Compact Operators

(1) Properties:

Def: Linear operator $T : E \to F$. n.v.s. is cpt.

if $T(B_E)$ is relatively cpt in $(F, \|\cdot\|_F)$.

Rmk: $T \in \mathcal{L}(E,F)$. Otherwise $\exists (x_n)$. $\|x_n\| \le C$.

but $T(x_n) \to \infty$. Contradict!

e.g. Finite-rank operator is cpt.

Denote: $K(E,F)$ is set of all cpt operators.

Prop. X, Y. n.v.s. Then $T \in K(X,Y) \Leftrightarrow \bar{T} \in K(\bar{X}, \bar{Y})$.

where $\bar{T}, \bar{X}, \bar{Y}$ are all completions of T, X, Y.

Pf: $(\Leftarrow)$ $\overline{T(B_X)} \subseteq \overline{\bar{T}(B_{\bar{X}})}$. cpt

$(\Rightarrow)$ $\bar{T}(B_{\bar{X}}) \subseteq \bar{T}(\bar{B}_{\bar{X}}) = \bar{T}(\bar{B}_X)$

$\subseteq \overline{\bar{T}(B_X)} = \overline{T(B_X)}$ cpt.

Thm. $K(E,F)$ is CLS of $\mathcal{L}(E,F)$. w.r.t $\|\cdot\|_{\mathcal{L}(E,F)}$

Pf: 1°) Linearity:

$$\begin{matrix} F \times F \longrightarrow F \\ (x, y) \longmapsto x+y \end{matrix} \quad \text{conti} \Rightarrow \quad \begin{matrix} \alpha \overline{T_1(B_E)} \times \overline{T_2(B_E)} \longmapsto \alpha \overline{T_1(B_E)} \\ + \overline{T_2(B_E)} \text{ is cpt} \end{matrix}$$

$\Rightarrow \alpha T_1 + T_2 \in K(E,F)$.

2°) By totally bounded:

If $(T_n) \subseteq K(E,F) \to T$. $\exists N$. $\|T_N - T\| \le \frac{\varepsilon}{2}$.

Since $\overline{T_N(B_E)} \subset \bigcup_i^n B(q_i, \frac{\varepsilon}{2}) \Rightarrow \overline{T(B_E)} \subseteq \bigcup_i^n B(q_i, \varepsilon)$

Cor. $(T_n) \subseteq \mathcal{L}(E,F)$. $\lim \mathrm{Rk}(T_n) < \infty$. If $\|T_n - T\|_{\mathcal{L}(E,F)} \to 0$

Then $T \in K(E,F)$.

Rmk: i) Not every $T \in K(E,F)$ can be approxi. by some seq of finite rank operators.

If F is Hilbert space, it will hold:

$\overline{T(B_E)} \subseteq \bigcup_i^n B(q_k, \varepsilon)$. Let $G = \mathrm{span}\{q_k\}_1^n$.

$T_\varepsilon = P_G \circ T$. $\Rightarrow \|T_\varepsilon - T\| \le 2\varepsilon$

Since. $\exists k$. $Tx \sim q_k = P_G q_k \sim T_\varepsilon x$.

ii) More general, it holds when F has Schauder basis. E is Banach space. e.g.

Lemma (Characterization of cpt operator in ℓ_p)

$Tx = (\lambda_1 x_1 \cdots \lambda_n x_n \cdots)$ is cpt operator of ℓ_p $(1 \le p < \infty)$ $\Leftrightarrow$ $\lambda_n \to 0$.

$\Rightarrow$ For $E = F = \ell_p$. $T_n x = (\lambda_1 x_1 \cdots \lambda_n x_n, 0, 0 \cdots 0 \cdots) \to Tx$

iii) Approxi. problem:

$T : X \xrightarrow{\text{conti}} F$. X is topo space. F is Banach.

If $T(X)$ is relatively cpt. Then T can be approxi. by nonlinear conti maps of finite rank.

Pf: $\overline{T(X)} \subset \bigcup_i^n B(f_i, \varepsilon)$. Let $T_\varepsilon = \dfrac{\sum_i^n q_i(x) f_i}{\sum_i^n q_i(x)}$

where $q_i(x) = \max\{\varepsilon - \|Tx - f_i\|, 0\}$

T_ε conti. $\|Tx - T_\varepsilon x\| = \left\| \dfrac{\sum q_i(x)(q_i - Tx)}{\sum q_i(x)} \right\| \le \varepsilon$

$\because q_i(x) \ne 0 \Leftrightarrow \|Tx - q_i\| \le \varepsilon$.

