Hilbert Space

(1) Preliminary:

For X is vector space on $\mathbb{K}$. $\langle\cdot,\cdot\rangle : X \times X \longrightarrow \mathbb{K}$. satisfy:

i) $\mathbb{K} = \mathbb{C}$. $\langle \alpha x + y, z\rangle = \alpha\langle x, z\rangle + \langle y, z\rangle$. $\langle x, y\rangle = \overline{\langle y, x\rangle}$.
for $\forall\, x. y. z \in X^{\mathbb{C}}$. $\forall \alpha \in \mathbb{C}$.

ii) $\mathbb{K} = \mathbb{R}'$. $\langle \alpha x + y. z\rangle = \alpha\langle x. z\rangle + \langle y, z\rangle$. $\langle x. y\rangle = \langle y. x\rangle$.
for $\forall\, x. y \in X^{\mathbb{R}}$. $\forall \alpha \in \mathbb{R}'$.

Def: $\|x\| = \sqrt{(x, x)}$. for $x \in X$. "norm" of X.

Prop. $\|\cdot\|$ is a norm on X. $\mathbb{K} = \mathbb{C}$ ($\mathbb{K} = \mathbb{R}'$ similar)

Pf: $\|x\| \geq 0$. $\|\alpha x\| = |\alpha|\,\|x\|$. are trivial.

Next, prove: $\|x+y\| \leq \|x\| + \|y\|$

Lemma. (Schwarz Inequility)

$$|(x, y)| \leq \sqrt{(x, x)(y, y)} \quad . \forall x. y \in X.$$

Pf: For $t \in \mathbb{R}$. $(x+ty. x+ty)$

$$= (x. x) + 2t\,Re(x, y) + t^2 (y. y) \geq 0.$$

Suppose $|(x, y)|\, e^{i\theta} = (x. y)$

Let $x = e^{-i\theta}\tilde{x}$. $\therefore$ From $\Delta \leq 0$.

$|(\tilde{x}, y)| \leq \sqrt{(\tilde{x}. \tilde{x})(y. y)}$. i.e.

$|(x, y)| \leq \sqrt{(x, x)(y. y)}$.

"=" holds when $x = \frac{\|x\|}{\|y\|} e^{i\theta} y$.

$\Rightarrow \|x+\eta\| = \sqrt{\langle x+\eta, x+\eta\rangle} = \sqrt{\|x\|^2 + 2\mathrm{Re}\langle x,\eta\rangle + \|\eta\|^2}$

$\leq \sqrt{\|x\|^2 + 2|\langle x,\eta\rangle| + \|\eta\|^2} \leq \|x\| + \|\eta\|$.

Rmk: i) $\|x\| = \max\limits_{\|\eta\|=1} |\langle x,\eta\rangle|$. Take $\eta = x/\|x\|$.

ii) If $(X, \|\cdot\|)$ satisfies parallelogram law:

$\|a+b\|^2 + \|a-b\|^2 = 2(\|a\|^2 + \|b\|^2)$. $\forall a.b \in X$. Then X is

a inner product space with $[x,\eta] = \frac{1}{2}(\|x+\eta\|^2 - \|x\|^2 - \|\eta\|^2)$

Over $\mathbb{C}$: $\langle x,\eta\rangle = [x,\eta] - i[ix,\eta]$.

pf: i) $[u,v] = [v,u]$. $[-u,v] = -[u,v]$. $[u,2v] = 2[u,v]$.

ii) $[u+v,w] = [u,w] + [v,w]$

iii) $[\lambda u, v] = \lambda [u,v]$. $\forall \lambda \in \mathbb{R}^1$.

① Hilbert space:

· If X is a vector space equipped with $\langle\, ,\rangle$.

Then X is said to be Hilbert Space $\Leftrightarrow \|\cdot\| = \sqrt{\langle \cdot,\cdot\rangle}$

is complete.

prop. If $(X, \langle\, ,\rangle)$ isn't complete. Then we can

embed $(X, \langle\, ,\rangle)$ into its completion $(\overline{X}, \langle\langle\, ,\rangle\rangle)$

s.t. $X \underset{\text{dense}}{\subseteq} \overline{X}$, $\langle\langle\, ,\rangle\rangle|_X = \langle\, ,\rangle$.

pf: Define $\langle\langle\, ,\rangle\rangle$ in $\overline{X}$: $\langle\langle x,\eta\rangle\rangle = \lim\limits_{n\to\infty} \langle x_n, \eta_n\rangle$

where $x_n \to x$. $\eta_n \to \eta$.

1°) It's well-def. for another pair:

$\tilde{x}_n \to x$. $\tilde{\eta}_n \to \eta$. $|\langle x_n,\eta_n\rangle - \langle \tilde{x}_n, \tilde{\eta}_n\rangle|$

$\leq |\langle x_n - \tilde{x}_n, \eta_n\rangle| + |\langle \tilde{x}_n, \eta_n - \tilde{\eta}_n\rangle| \to 0$

2°) The limit $\langle x_n, \eta_n \rangle$ exists:

since $(\langle x_n, \eta_n \rangle)$ is Cauchy. (easy to check)

3°) $\langle\langle\ ,\ \rangle\rangle$ satisfies the requires to be scalar product.

4°) Norm from $\langle\langle\ ,\ \rangle\rangle$ is: $|||x||| = \lim \|x_n\|$. $x_n \to x$.

prop. H is Hilbert space. Then H is uniformly convex.

Pf: By paralogram law from scalar product.

② **Projection:**

Thm. $K \subseteq H$. $K \neq \emptyset$. closed. convex. Then $\forall f \in H$.

