L^p Spaces

(1) <u>Preliminary</u>:

<u>Notation</u>: $1 \le p \le \infty$. its conjugate exponent p' satisfies:

$1/p + 1/p' = 1$.

① <u>Hölder Inequility</u>:

<u>Thm</u>. Suppose $f \in L^p(\Omega)$, $g \in L^q(\Omega)$. $1/p + 1/q = 1$. Then

$fg \in L^1(\Omega)$. Besides. $\|fg\|_{L^1} \le \|f\|_{L^p} \|g\|_{L^q}$.

<u>Pf</u>: 1°) $p = 1$ or ∞. it's trivial.

2°) $1 < p < \infty$. Apply Young's Inequility:

$ab \le \frac{1}{p} a^p + \frac{1}{p'} b^{p'}$. $\forall a, b \ge 0$.

(Pf: Take log. by Jensen inequility)

WLOG. Let $\|f\|_{L^p} = \|g\|_{L^q} = 1$.

<u>Cor</u>. $1/p = 1/p_1 + 1/p_2$. $f \in L^{p_1}$. $g \in L^{p_2}$. Then

$fg \in L^p$. Besides. $\|fg\|_{L^p} \le \|f\|_{L^{p_1}} \|g\|_{L^{p_2}}$

<u>pf</u>: $\|fg\|_{L^1} \le \|f\|_{L^{\frac{p_1}{p}}} \|g\|_{L^{\frac{p_2}{p}}}$. Let $f = f^p$. $g = g^p$

<u>Cor</u>. $1/p = \sum_1^n 1/p_i$. $f_i \in L^{p_i}$. $f = \prod_1^n f_i$ Then.

$f \in L^p$. $\|f\|_{L^p} \le \prod_1^n \|f_i\|_{L^{p_i}}$.

<u>Pf</u>: By induction on n: $f = \prod_1^{n-1} f_i \cdot f_n$.

Cor. $f \in L^p \cap L^q$. $1 \le p \le q \le \infty$. Then $f \in L^r$. $(0 < \alpha < 1)$

$\forall\ p \le r \le q$. Besides. $\|f\|_{L^r} \le \|f\|_p^{\alpha} \|f\|_q^{1-\alpha}$. $1/r = \alpha/p + {1-\alpha}/q$.

Pf: $|f| = |f|^{\alpha} |f|^{1-\alpha}$.

Cor. $|\Omega| < \infty$. Then $L^p \subseteq L^q$. $1 \le q \le p \le \infty$. $\longrightarrow$ e.g. $L^p_{loc} \subseteq L^q_{loc}$ if $p \ge q$.

Pf: $\|f\|_{L^q} \le \|f\|_{L^p} |\Omega|^{\frac{1}{q} - \frac{1}{p}}$.

② Jensen Inequility:

$|\Omega| < \infty$. $j : \mathbb{R} \longrightarrow (-\infty, +\infty]$. l.s.c. convex. $j \not\equiv +\infty$.

$f \in L^1(\Omega)$. $f \in D(j)$. a.e.x. And $j(f) \in L^1(\Omega)$. Then

: $j\left(\frac{1}{|\Omega|} \int_\Omega f \right) \le \frac{1}{|\Omega|} \int_\Omega j(f)\, dm$

Pf. Define $\langle \cdot , \cdot \rangle$ in $\mathbb{R}$: $\langle f(x), g(t) \rangle = f(x) g(t)$

$\therefore\ f(x) t \le j^*(t) + j(f(x))$. integrate on X.

Since $j\left(\frac{1}{|\Omega|} \int_\Omega f\right) = \sup_{t \in \mathbb{R}} \left\{ \frac{1}{|\Omega|} \int_\Omega f\, dm \cdot t - j^*(t) \right\}$

$(j^{**} = j)\ \le \frac{1}{|\Omega|} \int_\Omega j(f(x))\, dm$.

③ Basic properties of L^p space:

Thm. L^p is a vector space. $\|\cdot\|_p$ is a norm

for $1 \le p \le \infty$

Pf. Check by Minkovsky. Inequility.

