Banach Algebra

(1) Introduction:

Def: i) Algebra is a linear space with multiplication

ii) Banach Algebra is a Banach space equipped with multiplication. St. $\|AB\| \leq \|A\|\|B\|$. for $\forall A, B \in \mathcal{B}$.

Rmk: A Banach algebra $\mathcal{B}$ can be embedded into a Banach algebra $\hat{\mathcal{B}}$ containing an identity: $\mathcal{B} \xhookrightarrow{i} \hat{\mathcal{B}} = \mathcal{B} \times \mathbb{K}$ (suppose $\mathcal{B}$ is L.S. on $\mathbb{K}$.)

$b \longmapsto (b, 0)$

Define: $(a, \alpha)(b, \beta) = (ab + \alpha b + \beta a, \alpha\beta)$

$\|(a, \alpha)\|_{\hat{\mathcal{B}}} = \|a\|_{\mathcal{B}} + |\alpha|$.

$\Rightarrow \hat{\mathcal{B}}$ is Banach algebra with $id = (0, 1)$

e.g. X is Banach. $\mathcal{L}(X)$ is Banach algebra.

prop. If A is a algebra. $a, b \in A$. Satisfies: $ab = ba$. Then: ab is invertible $\Rightarrow$ so a, b are.

Pf: $\exists c$. st. $abc = cab = e$.

$\Rightarrow bca = bcaabc = bac = abc = e$

Rmk: $ab \neq ba$. Then it doesn't hold: e.g.

$A: \ell^2 \to \ell^2$. $A(x_n) = (x_2, x_3 \cdots x_n \cdots)$

$B: \ell^2 \to \ell^2$. $B(x_n) = (0, x_1 \cdots x_n \cdots)$

$BA = id$. But A, B aren't inevitable.

(2) Spectral Theory:

① Spectrum:

Lemma. For $a \in B$. $\lambda \in \sigma(a) \Rightarrow \lambda^n \in \sigma(a^n)$

Pf: $\lambda^n - a^n = (\lambda - a)(a^{n-1} + \cdots + \lambda^{n-1})$

$=: AB = BA$

if $\lambda^n \in \rho(a^n) \Rightarrow \lambda - a$ is inevitable

i.e. $\lambda \in \rho(a)$. Contradiction!

Lemma. For $a \in B$. P is polynomial. Then:

$p(\sigma(a)) \subset \sigma(p(a))$

Pf: Suppose $\lambda \in \sigma(a)$. if $p(\lambda) \in \rho(p(a))$

$\Rightarrow p(a) - p(\lambda) = q(\lambda)(a-\lambda) = (a-\lambda)q(a)$

i.e. Similarly $a-\lambda$ is inevitable.

Next, we consider B is on $\mathbb{C}$:

prop. $p(x) \in \mathbb{C}[x]$. $a \in B$. Then $p(\sigma(a)) = \sigma(p(a))$

Pf: If $\lambda \in \sigma(p(a))$. $p(x) - \lambda = c\prod_i^n (x - z_i)$

1°) $c = 0$. It's trivial

2°) $c \neq 0$. Then at least exist i_0 st.

$z_{i_0} \in \sigma(a)$. $\therefore \lambda = p(z_{i_0}) \in p(\sigma(a))$

Lemma. $\forall a \in B$. $\sigma(a) \subseteq \mathbb{C}$ is cpt.

Pf: 1°) $\sigma(a)$ is bdd:

$\lambda - a = \lambda(1 - \frac{a}{\lambda})$. if $\|\frac{a}{\lambda}\| < 1$.

then: $\lambda - a$ is inevitable by expansion.

2°) $\rho(a) = \mathbb{C} / \sigma(a)$ is open:

For $|z| < \|\lambda - a\|/2$:

$(\lambda + z - a) = (\lambda - a)(1 + z(\lambda - a)^{-1})$

is inevitable.

Rmk: In particularly. $\mathcal{L}^{\times}(E) = \{T \in \mathcal{L}(E) \mid$

T is inevitable$\}$ is open set.

② Holomorphic on B:

Def: i) $f : D(f) \subseteq \mathbb{C} \longrightarrow B^{\mathbb{C}}$ (Banach space)

strongly analytic if $\forall\, x_0 \in D(f)$

$\exists\, B(x_0, r) \subset D(f)$. s.t. $\exists\, (a_n) \subseteq B$.