$\exists u \in K$, s.t. $|f-u| = \min_{v \in K} |f-v| = \mathrm{dist}(f, K)$.

u is characterized by: $u \in K$. $(f-u, v-u) \leq 0 \ \forall v \in K$.

Pf: 1°) $\varphi(u) = |f-u|$. convex BLF. $\lim_{\|u\| \to \infty} \varphi(u) = \infty$.

$\therefore \varphi$ attain minimum in K.

2°) $\forall v \in K$. since $u \in K$. $\therefore tu + (1-t)v \in K$.

$|f-u|^2 \leq |f - tu - (1-t)v|^2 = |f-u-t(v-u)|^2$.

Let $t \to 1^+$. attain $(f-u, v-u) \leq 0$.

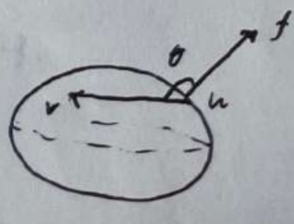
Conversely, $|u-f|^2 - |v-f|^2 = 2(f-u, v-u) - |u-v|^2 \leq 0$.

3°) Uniqueness: from characterization. Let $v = u_1, u_2$.

Remark: The characterization means that

$\theta = \langle \vec{uf}, \vec{uv} \rangle \geq \frac{\pi}{2}$. u satisfies:

$|u-f|^2 + |u-v|^2 \leq |v-f|^2$. as well.



Cor. Replace Hilbert space by uniformly convex Banach space. the thm still holds:

pf: since it's reflexive space.

Denotion: $u \triangleq P_K f$. if $|f-u| = \min_{v \in K} |f-v|$.

prop. $K \subseteq H$. convex. closed set. Then P_K doesn't increase distance: $|P_K f_1 - P_K f_2| \le |f_1 - f_2|$.

pf: By characterization of projection.

$v = P_K f_1, P_K f_2 \quad \therefore |P_K f_1 - P_K f_2| \le (f_1 - f_2, P_K f_1 - P_K f_2)$

Cor. $M \subseteq H$. closed linear subspace. $f \in H$. $u = P_M f$. is characterized by $(f-u, v) = 0$. $\forall v \in M$

Besides, $P_M(\cdot)$ is linear operator.

pf: 1°) $(f-u, tv-u) \le 0$. divide t. let $t \to +\infty / -\infty$.

2°) Conversely. $(f-u, v-u) = 0$. since $u \in M$.

③ Linear Span in Hilbert Space:

prop. $(x_\theta)_{\theta \in I} \subseteq H$. Then $z \in CLS\{x_\theta\}_{\theta \in I} \iff$ For $x \in H$. $\langle x, x_\theta \rangle = 0 \ \forall \theta \in I$, conclude $\langle z, x \rangle = 0$.

Pf: $Y \triangleq CLS\{x_\theta\}_{\theta \in I}$. Then $H = Y \oplus Y^\perp$.

Lemma. $(x_\theta)_{\theta\in I}$ Orthonormal vectors family.

For $x \in H$. Denote $\alpha_\theta = \langle x, x_\theta \rangle$. Then.

i) $\{\theta \in I \mid \alpha_\theta \neq 0\}$ is countable.

ii) $\sum_{\theta\in I} |\alpha_\theta|^2 \leq \|x\|^2$. (Bessel Inequality)

Pf: 1') For $J \subseteq I$. countable subset.

Denote $J = (\theta_i)_{i\in\mathbb{Z}^+}$.

Since $\|\sum_1^m \alpha_{\theta_i} x_{\theta_i} - x\|^2 \geq 0$, we obtain:

$\|x\|^2 \geq \sum_1^m |\alpha_{\theta_i}|^2$. let $m \to \infty$. $\therefore \|x\|^2 \geq \sum_J |\alpha_{\theta_i}|^2$

2') Consider $J_m = \{\theta \in I \mid |\alpha_\theta| \geq \frac{1}{m}\}$.

Claim it's a finite set. For $\forall (\alpha_k)_1^N \subseteq J_m$

$\frac{N}{m^2} \leq \sum_1^N |\alpha_k|^2 \leq \|x\|^2 = \frac{C}{m^2}$. for some $C > 0$.

$\therefore \{\theta \in I \mid \alpha_\theta \neq 0\} = \bigcup_1^\infty J_m$ countable set.

3') From 1'). 2'). obtain Bessel Inequality

Remark: "$\leq$" can hold strictly. since $(x_\theta)_{\theta\in I}$ may not be orthonormal Basis.

prop. $(x_\theta)_{\theta\in I}$ orthonormal set. Then $CLS\{x_\theta\}_{\theta\in I}$ has form: $\{\sum_{i\geq 1} a_i x_{\theta_i} \mid \sum_{i\geq 1} |a_i|^2 < \infty\}$.

Pf: 1') $\{\sum a_i x_{\theta_i} \mid \cdots\}$ is linear

2') $\{\sum a_i x_{\theta_i} \mid \cdots\}$ is closed:

For $z_n \to z$, $(z_n) \subseteq \{\sum a_i x_{\theta_i} \mid \cdots\}$.

prove: $z \in \{\sum \alpha_i x_{\theta_i} \mid \cdots\}$.

Suppose $z_n = \sum_{j\geq 1} \alpha_j^n \chi_{\theta_j^n}$. $\sum_j |\alpha_j^n|^2 < \infty$. $\forall n$.

Denote $J_n = \{\theta_j^n \mid \alpha_j^n \neq 0\}$. $\therefore J = \bigcup_{n\geq 1} J_n$ countable.