Remark: We don't discuss about L^p space when $0<p<1$. Because $\|\cdot\|_{L^p}$ isn't a norm (It doesn't satisfy triangle inequality)

e.g. $p=\frac{1}{2}$. $(a+b)^2 \not\leq a^2+b^2$. but $2(a^2+b^2)$.

Moreover, there're no BLF's on L^p, $0<p<1$, when $\Omega = \mathbb{R}^1$. (except $l \equiv 0$).

Pf: If l is nontrivial BLF on $L^p(\mathbb{R}^1)$.

Let $F(x) = l(\chi_{[0,x]})$.

$|F(x)-F(y)| = |l(\chi_{[y,x]})| \leq M \|\chi_{[y,x]}\|_{L^p}$

$= M|x-y|^{\frac{1}{p}}$. since $\frac{1}{p}>1$.

$\therefore |F'(x)| = 0$. $\forall x \geq 0$. symmetrically, $|F(x)| \equiv 0$. $\forall x$.

$\therefore l(\chi_{[x,y]}) \equiv c$. $c\equiv 0$ by linearity.

$\forall f \in L^p(\mathbb{R}^1)$. Approxi by step functions

$\therefore l(f) \equiv 0$. $\forall f \in L^p(\mathbb{R}^1)$ $\therefore l \equiv 0$.

prop. $f \in L^\infty(\Omega)$, supports on a set E with finite measure. Then $f \in L^p(\Omega)$, $\forall 0<p\leq\infty$. Besides, $\|f\|_{L^p} \to \|f\|_{L^\infty}$.

Pf. 1°) $\|f\|_{L^p} \leq \|f\|_{L^\infty} \cdot m(E)^{\frac{1}{p}}$. Take $\overline{\lim_{p\to\infty}}$.

2°) $\forall \varepsilon>0$. $\exists \delta>0$. s.t. $m(\{|f| > \|f\|_{L^\infty} - \varepsilon\}) > \delta > 0$.

$\therefore \|f\|_{L^p}^p \geq \delta(\|f\|_{L^\infty} - \varepsilon)^p$.

Then take $\underline{\lim}_{p\to\infty}$ after $(\cdot)^{\frac{1}{p}}$.

Thm. $L^p(\Omega)$ is Banach space. $\forall 1 \le p \le \infty$.

Pf: 1) $p = \infty$: $|f_m - f_n| \le \frac{1}{k}$ on Ω / E_k. $\mu(E_k) = 0$.

Let $E = \cup E_k$. $\therefore (f_m)$ converges in $L^\infty(\Omega / E)$.

2) $1 \le p < \infty$: Extract $\{f_{n_k}\}$. $\| f_{n_k} - f_{n_{k+1}} \|_{L^p} \le \frac{1}{2^k}$.

Let $g_N(x) = \sum_1^N |f_{n_{k+1}} - f_{n_k}| \in L^p(\Omega)$. $\forall N \in \mathbb{Z}^+$.

Besides, $(\int |g_N|^p d\mu)^{\frac{1}{p}} \le 1$. $\therefore g_N \to g$. a.e. $g \in L^p$.

Check $|f_m - f_n| \le \varepsilon$. (f_n) Cauchy $\to f$. a.e.

$\| f - f_n \|_{L^p} \to 0$ by Dominated Convergence Thm.

Thm. $f_n \to f$ in L^p. $\{f_n\} \cup \{f\} \subseteq L^p(\Omega)$. $1 \le p \le \infty$.

Then exists $(f_{n_k}) \subseteq (f_n)$. $h \in L^p(\Omega)$. s.t.

i) $f_{n_k} \to f$. a.e. ii) $|f_{n_k}| \le h$. $\forall k$. a.e.

Pf: 1) $p = \infty$ is trivial.

2) We have showed before: $\exists (f_{n_k})$

$f_{n_k} \to f^*(x)$. a.e. $f^* = f$. a.e.

Let $g + |f^*| = h$. $g = \lim_N \sum_1^N |f_{n_k} - f_{n_{k+1}}|$

(2) Dual of $L^p(\Omega)$ spaces:

① $1 < p < \infty$:

Thm. L^p is reflexive for $1 < p < \infty$.