Prop. E, F, G are Banach spaces. $T_1 \in \mathcal{L}(E,F)$. $T_2 \in K(E,F)$

$S_1 \in K(F,G)$. $S_2 \in \mathcal{L}(F,G)$ Then $S_1 \circ T_1$, $S_2 \circ T_2 \in K(E,G)$

Pf: 1) $|T_1(B_E)| \le \|T_1\|$. $\overline{S_1 \circ T_1(B_E)}$ closed in cpt set

2) $\overline{T_2(B_E)} \subseteq \bigcup_i^N B(g_i, \varepsilon)$ $\therefore$ $S_2 \circ T_2(B_E) \subseteq \bigcup_i^N B(S_2(g_i), \varepsilon\|S_2\|)$

$\therefore$ $\overline{S_2 \circ T_2(B_E)} \subseteq \bigcup_i^N B(S_2(g_i), 2\varepsilon\|S_2\|)$.

Thm. $T \in K(E,F) \iff T^* \in K(F^*, E^*)$.

Pf: $(\Rightarrow)$. Test with $(v_n) \subseteq B_{F^*}$. let $K = \overline{T(B_E)}$. cpt.

Denote $\mathcal{H} = \{ \varphi_n : x \in K \longrightarrow \langle v_n, x \rangle \}$.

$\mathcal{H}$ satisfies Ascoli Thm. $\therefore \overline{\mathcal{H}}$ is cpt.

$\exists\ \varphi_{n_k} \longrightarrow \varphi$ on K. $\iff (T^* v_{n_k})$ is Cauchy.

$(\Leftarrow)$ Then $T^{**} \in K(E^{**}, F^{**})$

$T^{**}(B_E) = J_F \circ T \circ J_E^{-1}(B_E) = T(B_E)$

has cpt closure in F^{**}. $(B_E \subseteq B_{E^{**}})$

$\therefore F \underset{\text{closed}}{\subseteq} F^{**}$. $\therefore$ $\overline{T(B_E)}$ cpt in F.

Criteria:

$T \in \mathcal{L}(E,F)$. If $T \in K(E,F)$. Then, $u_n \rightharpoonup u$. in $\sigma(E, E^*)$.

$\Longrightarrow$ we have $Tu_n \longrightarrow Tu$ in F.

Pf: $(\Rightarrow)$. $|\langle T^* v, u_n - u \rangle| \to 0$. $\forall v \in F^*$.

$\therefore$ $|\langle v, Tu_n - Tu \rangle| \to 0$. $Tu_n \rightharpoonup Tu$ in $\sigma(F, F^*)$

$\|Tu_n\| \le C \Rightarrow$ Since $T \in K(E,F)$. $\exists (Tu_{n_k})$

$Tu_{n_k} \longrightarrow a$. in F. $\therefore a = Tu$.

If $Tu_n \not\to Tu$. Then $\exists (u_{n_k})$. $\|Tu_{n_k} - Tu\| \geq \varepsilon_0 > 0$

But $\exists (Tu_{n_k'}) \subseteq (Tu_{n_k})$. $Tu_{n_k'} \to Tu$. Contradicts!

($\Leftarrow$) Converse holds when E is reflexive.

since for $(u_n) \subseteq B_E$. $\exists (u_{n_k})$ weakly converges.

<u>Prop.</u> $T \in K(E,F)$. $R(T)$ is closed. Then T is finite rank.

<u>Pf.</u> $T: E \to R(T)$. (Banach) surjective BLO

$\therefore T(B_E(0,1)) \supseteq B_{R(T)}(0,c)$. $\exists c > 0$.

Then: $\overline{B_{R(T)}(0,c)}$ is cpt. $\dim(R(T)) < \infty$.

(2) <u>Fredholm Alternative</u>:

For $T \in K(E)$. $\lambda \neq 0$.

① $N(\lambda I - T)$ is finite dimensional

② $R(\lambda I - T)$ is closed. $R(\lambda I - T) = N(\lambda I - T^*)^\perp$

③ $N(\lambda I - T) = 0 \iff R(\lambda I - T) = E$.