$f(z) = \sum_0^\infty a_n (z - x_0)^n$. $\forall\, z \in B(x_0, r)$ $\Updownarrow$ equi. actually

ii) $f : D(f) \subseteq \mathbb{C} \longrightarrow B^{\mathbb{C}}$ is weakly analytic

if $\forall\, l^* \in B^*$. $l^* \circ f$ is holomorphic.

Rmk: $\forall\, a \in \mathcal{B}$: $\begin{aligned} \rho(a) &\xrightarrow{\phi} \mathcal{B} \\ z &\longmapsto (z-a)^{-1} \end{aligned}$ is

strongly analytic.

Pf: For $z_0 \in \rho(a)$. $\forall\, z \in B(z_0, r)$

$(z - a)^{-1} = (z_0 - a + w)^{-1}$. $w \in B(0, r)$

$= (1 + w(z_0 - a)^{-1})^{-1} (z_0 - a)^{-1}$

$= \sum [-w(z_0 - a)^{-1}]^n (z_0 - a)^{-1}$

holds if $r < \frac{1}{2} \|z_0 - a\|$.

Lemma. If $|z| > \lim_n \|a^n\|^{\frac{1}{n}}$ (exists by Fekete Thm.)

Then $(z-a)^{-1} = \sum_1^{\infty} z^{-n} a^{n-1}$.

Pf: $\exists n_0$ s.t. $\|a^{n_0}\|^{\frac{1}{n_0}} < |z| \Rightarrow \|\frac{a^{n_0}}{z^{n_0}}\| < 1$

set $b = a/z$. $\therefore \|b^{n_0}\| < 1$.

$(z-a)^{-1} = z^{-1}(1-b)^{-1} = z^{-1}\sum b^i$

$\|(z-a)^{-1}\| \le |z^{-1}| \sum_{r=0}^{n_0-1} \sum_{l}^{\infty} \|b^{n_0}\|^{l} \|b^r\| \le C$.

$\therefore (z-a)^{-1} = \sum z^{-n} a^{n-1}$ is well-def.

Cor. $\sigma(a) \subseteq \{z \in \mathbb{C} \mid |z| \le \lim_n \|a^n\|^{\frac{1}{n}}\}$

Lemma. $a^n = \frac{1}{2\pi i} \oint_{C_a} \zeta^n (\zeta - a)^{-1} d\zeta$. C_a contour a.

Pf: $\frac{1}{2\pi i} \oint_{C_a} \zeta^n (\zeta-a)^{-1} d\zeta$

$= \frac{1}{2\pi i} \oint_{C_a} \zeta^n \sum_1^{\infty} \zeta^{-k} a^{k-1} d\zeta = a^n$

Thm. $\max_{\lambda \in \sigma(a)} |\lambda| = \lim_n \|a^n\|^{\frac{1}{n}}$. for $a \in \mathcal{B}$.

pf: 1') $\sigma(a) \ne \emptyset$. for a is nontrivial.

By contradiction: $\forall l \in \mathcal{B}^*$. $l \circ (z-a)^{-1} \in \mathcal{O}(\mathbb{C})$.

But $f(z) = l \circ (z-a)^{-1}$ is bdd

since for z is large enough: $\exists C > 0$.

$|(z-a)^{-1}| \le \sum \|a\|^{n-1}/z^n \le C$.

$\Rightarrow f(z) \equiv$ const $\therefore (z-a)^{-1} \equiv$ const.

2') prove: $r(a) \le \max_{\lambda \in \sigma(a)} |\lambda|$ (Note: converse holds)

$\forall \varepsilon > 0$. $a^n = \oint_C \frac{z^n (z-a)^{-1}}{2\pi i} dz$. $C = B(a, \widetilde{r(a)}+\varepsilon)$. $\widetilde{r(a)} = \max|\lambda|$.