Actually, L^p is uniformly convex. $1 < p < \infty$.

Pf: 1°) $2 \le p < \infty$:

Lemma (Clarkson's first inequility)

$$\left\|\frac{f+g}{2}\right\|_p^p + \left\|\frac{f-g}{2}\right\|_p^p \le \frac{1}{2}(\|f\|_p^p + \|g\|_p^p).\ \forall f,g \in L^p.$$

prove: $\alpha^p + \beta^p \le (\alpha^2+\beta^2)^{\frac{p}{2}}$. $\forall \alpha, \beta > 0$. first

2°) $1 < p \le 2$:

Lemma (Clarkson's Second Inequility).

$$\left\|\frac{f+g}{2}\right\|_p^{p'} + \left\|\frac{f-g}{2}\right\|_p^{p'} \le \left(\frac{1}{2}(\|f\|_p^p + \|g\|_p^p)\right)^{1/p-1}.\ \forall f,g \in L^p.$$

Thm. (Riesz Representation)

$1 < p < \infty$. $\forall \phi \in (L^p)^*$. Then exists a unique element $u \in L^{p'}$. s.t. $\langle \phi, f \rangle = \int_\Omega u f \, dm$. $\forall f \in L^p$. $\|u\|_{p'} = \|\phi\|_{(L^p)^*}$.

i.e. $(L^p)^* \underset{\text{isometry}}{\overset{T}{\cong}} L^{p'}$.

Pf: Consider $T: L^{p'} \longrightarrow (L^p)^*$ $\quad \langle Tu, f \rangle = \int uf$. $\forall f \in L^p$.
$\qquad u \longmapsto Tu$

Since $|\langle Tu, f \rangle| \le \|u\|_{L^{p'}} \|f\|_{L^p}$. $\therefore Tu \in (L^p)^*$. well-def.

Besides, $\|Tu\|_{(L^p)^*} \le \|u\|_{L^{p'}}$

Let $f_0 = |u|^{p'-2} u$ $\therefore \|u\|_{L^{p'}}^{p'} = |\langle Tu, f_0 \rangle| \le \|Tu\| \|f_0\|_{L^p}$.

$\therefore \|Tu\|_{(L^p)^*} \ge \|u\|_{L^{p'}}$ i.e. $\|Tu\| = \|u\|_{L^{p'}}$. T is isometry.

For surjective: $T(L^{p'})$ is closed. (T is isometry)

prove: $T(L^{p'})$ dense in $(L^p)^*$.

If $h \in (L^p)^{**}$. $\langle h, Tu \rangle = 0$. $\forall u \in L^{p'}$.

By reflective $h \in L^p$. $\therefore \langle Tu, h \rangle = 0$. choose $u = |h|^{p-2} h$. $h \equiv 0$.

Thm. $C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$. $1 \le p < \infty$.

Pf: 1°) For $f \in L^p(\mathbb{R}^n)$. $\forall \varepsilon > 0$. $\exists g \in L^\infty(\mathbb{R}^n)$

$K = \operatorname{supp} g$ is cpt. st. $\|f - g\|_{L^p} < \varepsilon$

Let $g = \chi_{\overline{B(0,n)}} T_n(f)$. truncation of $f(x)$.

2°) $\exists g_1 \in C_c(\mathbb{R}^n)$. st. $\|g_1 - g\|_{L^1} < \delta$.

Since $g \in L^1(\mathbb{R}^n)$. $C_c(\mathbb{R}^n)$ dense in $L^1(\mathbb{R}^n)$.

suppose $\|g_1\|_\infty \le \|g\|_\infty$. or Let $g_1 = T_{\|g\|_\infty}(g_1)$.

3°) Check $\|g - g_1\|_{L^p} \le \|g - g_1\|_{L^1}^{\frac{1}{p}} 2\|g\|_{L^\infty} \le C\delta$.

$\therefore g \overset{\varepsilon}{\sim} g_1$. $g_1 \overset{\varepsilon}{\sim} f$. choose δ small enough.

