hat{L}$.

Show $\hat{L} = L \iff \hat{R}_\lambda = R_\lambda$. for some λ.

choose b, λ large enough. λ is enclosed in $\gamma_{\varphi,b}$.

$\hat{R}_\lambda = \int_0^\infty e^{-\lambda t} S(t) \, dt = \frac{1}{2\pi i} \int_{\gamma_{\varphi,b}} \int_0^\infty e^{t(z-\lambda)} dt \, R_z \, dz = R_z$

For analytic : S_t is also indept with choice of φ.

Set φ closed to θ. $\therefore t \longmapsto S_t$ is analytic. $|\arg t| < \theta$.

<u>Thm.</u> (Perturbation)

L_0 is generator of analytic semigroup. $B : D(B) \to \mathcal{B}$.

st. $D(B) \supseteq D(L_0)$. $\forall \varepsilon > 0$. $\exists c > 0$. st. $\|Bx\| \le \varepsilon \|L_0 x\| + c\|x\|$.

for $\forall x \in D(L_0)$.

Then $L = L_0 + B$ with $D(L) = D(L_0)$ is generator of some analytic semigroup as well.

<u>Pf:</u> Apply Yosida Thm : Find $S_{\theta,a}$.

1°) $\exists S_{\theta,a}$, $c < 1$. st. $\|B R_\lambda^0\| \le c$. $\forall \lambda \in S_{\theta,a}$.

Note $\|B R_\lambda^0 x\| \le \varepsilon \|L_0 R_\lambda^0 x\| + c \|R_\lambda^0 x\|$.

$\exists S_{\tau,b}$. st. $\|R_\lambda^0\| \le M d^{-1}(\lambda, S_{\tau,b})$. $L_0 R_\lambda^0 = \lambda R_\lambda^0 - 1$.

$\therefore \|B R_\lambda^0\| \le \dfrac{(\varepsilon|\lambda| + c) M}{d(\lambda, S_{\tau,b})} + \varepsilon$. Find θ, a.

2') Consider $y = (\lambda I - L)x$. $x \in D(L_0)$.

$\exists z$. $x = R_\lambda^\circ z$. $\therefore y = (I - BR_\lambda^\circ)z$

$$\|R_\lambda y\| = \|x\| = \|R_\lambda z\| = \|R_\lambda^\circ (I - BR_\lambda^\circ)^{-1} y\|$$

$$\le \|R_\lambda^\circ\| \frac{\|y\|}{1 - \|BR_\lambda^\circ\|} \lesssim \frac{\|y\|}{d(\lambda, S_{\sigma,b})}$$

Since $S_{\theta,\alpha} \supseteq S_{\sigma,b}$ $d(\lambda, S_{\theta,\alpha}) \le d(\lambda, S_{\sigma,b})$. $\forall \lambda \notin S_{\theta,\alpha}$

(4) Interplation Space:

- Consider analytic semigroup $S(t)$ with generator L.

St. $\exists M, w > 0$. $\|S(t)\| \le Me^{-wt}$. So $\sigma(L) \subseteq \mathbb{R}_{<0}$

Def: $(-L)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} S(t)\,dt$. negative. fractional power of L. $\alpha > 0$. $(-L)^\alpha$ is inverse of $(-L)^{-\alpha}$. $D(-L)^\alpha = R(-L)^{-\alpha}$.

Rmk: $(-L)^{-\alpha}(-L)^{-\beta} = (-L)^{-\alpha-\beta}$. can be checked directly.

prop. $(-L)^{-\alpha}$ is injective. for $\forall \alpha > 0$.

Pf: $(-L)^{-n} = (-L)^{-n+\alpha}(-L)^{-\alpha}$. $n \in \mathbb{Z}^+$.

But $(-L)^{-n}$ is bijective $(0 \in \rho(L))$

Def: Interplation space B_α is $(D((-L)^\alpha), \|x\|_\alpha)$ where.

$\|x\|_\alpha = \|(-L)^\alpha x\|$.

Rmk: Actually, we don't need assumption at begin.

Since we can replace $-L$ by $\lambda - L$ for large fix λ. And $\|(-L)^\alpha x\| \sim \|(\lambda - L)^\alpha x\|$.

prop. i) $B_\alpha \subseteq B_\beta$. $\forall \alpha \geq \beta$. ($(-L)^{-p}$ is bdd. $\forall p>0$)

ii) $(-L)^\tau x = \frac{\sin(\alpha\pi)}{\pi} \int_0^\infty t^{\tau-1} (t-L)^{-1} (-L) x \, dt$

for $\tau \in (0,1)$. $x \in D(L)$.

iii) $\forall \tau \in (0,1)$. $\exists C>0$. st. $\|(-L)^\tau x\| \leq C \|Lx\|^\tau \|x\|^{1-\tau}$

holds for $\forall x \in D(L)$.