Denote $z_n = \sum_{j\geq 1} \beta_j^n \chi_{\hat{\theta}_j} = \lim_{N\to\infty} \sum_{j\geq 1}^{N} \beta_j^n \chi_{\hat{\theta}_j}$. where $J \triangleq (\hat{\theta}_j)_{j\in\mathbb{Z}^+}$.

Embed $\{\sum_{j\geq 1} \alpha_j \chi_{\hat{\theta}_j} \mid \sum |\alpha_j|^2 < \infty\}$ into l^2:

$\{\sum \alpha_i \chi_{\hat{\theta}_i} | \cdots\} \xrightarrow{T} l^2$. T is isometry. Besides,

$\sum \alpha_i \chi_{\hat{\theta}_i} \longmapsto (\alpha_i)$ $\quad (z_n) \subseteq \{\sum \alpha_i \chi_{\hat{\theta}_i} | \cdots\}$.

$\therefore (z_n)$ Cauchy $\to (Tz_n)$ Cauchy. Know that l^2 is Banach

$\therefore Tz_n \to (\beta_k)$. $\therefore z_n \to \sum \beta_k \chi_{\hat{\theta}_k} \in \{\sum \alpha_i \chi_{\theta_i} | \cdots\}$.

Since $\{\sum \alpha_i \chi_{\theta_i} | \cdots\} \subseteq CLS\{\chi_\theta\}_{\theta\in\mathbb{Z}}$ $\quad$ we're done.

$\{\chi_\theta\}_{\theta\in\mathbb{Z}} \subseteq \{\sum \alpha_i \chi_{\theta_i} | \cdots\}$ CLS

(2) <u>Dual Space</u>:

① <u>Thm</u>. (Riesz Representation)

$\forall \varphi \in H^*$. Then exists a unique $f \in H$. s.t.

$\varphi(u) = (f, u)$. $\forall u \in H$. $|f| = \|\varphi\|_{H^*}$.

<u>Pf</u>: For $\varphi \neq 0$. $\exists g_0 \notin N_\varphi$. Set $g = \dfrac{g_0 - P_{N_\varphi} g_0}{\|g_0 - P_{N_\varphi} g_0\|}$

$\therefore \|g\| = 1$. $(g, v) = 0$. $\forall v \in N_\varphi$.

Note that $\forall u \in H$. $u = \dfrac{\varphi(u)}{\varphi(g)} g + u - \dfrac{\varphi(u)}{\varphi(g)} g$

$\therefore (g, u) = \dfrac{\varphi(u)}{\varphi(g)}$. since $u - \dfrac{\varphi(u)}{\varphi(g)} g \in N_\varphi$.

$\therefore \varphi(u) = (g, u)\varphi(g)$. $\forall u$. $f = \varphi(g) g$.

Remark: The ideal is find a vector g orthonormal to N_φ. since $(g, u) = \varphi_g(u) \in H^*$. $N_\varphi = N_{\varphi_g}$

From conclusion before, $\varphi(u) = c\,(g, u)$.

Def: From Riesz Thm. we obtain $H^* = H$.

Define $M^\perp = \{u \mid (u, v) = 0.\ \forall v \in M\}$.

prop. $H = M \oplus M^\perp$. $M^{\perp\perp} = M$. $M^\perp$ is closed linear space.

if $M \subseteq H$. closed linear space.

Pf: It's easy to check $M^\perp$ is CLS. $M^{\perp\perp} = M$.

Note that $M \cap M^\perp = \{0\}$. For $\forall x \in H$.

$x = P_M x + x - P_M x \in M \oplus M^\perp$.

Cor. $G \subseteq H$. linear subspace. equip with norm of H.

F is Banach space. If $S : G \longrightarrow F$. BLF.

Then exists BLF : $T : H \longrightarrow F$. st. $\|H\| = \|F\|$.

Pf: $H = \bar{G} \oplus G^\perp$. extend S to $\bar{S}$ on $\bar{G}$. certily.

Let $T : H = \bar{G} \oplus G^\perp \longrightarrow F$

$(g, h) \longmapsto \bar{S}(g)$.

② **Triplet** : $V \subseteq H \subseteq V^*$:

. Note that by Riesz Representation Thm. H can be identified as H^*. Consider $V \subset H$. dense linear subspace. with its norm $\|\cdot\|$. And $(V, \|\cdot\|)$ is Banach.

If we have a injection (conti): $V \overset{}{\underset{i}{\hookrightarrow}} H$.

since we can construct $T: H^* \longrightarrow V^*$. where

$T\varphi = \varphi|_V$. then T is conti injection (V is dense)

In sum: $V \underset{i}{\hookrightarrow} H \underset{I_1}{\cong} H^* \underset{T}{\hookrightarrow} V^*$.

However if V is Hilbert space with its own scalar product. We have $V \underset{I_2}{\cong} V^*$. by Riesz.

since $V \subsetneq H \subsetneq V^*$. it will be absurb to identify V with V^*. (Then $V \subsetneq V$)

<u>Remark</u>: i) In that case. Only when $\dim V = \dim H$.

Note that even $H_1 \subsetneq H_2$. we can construct $H_1 \underset{T}{\cong} H_2$. e.g. $H_1 = \mathbb{Z} \oplus \mathbb{Z}$.

$H_2 = 2\mathbb{Z} \oplus 2\mathbb{Z}$. with scalar products:

$\langle (a,b), (c,d) \rangle_1 = ac + bd$. $\langle (a,b), (c,d) \rangle_2 = ac + bd/4$

Then $T(a,b) = (2a, 2b)$. surjective isometry.

ii) $T \circ I_1 \circ i \neq I_2$. The norms used in the two case are different.