④ $\dim N(\lambda I - T) = \dim N(\lambda I - T^*)$

<u>Remark</u>: Consider the equation: $\lambda u - Tu = f$.

① is saying: The eigenvalue λ of T is finitely multiple. i.e. $\dim\{u \mid Tu = \lambda u\} < \infty$.

② is saying the equation $\lambda u - Tu = f$ is solvable. iff $f \in N(\lambda I - T^*)^\perp$. For E is Hilbert space. $\Leftrightarrow$ $f \in N_\lambda^\perp = \{u \mid Tu = \lambda u\}^\perp$.

③ is saying if for $\forall f \in E$. The equation has solution. Then the solution is unique. We can denote it by $u = (\lambda I - T)^{-1} f$. formly. In the case $(\lambda I - T)^{-1}$ is BLO. By the openning mapping Thm. $\lambda I - T$ bijective. conti.

Pf: WLOG. Let $\lambda = 1$. Since $\frac{T}{\lambda} \in K(E)$.

① Denote $E_1 = N(I-T)$. $\therefore T(B_{E_1}) = B_{E_1}$

$\because B_{E_1} \subseteq B_E$ $\therefore$ $B_{E_1} = T(B_{E_1}) \subseteq T(B_E)$.

$\overline{B_{E_1}}$ cpt in $N(I-T)$ $\therefore$ $\dim(N(I-T)) < \infty$

② Consider $f_n = u_n - Tu_n \to f$.

Prove (u_n) won't be so far from $N(I-T)$.

Since $N(I-T)$ is reflexive $\therefore \exists v_n$. for $\forall u_n$. st.

$d(u_n, N(I-T)) = \|u_n - v_n\|$. (convex)

prove: $(u_n - v_n)$ is bounded.

Then apply $T \in K(E)$. Find $y \in E$. $y - Ty = f$.

③ ($\Rightarrow$) By contradiction: $E_1 = (I-T)(E) \subsetneq E$.

$\therefore E_2 = (I-T)(E_1) \subsetneq E_1$. Denote $E_n = (I-T)^n E$.

We have: $E_n \subsetneq E_{n-1}$. $\{E_n\}$ closed. LS.

By Riesz. Lemma $\exists (u_n) \subseteq E$. $u_n \in E_n$. st.

$\|u_n\| = 1$. $\forall n$. $d(u_n, E_{n-1}) \geq \frac{1}{2}$. $\forall n$.

prove: (Tu_n) won't converge in E.

($\Leftarrow$) Since $R(I-T)$ is closed. $\therefore N(I-T^*) = \{0\}$.

$T^* \in K(E^*)$ $\therefore R(I-T^*) = E^*$. $\therefore N(I-T) = \{0\}$.

④ First prove: $d^* \leq d$. $d^* = \dim N(I-T^*)$. $d = \dim N(I-T)$

By contradiction: $d^* > d$.

Since $R(I-T) = N(I-T^*)^\perp$. $\therefore$ it has codim $= d^*$.

$\therefore \exists$ complement F, G. st. $N(I-T) \oplus F = R(I-T) \oplus G = E$.

$\because \dim G = d^* > d$. $\therefore \exists \Lambda : N(I-T) \longrightarrow G$. injection.

Set $P: E \xrightarrow[\text{proj}]{} N(I-T)$. $S = T + \Lambda \circ P \in K(E)$. Since $\dim(R(\Lambda \circ P)) < \infty$.

Claim: $N(I-S) = \{0\}$. (check) $\Rightarrow$ $R(I-S) = E$.

Check if $f \in G$. $f \notin R(\Lambda)$. then $f \notin R(I-S)$. contradict!

$\therefore d^* \leq d$. From $N(I-T)^{\perp\perp} = N(I-T^{**}) \supseteq N(I-T)$. $\therefore d^{**} \geq d$.

Then $d = d^*$. We're done.

<u>Remark</u>: If E is separable Hilbert space. for: solve

$\lambda u - Tu = f$. we can apply Spectral Decomposition on T. $u = \Sigma \langle u, e_i \rangle e_i$. $f = \Sigma \langle f, e_i \rangle e_i$

solve $\langle u, e_i \rangle$. Check it's convergent (by Bessel)

(3) Spectrum of Cpt Operators:

<u>Def</u>: For $T \in \mathcal{L}(E)$. $E = E^{\mathbb{R}}$.

i) Resolvent set: $\rho(T) = \{\lambda \in \mathbb{R} \mid (T - \lambda I) \text{ is bijection}\}$.

ii) Spectrum $\sigma(T) = \mathbb{R} / \rho(T)$.

iii) $EV(T) = \{\lambda \in \mathbb{R} \mid N(\lambda I - T) \neq \{0\}\}$.