Pf. i) $\|(-L)^\beta x\| = \|\int_0^\infty t^{\alpha-\beta-1} S_t (-L)^\alpha x \, dt\| \lesssim \|(-L)^\tau x\|$.

ii) $(t-L)^{-1} = \int_0^\infty e^{-ts} S(s) ds$. Transf. variables.

iii) Apply ii). Note: $(t-L)^{-1}(-L) = 1 - t(t-L)^{-1}$

$\int_0^k \square \lesssim k^\alpha \|x\|$. $\int_k^\infty \square \lesssim \int_k^\infty t^{\alpha-2} \|Lx\|$

optimize k.

prop. $\forall t>0$. $k \in \mathbb{Z}^+$. $S(t)$ maps B to $D(L^k)$. And $\exists C_k$.

st. $\|L^k S(t) x\| \leq \frac{C_k}{t^k} \|x\|$. $\forall x \in B$. $t \in (0,1]$.

Pf. Note: $L R_\lambda = \lambda R_\lambda - 1$. $\int_{\gamma_{\varphi,b}} e^{tz} dz = 0$. $|\arg z| < \varphi$.

$L S(t) = \int_{\gamma_{\varphi,b}} \frac{1}{2\pi i} L R_z e^{zt} dz. = \int_{\gamma_{\varphi,b}} \frac{1}{2\pi i} z R_z e^{zt} dz$

$\therefore L^k S(t) = \frac{1}{2\pi i} \int_{\gamma_{\varphi,b}} z^k e^{tz} R_z dz$. Directly estimate.

prop. $\forall t>0$. $\tau>0$. $S(t)$ maps B to B_τ. And $\exists C_\alpha$. st.

$\|(-L)^\alpha S_t x\| \leq \frac{C_\tau}{t^\alpha} \|x\|$. $\forall t \in (0,1]$

Pf. $\exists n \in \mathbb{Z}^+$. $B_n = D(L^n) \subseteq B_\alpha$. $S(B) \subseteq \bigcap D(L^k)$.

$\Rightarrow S(B) \subseteq B_\tau$. (Moreover $S(B) \subseteq \bigcap_{\tau \in \mathbb{R}} B_\tau$).

Note: $(-L)^\alpha = (-L)^{\tau - [\tau] - 1} (-L)^{[\tau]+1}$

$$\Rightarrow (-L)^{\alpha} S(t) = \frac{(-1)^{[\alpha]+1}}{\Gamma([\alpha]-\alpha+1)} \int_0^{\infty} s^{[\alpha]-\alpha} L^{[\alpha]+1} S(t+s)\, ds$$

Apply the previous estimate for $k = [\alpha]+1$

$$\int_0^{\infty} s^{[\alpha]-\alpha} \| L^{[\alpha]+1} S(t+s) \| \lesssim \int_0^{\infty} s^{[\alpha]-\alpha} \left[\sum_{k=1}^{[\alpha]} \frac{1}{(t+s)^k} + \frac{e^{-w(t+s)}}{(t+s)^{[\alpha]+1}} \right] ds$$

$$\leq C_\alpha \left(\sum_{k=1}^{[\alpha]} t^{-\alpha+k} \right) \leq C_\alpha t^{-\alpha}$$

Cor. i) $S(t)$ maps B_α into B_β. $\forall \alpha, \beta \in \mathbb{R}'$, $\beta > \alpha$

i.e. $\| S(t) x \|_{B_\beta} \leq C(\alpha, \beta) \| x \|_{B_\alpha} t^{\alpha-\beta}$. $\forall t \in (0,1]$

ii) $\forall \alpha \in \mathbb{R}'$, $\forall \beta \in [\alpha, \alpha+1)$. Then $\exists C > 0$. s.t.

$\| (t-L)^{-1} x \|_{B_\beta} \leq C (1+t)^{\beta-\alpha-1} \| x \|_{B_\alpha}$. $\forall t \geq 0$.

Pf. i) $(-L)^{\gamma}$ commutes with $S(t)$.

ii) $\| R_t(L) x \|_{B_\beta} = \| \int_0^{\infty} e^{-ts} S(s) x\, ds \|_{B_\beta}$

$$\leq \int_0^{\infty} e^{-ts} \| S(s) x \|_{B_\beta}\, ds$$

Then apply i). Directly.

prop. (Speed of Convergence)

For $\forall \alpha \in (0,1)$. $\exists C_\alpha$. s.t. $\| S(t) x - x \| \leq C_\alpha t^{\alpha} \| x \|_{B_\alpha}$

for every $x \in B_\alpha$. $t \in (0,1]$.