(3) Representation of bilinear Function:

<u>Def</u>: bilinear $a(\cdot,\cdot): H \times H \longrightarrow \mathbb{R}$.

i) a is conti if $\exists C > 0$. s.t. $|a(u,v)| \leq C\|u\|\|v\|$. for every $u, v \in H$.

ii) a is coerice if $\exists\, \alpha > 0$. s.t. $a(v,v) \geq \alpha |v|^2$.

for any $v \in H$.

① <u>Stampacchia Thm</u>:

If $a(\cdot,\cdot) : H \times H \longrightarrow \mathbb{R}$. conti. coerice. $K \subseteq H$.

$K \neq \emptyset$. convex. closed. Then $\forall\, \varphi \in H^*$. exists unique

$u \in K$. s.t. $a(u, v-u) \geq \varphi(v-u)$. $\forall\, v \in H$.

Moreover. if a is symmetric. then u can be characterized:

$u \in K$. $\frac{1}{2} a(u,u) - \varphi(u) = \min\limits_{v \in K} \{ \frac{1}{2} a(v,v) - \varphi(v) \}$.

<u>Pf</u>: <u>Lemma</u>. (Banach Fixed Point Thm)

$X \neq \emptyset$. complete metric space. $S: X \longrightarrow X$ is a strictly contraction. i.e. $d(S(v_1), S(v_2)) \leq k\, d(v_1, v_2)$.

$k < 1$. for $\forall\, v_1, v_2 \in X$. Then:

$\exists$ unique $u \in X$. s.t. $Su = u$.

<u>Pf</u>: 1°) Existence:

For $u_0 \in X$. Denote $Su_n = u_{n+1}$. $n \geq 0$.

$\therefore d(u_{n+1}, u_n) \leq k^n d(u_1, u_0)$ $\therefore (u_n)$ is Cauchy

$\therefore u_n \to u$. Since $d(u_{n+1}, Su) \leq k\, d(u_n, u) < \varepsilon$

$\therefore u_n \longrightarrow Su$. $\therefore u = Su$.

2°) Uniqueness:

For $u_1 = Su_1$. $u_2 = Su_2$. $d(u_1, u_2) \leq k\, d(u_1, u_2)$

$\therefore d(u_1, u_2) = 0$. $u_1 = u_2$.

$\Rightarrow$ 1°) Fix u. $a(u,v)$ is conti $\therefore a(u,v) = (Au, v)$

$|Au| \leq |M|$. $(Au, u) \geq \alpha |u|^2$. A is linear $: H \longrightarrow H$.

2°) $\exists f \in H$. s.t. $\varphi(w) = (f, w)$ By Riesz Thm.

It suffices to find $u \in K$. s.t. $(Au, v-u) \geq (f, v-u)$

$\Leftrightarrow$ $((\rho f + u - \rho Au) - u, v-u) \leq 0$. $\rho > 0$. $\Leftrightarrow u = P_K(\rho f + u - \rho Au)$

Suppose $S: H \longrightarrow H$. $Sv = P_K(\rho f + v - \rho Av)$.

Find ρ. s.t. S is strict contraction

For $a(u,v)$ is symmetric. it defines a new scalar product on H. Let $|u| = a(u,u)^{\frac{1}{2}}$.

By Riesz Thm. $\exists g \in H$. s.t. $\varphi(v) = a(g, v)$.

Then u is characterized by projection on K.

<u>Remark</u>: $a(u,v) = (Au, v)$. $R(A)$ is closed and dense

$N(A) = \{0\}$. $\Rightarrow A: H \underset{iso}{\simeq} H$.

<u>Cor.</u> (Lax-Milgram)

$a(\cdot,\cdot)$ is conti. coerciv. bilinear. $H \longrightarrow H$.

Then $\forall \varphi \in H^*$. $\exists$ unique $u \in H$. s.t.

$a(u,v) = \varphi(v)$. $\forall v \in H$.

Moreover, if $a(\,,\,)$ is symmetric. then

u is characterized by: $u \in K$ and

$$\frac{1}{2} a(u,u) - \varphi(u) = \min_{v \in K} \{\frac{1}{2} a(v,v) - \varphi(v)\}.$$

<u>Pf</u>: Let $K = H$. We have: $a(u, tv-u) \geq (\varphi, tv-u)$

Divide t. Let $t \to +\infty / -\infty$.

(4) Hilbert Sum and Orthonormal Bases:

① **Def:** $(E_n)_{n\in\mathbb{Z}^+}$ seq of closed subspaces of H.

$H = \bigoplus_{n\in\mathbb{Z}^+} E_n$, Hilbert Sum if:

i) $(u,v)=0$. $\forall u \in E_m$. $v \in E_n$, $n \neq m$.

ii) $\text{Span}(\bigcup_{n\in\mathbb{Z}^+} E_n)$ is dense in H.

Thm. $H = \bigoplus E_n$. $\forall u \in H$. Denote $u_n = P_{E_n} u$. $S_n = \sum_1^n u_k$.

Then. $\lim_{n\to\infty} S_n = u$. and $\sum |u_n|^2 = |u|^2$.

pf: 1°) $(S_n, u) = |S_n|^2 \Rightarrow |u| \geq |S_n|$. $\forall n \in \mathbb{Z}^+$

$\therefore \sum_1^\infty |u_n|^2$ converges.

2°) $|S_n - S_m|^2 = \sum_r^m |u_k|^2$ $\therefore (S_n)$ is Cauchy

$\lim_n S_n$ exists. Denote S.