<u>Remark</u>: $EV(T) \subseteq \sigma(T)$. If T is cpt operator

Then $EV(T) = \sigma(T)$. But it will also

happen $EV(T) \subsetneqq \sigma(T)$. e.g. $l^2 \xrightarrow{T} l^2$. st.

$T(u_1, u_2 \cdots u_n, \cdots) = (0, u_1, u_2 \cdots u_n, \cdots)$. $u \in l^2$

Then $EV(T) = \emptyset$. T isn't bijection. $0 \in \sigma(T)$.

① <u>Spectral Radius</u>:

· <u>Def</u>: $r(T) = \lim\limits_{n\to\infty} \|T^n\|^{\frac{1}{n}}$ is spectral radius of T.

For E is Banach Space. $T \in \mathcal{L}(E)$. then

we have: $r(T)$ exists. and $r(T) \le \|T\|$

<u>Pf</u>: For $a_n = \log \|T^n\|$. we have $a_i + a_j \ge a_{i+j}$

$\forall m, n.\ (n \ge m)$. $\frac{a_n}{n} = \frac{a_{mq+r}}{mq+r} \le \frac{q a_m + a_r}{mq+r}$

$\therefore \frac{a_n}{n} \le \frac{a_m}{m} + \frac{a_r}{n}$. Take $\sup\limits_{n \ge m}$: (fix m)

$\therefore \sup\limits_{n \ge m} \frac{a_n}{n} \le \frac{a_m}{m} + \sup\limits_{n \ge m} \frac{a_r}{n}$. Take $\inf\limits_{m \ge 1}$:

$\therefore \overline{\lim} \frac{a_n}{n} \le \inf \frac{a_m}{m} \le \sup \inf \frac{a_k}{k} = \underline{\lim} \frac{a_n}{n}$

$\therefore \lim \frac{a_n}{n} = \inf \frac{a_m}{m}$. $\therefore \lim \|T^n\|^{\frac{1}{n}} \le \|T\|$.

① Polynomial on Operators:

For $T \in \mathcal{L}(E)$. $a(t) = \sum_{0}^{p} a_k t^k$.

Then i) $a(Ev(T)) \subseteq Ev(a(T))$

ii) $a(\sigma(T)) \subseteq \sigma(a(T))$

If E is Hilbert space. (rather than Banach)

Then iii) $a(Ev(T)) = Ev(a(T))$

iv) $a(\sigma(T)) = \sigma(a(T))$

Pf: i) For $\forall \lambda \in Ev(T)$. $a(\lambda)I - a(T) = \tilde{a}\cdot(\lambda I - T)$.

$N(\lambda I - T) \neq 0 \Rightarrow N(\tilde{a}(\lambda I - T)) \neq 0$.

ii) Similarly. $a(\lambda)I - a(T) = (\lambda I - T)\cdot\tilde{a} = \tilde{a}(\lambda I - T)$

$(\lambda I - T)$ isn't bijection. So $a(\lambda)I - a(T)$.

iii) Lemma. $\tilde{a}(t)$ has no real root. Then $\tilde{a}(T)$ is bijection.

Pf: $\tilde{a}(t) = \prod_{1}^{k}(a_i t^2 + b_i)$. $b_i \neq 0$. $\forall 1 \le i \le k$.

For $a_i T^2 + b_i = \varphi_i$. By Lax-Milgram:

$\langle \varphi_i(u), v \rangle \le (a_i \|T\|^2 + b_i)\|u\|\|v\|$.

$\langle \varphi_i(v), v \rangle \geq |b_i|\|v\|^2$ $\therefore \varphi_i$ is bijection.

For $\lambda \in Ev(a(T))$. $a(t) - \lambda$ will have real root.

Otherwise $a(T) - \lambda I$ is bijection $\therefore \lambda \notin Ev(a(T))$

$\therefore a(T) - \lambda I = \tilde{a}(T)\prod_{1}^{q}(T - t_i I)$. $\exists i$. $N(T - t_i I) \neq \{0\}$.