Def: measure space $(\Omega, \mathcal{M}, \mu)$ is separable. if $\mathcal{M}$ is countably generated. If Ω is metric space and $\mathcal{M}$ consists of Borel sets, call it separable measure space.

Thm. If Ω is separable measurable space. Then $L^p(\Omega)$ is separable. $1 \le p < \infty$.

Pf. Only consider $\Omega = \mathbb{R}^n$. Then $\mathcal{M} = \sigma(\{R = \prod_1^n (a_i, b_i)$

$\mid a_i, b_i \in \mathbb{Q}. \forall 1 \le i \le n\}) \triangleq \sigma(\mathcal{R})$.

Claim: $\mathcal{E} = \{\chi_R \mid R \in \mathcal{R}\}$ dense in $L^p(\Omega)$.

First $\exists g \in C_c(\mathbb{R}^n) \sim f \in L^p(\Omega)$

Then approxi g by $\mathcal{E}$.

① $p=1$:

Thm. (Riesz Representation)

If $\phi \in (L^1)^*$. There exists unique $u \in L^\infty$. s.t.

$\langle \phi, f \rangle = \int_\Omega u f \, d\mu$. $\forall f \in L^1$. Besides. $\|u\|_{L^\infty} = \|\phi\|_{(L^1)^*}$.

i.e. $(L^1)^* \underset{\text{isomery}}{\overset{T}{=}} L^\infty$.

Pf: Suppose Ω is σ-measurable. $\Omega = \bigcup \Omega_n$.

Denote $\chi_n = \chi_{\Omega_n}$. $|\Omega_n| < \infty$. $\forall n$.

1°) Uniqueness:

$\int_\Omega (u_1 - u_2) f \, d\mu = 0$. Let $f = [\operatorname{sgn}(u_1 - u_2)] \chi_n$.

2°) Existence:

i) Construct $\theta(x) \in L^2(\Omega)$. Choose $\{\alpha_n\}_{n \in \mathbb{Z}^+}$:

Let $\theta = \alpha_1$. $x \in \Omega_1$. $\theta = \alpha_n$. $x \in \Omega_n / \Omega_{n-1}$.

It's for $\forall f \in L^2(\Omega) \Rightarrow \theta f \in L^1(\Omega)$.

ii) $\varphi_\phi(f) = \langle \phi, \theta f \rangle$ is BLF on $f \in L^2(\Omega)$.

By Riesz Representation on $p=2$.

$\langle \phi, \theta f \rangle = \int u f$. $\exists u \in L^2(\Omega)$. Set $v = \frac{u}{\theta}$

$\therefore \langle \phi, \theta f \rangle = \int v \theta f$. Let $f = q \chi_n / \theta$. $q \in L^\infty(\Omega)$.

$\therefore \langle \phi, q \chi_n \rangle = \int v q \chi_n \, d\mu$. $\forall q \in L^\infty(\Omega)$.

iii) Claim: $v \in L^\infty(\Omega)$. $\|v\|_\infty \le \|\phi\|_{(L^1)^*}$.

$\Leftrightarrow$ Prove $A = \{ v(x) > c > \|\phi\| \}$ is μ-null.

Test with $q = \chi_A$. for $\forall n$.

iv) Claim: $\langle \phi, h \rangle = \int v h \, d\mu$ conti on $\forall h \in L^1(\Omega)$

by truncation: $q = \chi_n T_n(h) \longrightarrow h$ in L^1

Besides $\|\phi\|_{(L^1)^*} \le \|v\|_{L^\infty}$ $\therefore \|\phi\| = \|v\|$.

Remark: $L^1(\Omega)$ is never reflexive except where Ω consists of finite number of atoms, in that case $L^1(\Omega)$ is finite dimensional.