3°) Claim: $S = P_{\overline{\text{CLS}\{u_n\}_{n\in\mathbb{Z}}}} u$. $S = u$.

Since $(u - S_m, v) = 0$. $\forall v \in E_n$. $n \leq m$.

Let $m \to \infty$. $\therefore (u - S, v) = 0$. $\forall v \in E_n$.

$\therefore S = P_{\square} u$. and $u - S \in \text{CLS}\{u_n\}_{n\in\mathbb{Z}}^{\perp} = \{0\}$.

② **Orthonormal Bases:**

i) H is separable:

Def: $(e_n)_{n\geq 1}$ is Hibert bases of H (or say complete basis) if:

i) $(e_n, e_m) = 0$. $\forall n \neq m$. $|e_n| = 1$. $\forall n$.

ii) $\text{Span}\{e_n\}_{n\geq 1}$ is dense in H.

Thm. Every separable Hilbert space has countable orthonormal basis.

Pf: Apply Gram-Schmidt method on l.i set:

suppose $D = (u_k)_{k\geq 1}$. dense. $J_k = \text{span}\{u_i\}_1^k$. $1 \leq k$.

Find l.i set $\{v_k\}_1^{n_k}$ on J_k. then extend to $\{v_k\}_1^\infty$

properties:

(a) For every $u \in H$. $u = \sum_1^\infty (u, e_k) e_k$ and $|u|^2 = \sum |(u, e_k)|^2$. Conversely. if $\sum_1^N \alpha_k e_k \to u$. then $\alpha_k = (u, e_k)$. $|u|^2 = \sum_1^\infty \alpha_k^2$

Pf: Let $E_n = \mathbb{R} e_n$. then $P_{E_k} u = (u, e_k) e_k$.

Remark: Separable Hilbert Space $\underset{\text{isomorphic}}{\simeq} l^2$.

(b) $e_n \rightharpoonup 0$

pf: $(u, e_n) = \alpha_n$. if $u = \sum \alpha_n e_n$

$\therefore |\alpha_n|^2 \to 0$. i.e. $|\alpha_n| \to 0$

Remark: For (a_n) bounded. $u_n = \sum_1^n a_k e_k / n$

$|u_n| \to 0$. $\sqrt{n}\, u_n \rightharpoonup 0$. (Test with e_k. since $\sum_1^n \frac{|a_n|}{n} = \sum |(u_n, e_k)| \leq C$)

Note that $l^1 \subsetneq l^2$. $\sum |(u, e_k)|$ may not converge!

prop. $D \subseteq H$. $\mathrm{span}(D)$ is dense in H. If $(E_n)_{n\geq 1}$

is seq of closed subspace mutually orthogonal.

Besides $|u|^2 = \sum |P_{E_n} u|^2$. $\forall u \in D$. Then $H = \bigoplus E_n$

Pf: Denote $F = \overline{\cup E_n}$. $\therefore H = F \oplus F^\perp$.

$|u|^2 = |P_F u|^2 + |P_{F^\perp} u|^2 = \sum |P_{E_n} u|^2 = |P_F u|^2$

$\therefore P_{F^\perp} u = 0$. $\forall u \in D$. $\Rightarrow \forall v \in \mathrm{span}(D)$.

$\therefore P_{F^\perp} f = 0$. $\forall f \in \overline{\mathrm{span}(D)}$. i.e. $H = F$.

ii) <u>H isn't separable:</u>

· Then H may have an orthonormal Basis $(e_i)_{i\in I}$. s.t.

$|I| > \aleph_0$.

Thm. All Hilbert space has an orthonormal basis.

Pf: Suppose $\Lambda = \{ (x_\theta)_{\theta\in I} \mid x_\theta$ are orthonormal$\}$.

with "$\preccurlyeq$". $(x_\theta)_{\theta\in I} \preccurlyeq (x_\beta)_{\beta\in P} \Leftrightarrow (x_\theta)_{\theta\in I} \subseteq (x_\beta)_{\beta\in P}$.

Apply Zorn's Lemma. exists maximal Λ_0. (check chain)

If $F = \overline{\mathrm{span}(\Lambda_0)} \neq H$. Then $H = F \oplus F^\perp$.

$\exists x \in F^\perp$. $\Lambda_0 \cup \{x\} \supsetneq \Lambda_0$. contradiction!

Thm. Any two different orthonormal bases $(x_\theta)_{\theta\in I}$.

$(y_\beta)_{\beta\in P}$. satisfies: $|P| = |I|$.

Pf: $\forall \theta \in I$. $J_\theta = \{ \beta \in P \mid \langle y_\beta, x_\theta \rangle \neq 0 \}$ is countable

$\therefore |\bigcup_{\theta\in I} J_\theta| = |I| \cdot \aleph_0 = |I|$. with $\cup J_\theta \subseteq P$.

$\Rightarrow |I| \leq |P|$. By Symmetry. $|I| = |P|$.

(6) Normal Operators:

Next, we consider Hilbert $\mathcal{H}$ on $\mathbb{C}$.

① Preworks:

Thm. $B : V^{\mathbb{C}} \times V^{\mathbb{C}} \to \mathbb{C}$. sesquilinear form.

Then: $B(u,v) = \frac{1}{4} \sum_{0}^{3} i^m B(u+i^m v, u+i^m v)$

Pf: Check directly: $B(u+i^m v, u+i^m v) =$

$B(u,u) + \overline{i^m} B(u,v) + i^m B(v,u) + B(v,v)$

Rmk: It means $B(u,v)$ is determined by its diagonal form: $B(u,u)$.