$\therefore t_i \in Ev(T)$. $\therefore \lambda = a(t_i) \in a(Ev(T))$

iv) Similarly argument:

$$\alpha(T)-\lambda I = \prod_1^2 (T-t_i I)\,\tilde{a} = \tilde{a}\prod_1^2 (T-t_i I)$$

③ Spectrum $\sigma(T)$:

prop. $T\in\mathcal{L}(E)$. Then $\sigma(T)$ is cpt. Besides.

$\sigma(T)\subseteq[-\|T\|,\|T\|]$. More precisely, we have:

$\sigma(T)\subseteq[-r(T),r(T)]$. (if $\sigma(T)\neq\emptyset$)

Pf: 1°) For $|\lambda|>\|T\|$. $\lambda u - Tu = f$ has solution:

$$\Leftrightarrow u = \frac{T}{\lambda}u + \frac{1}{\lambda}f \ . \ \forall f\in E.$$

Denote $Sv = \dfrac{T}{\lambda}v + \dfrac{f}{\lambda}$. is a contraction.

2°) $\ell(T)$ is open:

$\forall \lambda_0\in\ell(T)$. For $\lambda\in\mathbb{R}$. $f\in E$.

$Tu-\lambda u = f$ has solution $\Leftrightarrow$ $Tu-\lambda_0 u = f+(\lambda-\lambda_0)u$.

i.e. $u = (T-\lambda_0)^{-1}(f+(\lambda-\lambda_0)u)$

$\therefore$ If $\|(T-\lambda_0 I)^{-1}\|\,|\lambda-\lambda_0|<1$. $(*)$ (Neigbour of λ_0)

By contraction. then it has solution. $\lambda\in\ell(T)$.

3°) For $\alpha(T)=T^n$. Since $\sigma(T)^n\subseteq\sigma(T^n)$.

$\therefore$ $\sigma(T)^n\subseteq[-\|T^n\|,\|T^n\|]$.

i.e. $\sigma(T)\subseteq[-\|T^n\|^{\frac{1}{n}},\|T^n\|^{\frac{1}{n}}]$. let $n\to\infty$.

Cor. $\operatorname{dist}(\lambda_0,\sigma(T))\geq\dfrac{1}{\|(T-\lambda_0 I)^{-1}\|}>0$. $\forall\lambda_0\in\ell(T)$.

From $(*)$.

Thm. $T \in K(E)$. $\dim E = \infty$. Then

i) $0 \in \sigma(T)$

ii) $\sigma(T)/\{0\} = EV(T)/\{0\}$.

iii) One of following cases will happen:

(a) $\sigma(T) = \{0\}$. (b) $\sigma(T)/\{0\}$ is finite.

(c) $\sigma(T)/\{0\}$ is countable, tend to 0. (All can conclude: $\lambda_n \to 0$)

Pf: i) If $0 \in \rho(T)$. Then T is bijection.

$\therefore I = T \circ T^{-1} \in K(E)$. B_E is cpt. $\Rightarrow \dim E < \infty$.

ii) For $\lambda \neq 0$. Apply Fredholm Alternative.

iii) Lemma. $T \in K(E)$. $(\lambda_n) \subseteq \sigma(T)/\{0\}$. distinct.

$\lambda_n \to \lambda$. Then $\lambda = 0$. (i.e. $\forall \lambda \in \sigma(T)/\{0\}$. λ is isolated point)

Pf: Find (e_n). st. $Te_n = \lambda_n e_n$. for each λ_n.

Denote $E_n = \text{span}\{e_k\}_1^n$. closed. $E_n \subsetneq E_{n+1}$.

Apply Riesz Lemma. $\exists (u_n)$. $\|u_n\| = 1$. $d(u_n, E_{n-1}) \geq \frac{1}{2}$

$\because (T - \lambda_n I) E_n \subseteq E_{n-1}$. Check: $\left\| \frac{Tu_n}{\lambda_n} - \frac{Tu_m}{\lambda_m} \right\| \geq \frac{1}{2}$

Since $\left(\frac{u_n}{\lambda_n}\right)$ will be bounded. if $\lambda \neq 0$.

then $\left(\frac{Tu_n}{\lambda_n}\right)$ admits a convergent subseq.

$\Rightarrow \sigma(T) \cap \{|\lambda| \geq \frac{1}{n}\}$ is finite or empty.