Pf. By density, prove it holds for $x \in D(L)$.

$\| S(t) x - x \| = \| \int_0^t S(s) L x\, ds \| = \| \int_0^t (-L)^{1-\alpha} S(s) (-L)^{\alpha} x\, ds \|$

$$\lesssim \|x\|_{B_\tau} \int_0^t s^{\tau-1} ds = t^\tau \|x\|_{B_\tau}.$$

<u>prop.</u> (Perturbation)

L_0 is generator of analytic semigroup $S(t)$ on B.

Denote B_γ° its interplation space. Let $B: B_\tau^\circ \to B$.

bdd for some $\tau \in [0,1)$. $L = L_0 + B$.

Then, for interplation B_γ of L. $B_\gamma = B_\gamma^\circ$. $\forall \gamma \in [0,1]$.

<u>Pf:</u> 1°) $\gamma = 0, 1$. is trivial.

2°) Next, show: $c^{-1} \|(-L_0)^\gamma x\| \le \|(-L)^\gamma x\| \le c\|(-L_0)^\gamma x\|$.

$\exists\, c > 0$. for $\forall x \in D(L_0)$. $0 < \gamma \le \alpha$.

3°) Note: $D(L) = D(L_0)$. BR_t is bdd. $\forall t$.

And $R_t = R_t^\circ + R_t^\circ B R_t$. ⋯ ($\Delta$)

$$\|BR_t x\| \le C\|R_t x\|_{B_\tau^\circ} \le C\|R_t^\circ x\|_{B_\tau^\circ} + C\|R_t^\circ B R_t x\|_{B_\tau^\circ}$$

$$\le C(1+t)^{\tau-\gamma-1}\|x\|_{B_\gamma^\circ} + C(1+t)^{\alpha-1}\|BR_t x\|$$

for $\forall x \in B_\gamma^\circ$. $\gamma \le \alpha < 1 < 1+\gamma$

Then if t is large enough: $C(1+t)^{\alpha-1} < 1$

$\therefore \|BR_t x\| \lesssim (1+t)^{\tau-\gamma-1}\|x\|_{B_\gamma^\circ}$ ⋯ (*)

Extend to $\forall s > 0$. by $R_s = R_t + (t-s)R_t R_s$

4°) $\|x\|_{B_\gamma} = \| \frac{\sin \pi\gamma}{\gamma} \int_0^\infty t^{\gamma-1}(t-L)^{-1}(-L)x \|$

$$\lesssim \|\int_0^\infty t^{\gamma-1} L_0 R_t^\circ x\, dt\| + \int_0^\infty t^{\gamma-1}\|(L_0 R_t^\circ + 1) B R_t x\| dt$$

$$\lesssim \|x\|_{B_\gamma^\circ} + \int_0^\infty t^{\gamma-1}\|BR_t x\| dt$$

$$\lesssim \|x\|_{B_\gamma^\circ} \quad \text{by } (*).$$

5°) Conversely. $\|x\|_{B_\gamma^\circ} = \| \frac{\sin \pi\gamma}{\pi} \int_0^\infty t^{\gamma-1} (t-L_0)^{-1}(-L_0) x \, dt \|$

$$\overset{(\text{By } \Delta)}{\leq} \|x\|_{B_\gamma} + C \int_0^\infty t^{\gamma-1} \|BR_t x\| \, dt.$$

By resolvent equation: $\forall k > 0$

$$\int_0^\infty t^{\gamma-1} \|BR_t x\| \lesssim \int_0^\infty t^{\gamma-1} \|BR_{t+k} x\| \, dt + k \int_0^\infty t^{\gamma-1} \|BR_{t+k} R_t x\|$$

$$\lesssim \int_0^\infty t^{\gamma-1} (t+k)^{\tau-\gamma-1} dt \, \|x\|_{B_\gamma} + k \int_0^\infty \frac{t^{\gamma-1}}{1+t} dt \, \|x\|$$

$$\lesssim k^{\tau-1} \|x\|_{B_\gamma^\circ} + k\|x\|. \qquad \|x\| \lesssim \|x\|_{B_\gamma}$$

Set k large enough. st. $Ck^{\tau-1} < \frac{1}{2}$.

6°) For $\tau \leq \gamma$. $\|BR_t x\| \lesssim (1+t)^{-1} \|x\|_{B_\gamma^\circ}$

Since $B_\tau^\circ \supseteq B_\gamma^\circ$. by prop.

Note: $\int_0^\infty \frac{t^{\gamma-1}}{1+t} < \infty$. $\int_0^\infty t^{\gamma-1}(t+k)^{\tau-\gamma-1} \sim k^{\gamma-1}$

The proof 1°) — 5°) still holds!