Cor. $(u,v)_{\mathcal{H}} = \frac{1}{4} \sum_{0}^{3} i^m \|u+i^m v\|^2$.

Cor. $A, B \in \mathcal{L}(\mathcal{H})$. $(Ax,x) = (Bx,x)$. $\forall x \in \mathcal{H}$.

$\Rightarrow A = B$.

② Definitions:

Def. For $A \in \mathcal{L}(\mathcal{H})$.

i) A is self-adjoint $\Leftrightarrow A = A^*$

ii) A is normal $\Leftrightarrow AA^* = A^*A$.

iii) A is unitary $\Leftrightarrow AA^* = A^*A = I_{\mathcal{H}}$. Denote: $\mathcal{U}(\mathcal{H})$.

Prop. i) A is normal ii) $\|Ax\| = \|A^*x\|$. $\forall x \in \mathcal{H}$.

iii) $\exists U \in \mathcal{U}(\mathcal{H})$. s.t. $A = UA^*$. Then:

i), ii), iii) are equivalent.

Pf: i) $\Leftrightarrow$ ii) : $\|Ax\| = \|A^*x\| \Leftrightarrow (A^*Ax, x) = (AA^*x, x)$.

ii) $\Leftrightarrow$ iii) : ($\Leftarrow$) is trivial. by $A^*A = Au^*uA^* = AA^*$.

Conversely: 1°) $\|Ax\| = \|A^*x\|$. $\forall x \Rightarrow N(A) = N(A^*)$.

So: $\overline{R(A^*)} = \overline{R(A)}$ by $N(A)^\perp = \overline{R(A^*)}$.

Set $T_0 : R(A) \to \overline{R(A^*)}$. $T_0(Ax) = A^*x$. $\forall x \in \mathcal{H}$.

It's well-def ($Ax = Ay \Rightarrow A^*x = A^*y$). isometric.

Then extend T_0 on $R(A)$ to T on $\overline{R(A)}$

Note: $\mathcal{H} = N(A^*) \oplus N(A^*)^\perp = N(A^*) \oplus \overline{R(A)} = N(A) \oplus \overline{R(A^*)}$

Set: $U : \overline{R(A)} \oplus N(A^*) \to \mathcal{H}$ follows from

$u + v \longmapsto Tu + v$ the next prop.

Rmk: By Normal Calculus: $f(z) = \begin{cases} \bar{z}/z , & z \neq 0 \\ 1 , & z = 0 \end{cases}$. Then:

$f(z) z = z f(z) = \bar{z}$. $f(z)\overline{f(z)} = \overline{f(z)} f(z) = 1$. $\forall A$. normal.

$\Rightarrow f(A)$ is unitary. exchangable with A.

$A^* = A f(A) = f(A) A$. a stronger conclusion!

prop. $u \in \mathcal{U}(\mathcal{H}) \Leftrightarrow \|ux\| = \|x\|$.

prop. A is self-adjoint $\Leftrightarrow (Ax, x) \in \mathbb{R}^1$. $\forall x \in \mathcal{H}$.

Pf. Both are from diagonal determination.

③ Properties:

Def: $A \in \mathcal{L}(\mathcal{H})$ is unitarily diagnolizable $\Leftrightarrow \exists (v_i)_{i \in I}$ orthonormal basis of A. st. $Av_i = \lambda_i v_i$.

Rmk: Likewise the finite dimension case. in Matrix.

Lemma. $S \in \mathcal{L}(\mathcal{H})$. Then: $\|(S^*S)^k\|^{1/2k} = \|(SS^*)^k\|^{1/2k} = \|S\|$

Pf: $\|S\|^2 = \sup_{\|u\|=1} |(Su, Su)| = \sup |(S^*Su, u)| \le \|S^*S\|$

$\le \|S^*\|\|S\| = \|S\|^2$. So: $\|S^*S\| = \|S\|^2$.

Note: $(S^*S)^*(S^*S) = (S^*S)^2 \Rightarrow \|(S^*S)^{2^k}\| = \|S\|^{2^{k+1}}$.

To interplation by induction on k.

If $n < p$ holds for p: $2^k < p < 2^{k+1}$, then:

$\|(S^*S)^{2^k}\| = \sup_{\|u\|=1} |((S^*S)^p u, (S^*S)^{2^{k+1}-p} u)|$

$\le \|(S^*S)^p\|^{\frac{1}{2}} \|(S^*S)^{2^{k+1}-p}\|^{\frac{1}{2}} \le \|S\|^{2^{k+1}}$

prop. A is normal $\Rightarrow \|A^n\| = \|A\|^n$. $\forall n \in \mathbb{Z}^+$.

Pf: $n=2$ holds. By induction: $\|A^n\| \le \|A^{n+1}\|^{\frac{1}{2}} \|A\|^{\frac{n-1}{2}}$

Cor. $r_\sigma(A) = \max_{\lambda \in \sigma(A)} |\lambda| = \|A\|$.

Def: i) $S \subset \mathcal{H}$. CLS. is invariant subspace w.r.t $T \in \mathcal{L}(\mathcal{H})$.

if $T(S) \subset S$.

ii) $R \subset \mathcal{H}$. CLS. is reducing subspace w.r.t $T \in \mathcal{L}(\mathcal{H})$.

if $R, R^\perp$ are both invariant. w.r.t T.

Rmk: Interpretation in sense of block form:

$$T: \begin{array}{c|cc} & S & S^\perp \\ \hline S & T_{11} & T_{12} \\ S^\perp & T_{21} & T_{22} \end{array}$$

Note $\mathcal{H} = S \oplus S^\perp$. $Tx = T^S(x) + T^{S^\perp}(x)$

where $R(T^S) \subset S$. $R(T^{S^\perp}) \subset S^\perp$.