Otherwise. apply $\sigma(T)$ is cpt. $\exists \lambda_n \to \lambda \neq 0$.

Remark: $\forall (\alpha_n) \to 0$. We can construct $T \in K(E)$.

st. $\sigma(T) = (\alpha_n) \cup \{0\}$. e.g. Let $E = \ell^2$.

$T(u_1 \cdots u_n \cdots) = (\sigma_1 u_1, \sigma_2 u_2 \cdots \sigma_n u_n, \cdots) \in K(l^2)$

$T_n = (\sigma_1 u_1 \cdots \sigma_n u_n, 0, 0 \cdots)$, $\|T_n - T\| \to 0$ $\therefore T$ is cpt.

(4) Spectral Decomposition of Self-Adjoint Cpt Operators:

· Suppose $E = H^*$ Hilbert space on real.

Def: $T \in \mathcal{L}(H)$ is said to be self-adjoint if $T^* = T$. i.e. $(Tu, v) = (u, Tv)$. $\forall u, v \in H$.

① prop. $T \in \mathcal{L}(H)$ is self-adjoint. Let:

$m = \inf\limits_{\substack{u \in H \\ |u|=1}} (Tu, u)$. $M = \sup\limits_{\substack{u \in H \\ |u|=1}} (Tu, u)$. Then,

$\sigma(T) \subseteq [m, M]$. $m, M \in \sigma(T)$. $\|T\| = \max\{|m|, |M|\}$

Pf: 1) For $\lambda \in [m, M]^c$. Apply Lax-Milgram.

Check $\lambda I - T / T - \lambda I$. is bijection.

2) Consider $a(u, v) = (Mu - Tu, v)$.

Check it's linear. sym. $a(u,u) \geq 0$.

$|a(u,v)| \leq a^{\frac{1}{2}}(u,u)\, a^{\frac{1}{2}}(v,v)$. by Cauchy

since $a(\cdot,\cdot)$ is scalar product.

$\therefore |Tu - Mu| \leq C\,(Mu - Tu, u)^{\frac{1}{2}}$

If $M \in \rho(T)$. Consider (u_n). $(Tu_n, u_n) \to M$.

Then $u_n \to 0$. contradict. (m is analogous.)

3°) Denote $M = \max\{|M|, |m|\}$. $\|T\| \geq M$ is obvious.

Conversely. Consider paralogram law:

$$4\langle Tu, v\rangle = \langle T(u+v), u+v\rangle - \langle T(u-v), u-v\rangle$$

$$\therefore 4|\langle Tu,v\rangle| \leq M|u+v|^2 - m|u-v|^2 \leq 2M(|u+v|^2+|u-v|^2)$$

i.e. $|\langle Tu,v\rangle| \leq M(|u|^2+|v|^2)$. Let $u = \tau u$. $\tau = \frac{|v|}{|u|}$. (Optimal)

$\therefore |\langle Tu,v\rangle| \leq |u||v|M$ $\therefore M \geq \|T\|$. $M = \|T\|$.

Cor. $T \in \mathcal{L}(H)$. self-adjoint. st. $\sigma(T) = \{0\}$.

Then $T \equiv 0$

② Thm. Suppose H is separable Hilbert space. If T is self-adjoint cpt operator. Then there exists a Hilbert basis composed of eigenvectors of T.

Pf. 1°) $EV(T)$ is countable.

Otherwise, there exists $\{u_i\}_{i\in I}$. uncountable.

$Tu_i = \lambda_i u_i$. $\therefore \{u_i\}_{i\in I}$ is l.i.

2°) Suppose $EV(T) = (\lambda_n)_{n\in\mathbb{Z}^+}$. (distinct)

Denote $E_0 = N(T)$. $E_n = N(\lambda_n - T)$. $\dim(E_n) < \infty$. $n \neq 0$.

Check $(E_n)_{n\geq 0}$ are mutually orthogonal.

3°) $F = \mathrm{span}(E_n)_{n\geq 0}$ Check F is dense.

$H = \bar{F} \oplus F^\perp$. Check $T(F^\perp) \subseteq F^\perp$. Since $T(F) \subseteq F$.

Claim $\sigma(T|_{F^\perp}) = \{0\}$. So $T=0$ on $F^\perp$.