$\Rightarrow a^n = \frac{1}{2\pi}\int_0^{2\pi} (\widetilde{r(a)}+\varepsilon)^{n+1} e^{i(n+1)\theta} ((\widetilde{r(a)}+\varepsilon)e^{i\theta}-a)\, d\theta$

$\therefore \|a^n\|^{\frac{1}{n}} \le \left(\frac{(\widetilde{r(a)}+\varepsilon)^{n+1}}{2\pi}\right)^{\frac{1}{n}} \left(\int_0^{2\pi} \|(\widetilde{r(a)}+\varepsilon)e^{i\theta}-a\|\right) \le C^{\frac{1}{n}} (\widetilde{r(a)}+\varepsilon)^{\frac{n+1}{n}}$

Let $n \to \infty \Rightarrow r(a) \le \widetilde{r(a)}+\varepsilon$. $\forall \varepsilon > 0$

Rmk: $\|f(a)\| = \sup_{\|l\|=1} |\langle l, \frac{1}{2\pi i}\oint_I f(\zeta)(\zeta-a)^{-1} d\zeta \rangle| = \frac{1}{2\pi}|\oint_I \langle l, (\zeta-a)^{-1}\rangle f(\zeta) d\zeta|$
$\lesssim \oint_I \|\zeta-a\|^{-1}|f(\zeta)||d\zeta|$.

(3) Riesz Calculus:

Next, we consider Banach Algebra B is on $\mathbb{C}$.

Def: $Hol(\sigma(a)) = \{f \mid f$ is holomorphic in a open set U_f, $\sigma(a) \subset U_f\} \subseteq \{f : \mathbb{C} \longrightarrow \mathbb{C}\}$.

Rmk: For $f \in Hol(\sigma(a))$. Def: $f(a) := \frac{1}{2\pi i}\int_I f(\zeta)(\zeta-a)^{-1} d\zeta$.

I is union of Jordan curves contouring $\sigma(a)$ with winding number $= 1$. $I \subset U_f$ for $a \in B$.

Thm. For $a \in B$. $Hol(\sigma(a)) \xrightarrow{R_a} B$. defined by:

$R_a(f) = f(a)$. Then R_a is a homomorphism.

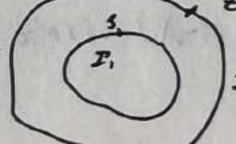
Pf. Check: $R_a(fg) = R_a(f) R_a(g)$. $(fg(\zeta) \overset{\text{def}}{=:} f(\zeta)g(\zeta))$

$$RHS = \left(\frac{1}{2\pi i}\right)^2 \oint_{I_1} \frac{f(\zeta)}{\zeta-a} d\zeta \oint_{I_2} \frac{g(z)}{z-a} dz \quad (\zeta, z \in \mathbb{C})$$

$$= \left(\frac{1}{2\pi i}\right)^2 \oint_{I_1}\oint_{I_2} \frac{f(\zeta)g(z)}{z-\zeta}\left[(\zeta-a)^{-1} - (z-a)^{-1}\right] dz\, d\zeta.$$

$$\overset{(Fubini)}{=} \frac{1}{2\pi i}\oint_{I_1} f(\zeta)g(\zeta)(\zeta-a)^{-1} d\zeta = fg(a)$$

$\overset{sym}{=} gf(a)$.



(Choose contours I_1. I_2. above)

Rmk: i) $Hol(\sigma(a)) \xrightarrow{R_a} B$. $R_a(z) = a$. $R_a(\mathbb{1}) = e$ is homomor.

ii) For $(f_n) \subseteq \theta(U)$. $\sigma(a) \subseteq U$. $f_n \xrightarrow{u.c.c} f$. in U.

$\Rightarrow f_n(a) \rightarrow f(a)$ as $n \rightarrow \infty$.

Characterization of R_a:

For map: $\ell : Hol(\sigma(a)) \rightarrow B$. st.

i) ℓ is algebra Homomorphism. ii) $\ell(\mathbb{1}) = e$. $\ell(z) = a$.

iii) $\forall U \supseteq \sigma(a)$. open. $(f_n) \subseteq \theta(U)$. $f \in \theta(U)$. $f_n \xrightarrow{u.c.c} f$

in $U \Rightarrow \ell(f_n) \rightarrow \ell(f)$. Then: $\ell = R_a$.

Thm If $f \in Hol(\sigma(a))$. $a \in B$. Then: $f(\sigma(a)) = \sigma(f(a))$

Pf: 1') $\forall \lambda \in \sigma(f(a))$. if $g(z) =: f(z) - \lambda \neq 0$. $\forall z \in \sigma(a)$.