Then $T_{11} = T^S|_S$. $T_{12} = T^{S^\perp}|_S$ and so on.

We obtain: S is T-Invariant $\Leftrightarrow T_{12} = 0$

S is T-reducing $\Leftrightarrow T_{12} = T_{21} = 0$

prop. A is normal. for $v \in \mathcal{H}$. $Av = \lambda v$. Then $\mathbb{C}v$ is a reducing subspace w.r.t. A. So. $A|_{(\mathbb{C}v)^\perp}$ is normal

Pf: $A(\mathbb{C}v) = \mathbb{C}v$ is trivial. for $w \perp v$.

check: $(Aw, v) = (w, A^*v) = (w, \bar{\lambda} v) = 0$.

Lemma. A is normal. Then $Av = \lambda v \Rightarrow A^*v = \bar{\lambda} v$.

Pf: Note $A - \lambda I$ is normal. (check)

So: $\|(A-\lambda I)v\| = \|(A^* - \bar{\lambda} I)v\|$.

So: $A|_{(\mathbb{C}v)^\perp} : (\mathbb{C}v)^\perp \longrightarrow (\mathbb{C}v)^\perp$. normal.

Thm. $A \in \mathcal{L}(\mathcal{H})$. normal. cpt. Then A is unitarily diag.

Pf: Fredolm Method:

WLOG. $A \neq 0$. $\exists \lambda_0 \in \sigma(A)$, st. $|\lambda_0| = \|A\|$. Then:

by Fredolm Alternative. $\exists v_0$. $\|v_0\| = 1$. $Av_0 = \lambda_0 v_0$.

Consider: $A_1 = A|_{(\mathbb{C}v_0)^\perp}$. $A_1 = 0$. end the process.

Otherwise. consider $|\lambda_1| = \|A_1\|$. find v_1. $Av_1 = \lambda_1 v_1$

If the process never stops:

Note: $\|A_k\| \geq \|A_{k+1}\|$. So: $\exists (\lambda_k)$: $|\lambda_0| \geq |\lambda_1| \cdots \geq |\lambda_n| \cdots > 0$

correspond (v_k). $\|v_k\| = 1$. $Av_k = \lambda_k v_k$.

Claim: $\lim\limits_{n \to \infty} |\lambda_n| = 0$.

By contradiction: $\exists \delta$. $|\lambda_n| \geq \delta > 0$. $\forall n$.

Note: (v_n) is orthonormal we can assume WLOG.

But: $\|Av_i - Av_j\| = \|\lambda_i v_i - \lambda_j v_j\| \geq 2\delta > 0$.

Contradict with A is cpt!

Claim: $A|_{(\text{span}(v_k)_{k\geq 0})^\perp} = 0$.

Lemma. If $R_1 \perp R_2$, both A-reducing. Then:

$R_1 \oplus R_2$ is A-reducing.

Pf: $(R_1 + R_2)^\perp = R_1^\perp \cap R_2^\perp$. by R_1, R_2 closed.

By contradiction: if $A|_{(\text{span}(v_k)_{k\geq 0})^\perp} \neq 0$.

$\exists \lambda'$. $|\lambda'| = \| A|_D \| > 0$. $\exists v$. $Av = \lambda' v$. $\|v\| = 1$.

Since $A|_D$ is also cpt. normal. by Lemma.

But $\exists N$. st. $|\lambda'| > |\lambda_N| \geq \cdots > 0$. Set $\tilde{A} = A|_D$.

So: $\|\tilde{A} v'\| = |\lambda'| > \| A|_{(\text{span}(v_k)_1^N)^\perp} \|$. contradict!

Rmk: A normal cpt operator on $\mathcal{H}$. supports on a separable CLS of $\mathcal{H}$.

Cor. Generally, cpt operator on $\mathcal{H}$ supports on a separable CLS of $\mathcal{H}$.

Pf: A is cpt $\Rightarrow A^*A$ is cpt. normal.

$\exists (v_i)_{i\geq 0}$. st. $N(A^*A) = (\text{span}(v_i)_{i\geq 0})^\perp$.

Lemma. $N(A) = N(A^*A)$. for $A \in \mathcal{L}(\mathcal{H})$

Pf: $A^*Ax = 0 \Leftrightarrow (A^*Ax, y) \overset{\forall y}{=} 0$. $Ax \in R(A)^\perp$.

$\Rightarrow \text{supp}(A) \subset \overline{\text{span}(v_i)_{i\geq 0}}$. separable.

Rmk: It can imply another method to prove cpt operator can be approxi. by finite dimension operator.

WLOG. Restrict $K \in K(H)$ on separable CLS.

(v_i) is o.n.b. $P_n = \sum_i^n v_i \otimes v_i^*$. $Q_n = I - P_n$. $n \in \mathbb{Z}^+$

$\Rightarrow \|Q_n x\| \to 0$. $\forall x \in H$. Check $P_n \circ K \to K$. $(n \to \infty)$.

(7) <u>Hilbert - Schmidt Class:</u>

<u>Def:</u> For H, K. Hibert spaces. $T \in S_2(H, K)$. the Hilbert-Schmidt class from H to K if $(v_i)_{i \in I}$ is o.n.b of H. we have : $\|T\|_{S_2} = (\sum_{i \in I} \|Tv_i\|^2)^{\frac{1}{2}} < \infty$. T defined on span $(v_i)_{i \in I}$.