$\therefore F^\perp \subset N(T) \subset F$. i.e. $F^\perp = \{0\}$.

4°) Construct orthonormal Basis for each E_n. $n\geq 0$.

(5) Application in integrable operators:

· Suppose $X = L^2[a,b]$. $k: [a,b]\times[a,b] \to \mathbb{R}$.

$\langle f, g\rangle = \int_a^b f(x) g(x)\,dx$, $\forall f, g \in L^2[a,b]$.

Consider $A: X \to X$. $A(f)(t) = \int_a^b k(s,t) f(s)\,ds$

(1) Properties:

① If k is conti. Then $A \in K(X)$.

Besides. $A(f) \in C[a,b]$. ($k \in L^2([a,b]^2)$. still hold)

② Under the condition above, if $k(s,t) = k(t,s)$. then A is self-adjoint.

Pf: 1°) $A(f) \in L^2[a,b]$ (A is well-def)

$|A(f)| \le \|k\|_{L^\infty} \int_a^b |f(t)|\,dt$

$\le \|k\|_\infty \|f\|_2 \sqrt{b-a} < \infty$.

2°) A is BLO:

$\|Af\|_2^2 = \int_a^b [\int_a^b k(s,t) f(s)\,ds]^2 dt$

$\le \int_a^b \int_a^b \|f\|_2^2 k^2(s,t)\,ds\,dt$

$\le C \|f\|_2^2$. Check A is linear.

3°) $A(f)$ is conti:

$|A(f)(t_1) - A(f)(t_2)| \le \int_a^b |k(s,t_1) - k(s,t_2)||f(s)|ds$

$\le \varepsilon \|f\|_2 \sqrt{b-a}$. since $[a,b]$ is cpt.

4°) $A \in K(X)$:

For $(\eta_n)_{n\in\mathbb{Z}^+}$. $\|\eta_n\|_2 \le C$.

Then $|A(\eta_n)| \le \|k\|_\infty \|\eta_n\|_2 \sqrt{b-a} < M < \infty$. $\forall n$.

$$|A(\eta_n)(t_1) - A(\eta_n)(t_2)| \le \sqrt{b-a}\,\|f\|_2 \sup_s |k(s,t_1) - k(s,t_2)|$$

$\therefore$ By Ascoli Thm. $(A\eta_n)$ admits a convergent subseq.

5°) If $k(s,t) = k(t,s)$:

$$\langle Ax, \eta\rangle = \int A(x)(t)\eta(t)\,dt = \int_a^b \int_a^b k(s,t)\, x(s)\, \eta(t)\, ds\, dt$$

$$= \int_a^b x(s) \int_a^b k(t,s)\,\eta(t)\,dt\,ds = \langle x, A\eta\rangle.$$ By Fubini Thm.

(2) Cor.

Under the assumption above (①. ②)

For a $x \in X$, $\exists z \in X$, st. $x = Az$. suppose $(\eta_n)_{n\in\mathbb{Z}^+} =$ $(x_k)_{k\in\mathbb{Z}^+} \cup (z_k)_{k\in\mathbb{Z}^+}$. st. $Ax_k = \lambda_k x_k$. $Az_k = 0$. Then.

$\sum_{i\ge 1} |\langle x, \eta_i\rangle \eta_i(t)|$ converges.

Pf: $\sum_n^m |\langle x,\eta_i\rangle \eta_i| = \sum_n^m |\langle Az, \eta_i\rangle \eta_i| = \sum_n^m |\langle z, A\eta_i\rangle \eta_i|$

$= \sum_n^m |\langle z,\eta_i\rangle \lambda_i \eta_i| = \sum_n^m |\langle z,\eta_i\rangle A\eta_i| \le (\sum_n^m |\langle z,\eta_i\rangle|^2)^{\frac{1}{2}} (\sum_n^m |A\eta_i|^2)^{\frac{1}{2}}$

$\le (\sum_n^m |\langle z,\eta_i\rangle|^2)^{\frac{1}{2}} \|k(t,s)\|_2 \le (\sum_n^m |\langle z,\eta_i\rangle|^2)^{\frac{1}{2}} \|k\|_\infty \sqrt{b-a}$. $\le C\varepsilon$.

By Bessel. $\sum_i^\infty |\langle z,\eta_i\rangle|^2$ converges.

Remark: $R(A)$ is set of conti Func. Its decomposition can be l^1 converges.