Then: $g^{-1}(z) \in Hol(\sigma(a))$.

But $e = R_a(g^{-1}(z) g(z)) = R_a(g^{-1}(z)) R_a(g(z))$

$\Rightarrow e = \frac{1}{f(z) - \lambda}(a)(f(a) - \lambda)$ Contradict!

$\therefore \exists \mu \in \sigma(a)$. st. $\lambda = f(\mu)$

2') $\forall \mu \in f(\sigma(a))$. i.e. $\exists \lambda \in \sigma(a)$. $\mu = f(\lambda)$.

$$h(z) = \frac{f(z) - f(\lambda)}{z - \lambda} \in Hol(\sigma(a))$$

def from expanding by series.

$\Rightarrow f(a) - f(\lambda) = h(a)(a - \lambda)$.

Note $(a - \lambda)$ isn't invertible. $\Rightarrow$ So $f(a) - f(\lambda)$

<u>Rmk</u>: If $f \in Hol(\sigma(a))$, $g \in Hol(\sigma(f(a)))$

Then $g \circ f \in Hol(\sigma(a))$. $h_a(g \circ f)$ is well-def.

(4) Commutative Banach Algebra:

<u>Def</u>: B is commutative if $\forall a, b \in B$

<u>eg</u>. $L^1(X, A, M)$ with multiplication is convolution $*$.

<u>Rmk</u>: It's a ring on $\mathbb{C}$. Ideal in B is defined.

<u>Lemma</u>. Every proper ideal I is contained in some maximal ideal.

<u>Pf</u>: Apply Zorn's Lemma.

<u>Cor</u>. Every element a which isn't inevitable is contained in some maximal ideal

<u>Pf</u>: $\langle a \rangle$ is an ideal.

<u>Thm</u>. Any maximal ideal in B is closed.

<u>Pf</u>: 1°) $\bar{I}$ is an ideal.

for $a \in \bar{I}$, $\exists a_n \in I \to a$.

$\forall b \in B$, $a_n b \in I \to ab$. $\therefore ab \in \bar{I}$

2°) Since $I \subset B/B^{\times}$ ($B^{\times}$ is set of inevitable element, which is open) $\therefore \bar{I} \subset B/B^{\times}$.

$\Rightarrow I \subset \bar{I} \subsetneqq B$ $\therefore I = \bar{I}$.

Thm. Proper ideal I is maximal $\Leftrightarrow$ $B = I \oplus \mathbb{C}e$.

Pf: ($\Leftarrow$) if $I \subsetneq J$ (ideal). Then $\exists\, b \in I^c \cap J$.

$b = a + \lambda e$. $a \in I$. $\lambda \neq 0$. $\therefore \lambda e \in J$.

i.e. $J = B$.

($\Rightarrow$) 1°) B/I is Banach Algebra.

check: $[a][b] = [ab]$ is well-def

$\|[ab]\| \leq \|[a]\| \, \|[b]\|$

where $\|[x]\| = d(x, I)$

2°) B/I is a field.

If $\exists\, [a]$ isn't invertible. $a \neq 0$.

Then $\exists$ maximal ideal $J \subsetneq B/I$. St.

$[a] \subset J$. ($I + aB/I \subset J$)

But $I + aB$ is a ideal contain I.

$\therefore I + aB = B$. $\Rightarrow B/I \subset J$. contradict!

3°) $B/I \simeq \mathbb{C}$.

$\forall\, [a] \neq [0]$. $\sigma([a]) \neq \emptyset$. $\exists \lambda \in \sigma([a])$

$[a] - \lambda[e] = [a - \lambda e] = [0]$.

$\therefore \exists\, b \in I$. St. $a = \lambda e + b$. $\forall a \in B$.

① **Def:** LF m is multiplicative function on B if $m \neq 0$.

$\forall\, a, b \in B$. $m(ab) = m(a)\, m(b)$. Denote the set of such functions by $M(B)$

Thm. $M(B) \xrightarrow{\phi} \{\text{maximal ideal in } B\}$. ϕ is bijection.

$m \longmapsto \ker m$

Pf: 1°) For $m \in M(B)$. check $\ker(m)$ is max ideal.

$\forall a \in \ker m$. $\forall b \in B$, $m(ab) = m(a)m(b) = 0$

$\therefore ab \in \ker m$. $\ker m$ is an ideal.