<u>Rmk:</u> $S_2(H, K)$ is Hilbert space. actually.

$$\langle T_1, T_2 \rangle_{S_2(H,K)} =: Tr(T_2^* T_1)$$
$$= \sum_{i \in I} \langle T_2^* T_1 v_i, v_i \rangle$$

for $T_1, T_2 \in S_2(H, K)$. It's well-def.

<u>Thm.</u> (v_i). (w_j) are two o.n.b of H. Then we have:

$\sum_{i \in I} \|Tv_i\|^2 = \sum_{j \in J} \|Tw_j\|^2$. for $T \in S_2(H, K)$.

<u>Pf:</u> $\forall$ $(h_\alpha)_{\alpha \in A}$ o.n.b of K. Then: consider:

$$\sum_{i \in I} \|Tv_i\|_K^2 = \sum_{i \in I} \sum_{\alpha \in A} |\langle Tv_i, h_\alpha \rangle|^2$$
$$= \sum_{j \in J} \sum_{\alpha \in A} |\langle w_j, T^* h_\alpha \rangle|^2$$
$$= \sum_{j \in J} \|Tw_j\|_K^2.$$

<u>Cor.</u> $\|T\|_{S_2(H,K)} = \|T^*\|_{S_2(K,H)}$

<u>Rmk:</u> $\|T\|_{S_2}$ is indept with choice of o.n.b. of H.

Next, consider measure space $(X, \mathcal{F}, \mu)$. μ is σ-finite. X is polish (i.e. metrizable, complete, separable). Set: $H = L^2(X, \mu)$, it's Hilbert separable, in fact.

Def: $k: X \times X \to \mathbb{C}$. $T_k(f)(x) = \int_X k(x,y) f(y) \mu(dy)$.

is operator on H.

prop. If $\int_{X^2} |k(x,y)|^2 d\mu(x) d\mu(y) < \infty$. Then $T_k \in S_2(H)$ and $\|T_k\|_{S_2} = \left(\int_{X^2} |k(x,y)|^2 \mu(dx) \mu(dy)\right)^{\frac{1}{2}}$

Pf: Suppose (φ_k) is o.n.b of H.

1°) $(\varphi_k \otimes \varphi_j)$ is o.n.b of $L^2(X \times X, \mu \otimes \mu)$.

$$\langle \varphi_k \otimes \varphi_j, \varphi_{k'} \otimes \varphi_{j'} \rangle = \int_{X \times X} \varphi_k(x) \varphi_j(y) \overline{\varphi_{k'}(x) \varphi_{j'}(y)} \mu(dx) \mu(dy)$$

$$= \delta_{kk'} \delta_{jj'}.$$

$\forall f \perp \operatorname{span}(\varphi_k \otimes \varphi_j)$. If $\langle f, \varphi_k \otimes \varphi_j \rangle = 0$.

Then: $\int_X \overline{\varphi_j(y)} \int_X f(x,y) \overline{\varphi_k(x)} d\mu(x) d\mu(y) = 0$. $\forall k, l$.

$\Rightarrow \int_X f(x,y) \overline{\varphi_k(x)} \mu(dx) = 0$. $\forall k \Rightarrow f \equiv 0$. μ^2-a.e.

2°) $\|T_k\|_{S_2}^2 = \sum \|T_k \varphi_k\|^2 = \sum \sum |\langle T_k \varphi_k, \varphi_i \rangle|^2$

$$= \sum_{i,j} |\langle k, \varphi_i \otimes \varphi_j \rangle_{L^2(X \times X, \mu^2)}|^2$$

$$= \int_{X \times X} |k(x,y)|^2 d\mu(x) d\mu(y).$$

prop. $S_2(H) \subset K(H)$. i.e. $\forall$ HS operator is cpt.

Pf: Suppose $(v_k)_{k \geq 0}$ is o.n.b of H.

Lemma. $\|T\| \le \|T\|_{S_2}$ for $T \in S_2(H)$.

Pf: Note $x = \sum (x, v_k) v_k$. $\|x\| = 1 \Rightarrow \sum (x, v_k)^2 = 1$

Lemma. $T \in S_2(H)$. $B \in \mathcal{L}(H)$. $\Rightarrow BT, TB \in S_2(H)$.

Pf: $\|BT\|_{S_2}^2 = \sum \|BTv_k\|^2 \le \|B\|^2 \sum \|Tv_k\|^2$

$$= \|B\|^2 \|T\|_{S_2}^2.$$

Besides. $\|TB\|_{S_2}^2 = \|B^* T^*\|_{S_2}^2 \le \|B\|^2 \|\tilde{T}\|_{S_2}$.

Rmk: $S_2(H)$ is a closed bilateral ideal

Moreover. $K(H)$ is the unique max one.

Cor. $A, B \in \mathcal{L}(H)$. $T \in S_2(H)$. Then, we have:

$$\|ATB\|_{S_2} \le \|A\| \|B\| \|T\|_{S_2}.$$

Rmk: These conclusions hold in nonseparable space. as well.

$\Rightarrow$ Return to the pf:

Set $P_n = \sum_{k=1}^{n} v_k \otimes v_k^*$. Prove: $\|T - TP_n\| \to 0$ $(n \to \infty)$

Note: $\|T - TP_n\|^2 \le \|T - TP_n\|_{S_2}^2 = \sum \|Tv_k - TP_n v_k\|^2$

$$= \sum_{k=1}^{n} + \sum_{n+1}^{\infty} \Longrightarrow$$

$$= \sum_{n+1}^{\infty} \|Tv_k\|^2 \longrightarrow 0 \ (n \to \infty)$$

So. T can be approxi. by finite dimension operators.