Note that $B/\ker m \cong \mathbb{C}$. ($m$ is LF).

$\therefore \ker m \oplus \mathbb{C}e = B$. $\ker m$ is maximal.

2°) For I is maximal ideal. $I \oplus \mathbb{C}e = B$.

Set $m(e) = 1$. $m(a) = 0$. $\forall a \in I$.

check $m \in M(B)$.

Rmk: $\forall m \in M(B)$. m is BLF and $\|m\| \le 1$. (No need unit)

Pf: By contradiction: $\exists a \in B$. $\|a\| < 1$. $|m(a)| = 1$.

$\Rightarrow 1 = |m^n(a)| = |m(a^n)|$. $\|a^n\| \to 0$. $\exists (a_k) \subset (a^n)$. $\begin{pmatrix} B \cong N_m \\ \oplus \mathbb{C}\tilde{e} \end{pmatrix}$ $m(\tilde{e}) = 1$

$m(a_k) \to \lambda$ ($|\lambda| = 1$). $a_k = b_k + m(a_k)\tilde{e}$. $b_k \to -\lambda\tilde{e} \in N_m$ closed $\therefore \lambda = 0$.

Note that if e exists in B. then $\|m\| = 1$. $\forall m \in M(B)$.

Since $|m(e)| = |m(e)|^2$. But $m \ne 0$. So: $|m(e)| = 1$.

Thm. $\forall a \in B$. $\lambda \in \sigma(a) \Leftrightarrow \exists m \in M(B)$. st. $m(a) = \lambda$.

Pf: $\lambda \in \sigma(a) \Leftrightarrow a - \lambda e$ isn't inevitable.

$\Leftrightarrow \exists I$ is maximal ideal. i.e. kernel of m

for some $m \in M(B)$. $\Leftrightarrow m(a - \lambda e) = 0$

i.e. $m(a) = \lambda$.

Rmk: $\sigma(a) = \{m(a) \mid m \in M(B)\}$. Characterization

② Joint Spectrum:

Def: $(a_1 \cdots a_n) \in B^n$. $(\lambda_1 \cdots \lambda_n) \in \mathbb{C}^n$ belongs to $\ell(a_1 \cdots a_n)$ if $\exists (b_1 \cdots b_n) \in B^n$. s.t. $\sum_{k=1}^{n} b_k (a_k - \lambda_k e) = e$.

Denote: $\sigma(a_1 \cdots a_n) = \mathbb{C}^n / \ell(a_1 \cdots a_n)$

prop. $\sigma(a_1 \cdots a_n) = \{ (m(a_1) \cdots m(a_n)) \mid m \in \mathcal{M}(B) \}$.

Pf: $\vec{\lambda} \in \sigma(a_1 \cdots a_n) \iff \{ \sum b_k (\lambda_k - a_k) \mid \vec{b} \in B^n \} \subset B / B^\times$

Note that $\sum (\lambda_k - a_k) B$ is an ideal.

So it's contained in some maximal ideal.

$\iff \exists m \in \mathcal{M}(B)$. st. $\sum (\lambda_k - a_k) B \subset \ker m$

i.e. $m(\sum (\lambda_k - a_k) b_k) = 0$

$\iff \sum (\lambda_k - m(a_k)) m(b_k) = 0$. $\forall \vec{b} \in B$.

which implies: $m(a_k) = \lambda_k$. $\forall 1 \le k \le n$.

Rmk: It's generalization of Thm. above in $n=1$. case.

prop. For $P(z_1 \cdots z_n)$ is polynomial. $(\lambda_1 \cdots \lambda_n) \in \sigma(a_1 \cdots a_n)$

Then. $|P(\lambda_1 \cdots \lambda_n)| \le \| P(a_1 \cdots a_n) \|$

Pf: $\exists m \in \mathcal{M}(B)$. $(\lambda_1 \cdots \lambda_n) = m(a_1 \cdots a_n)$

$p(\lambda_1 \cdots \lambda_n) = p \circ m(a_1 \cdots a_n) = m \circ p(a_1 \cdots a_n)$

$\Rightarrow |p(\lambda_1 \cdots \lambda_n)| \le \|m\| \, \| p(a_1 \cdots a_n) \|$