Pf: 1°) By contradiction: $L^1(\Omega)$ is reflexive.

i) $\forall \varepsilon > 0.\ \exists W \in M.$ st. $0 < \mu(W) < \varepsilon.$

$\exists (W_n).\ \mu(W_n) \downarrow 0.\ \mu(W_n) > 0.\ \forall n.$

Let $u_n = \frac{\chi_{W_n}}{\|\chi_{W_n}\|_{L^1}}$. $\exists (u_{n_k}).\ u_{n_k} \rightharpoonup u.$

Test with χ_{W_j}. By Dominated Convergence Thm.

ii) $\exists \varepsilon > 0.$ st. $\mu(W) \geq \varepsilon,\ \forall W \in M.\ \mu(W) > 0.$

Then Ω is atomic w.r.t. μ. with countable atoms (a_n). $\therefore L^1(\Omega) \cong \ell^1$.

But ℓ^1 isn't reflexive.

2°) Suppose $(a_i)_1^n$ is atoms. Then for $f \in L^1(\Omega)$. only consider values on $x = a_k.\ 1 \leq k \leq n.$

$\therefore f(x) = g(x)$. a.e. μ. if $f(a_k) = g(a_k)\ \forall 1 \leq k \leq n.$

③ $p = \infty$:

. Note that $L^\infty = (L^1)^*$.

properties: i) B_{L^∞} cpt in $\sigma(L^\infty, L^1)$

ii) $(f_n) \subseteq L^\infty$. $\exists (f_{n_k}) \xrightarrow{*} f$ in $\sigma(L^\infty, L^1)$ if (f_n) is bounded.

iii) L^∞ isn't reflexive except Ω consists of finite number of atoms.

iv) $L^\infty(\Omega)$ isn't separable except when Ω consists of finite number of atoms.

Def: $(\Omega, \mathcal{M}, \mu)$ is nonatomic. if $\forall A \in \mathcal{M}$. $\mu(A) > 0$.

$\exists B \subseteq A$. $B \in \mathcal{M}$. s.t. $0 < \mu(B) < \mu(A)$.

μ is conti on $\mathcal{M}$ if $\forall t$. $0 < t < \mu(\Omega)$. Then

$\exists W \in \mathcal{M}$. s.t. $\mu(W) = t$.

prop. μ is conti $\Leftrightarrow$ μ is nonatomic.

Pf: ($\Rightarrow$) It's trivial.

($\Leftarrow$) If $\exists c > 0$. no $E \in \mathcal{M}$. s.t. $\mu(E) = c$.

$A = \{ k \in \mathcal{M} \mid \mu(k) < c. \}$ with "$\leq_1$" s.t.

$k_1 \leq_1 k_2$. $\Leftrightarrow$ $k_1 \subseteq k_2$.

$B = \{ K \in \mathcal{M} \mid \mu(K) > c \}$. with "$\leq_2$" s.t.

$k_1 \leq_2 k_2$ $\Leftrightarrow$ $k_1 \supseteq k_2$.

Apply Zorn's Lemma on $(A, \leq_1)$. $(B, \leq_2)$.

We obtain max elements Q. R. $\mu(R/Q) > 0$.

But no $W \in \mathcal{M}$. s.t. $0 < \mu(W) < \mu(R/Q)$.

Otherwise. come into a contradiction.

Return to the pf:

Lemma. E is Banach space. If $\exists (O_i)_{i \in I}$. satisfies:

(a) I is uncountable (b) $O_i \cap O_j = \emptyset$. $\forall i \neq j \in I$.

(c). O_i open. nonempty. $\forall i \in I$. Then E isn't separable.

Pf: By contradiction:

Suppose (a_n) is countable dense.

$\exists a_{n_i} \in (a_n) \cap O_i$. $O_i \mapsto a_{n_i}$ Then I countable

which violates (a).

$\Rightarrow$ Consider to construct O_i. $i \in I$.

1°) Claim: $\exists (W_i)_{i\in I}$. s.t. I is uncountable. $W_i \in M$.

$\mu(W_i \Delta W_j) > 0$. $\forall i \neq j \in I$.

since $\Omega = \Omega_a \cup \Omega_d$. Ω_a is atomic. Ω_d is nonatomic.

If $\Omega_d \neq \varnothing$. Then $\forall t$, $0 < t < \mu(\Omega_d)$. $\exists W_t \in M$.

s.t. $\mu(W_t) = t$. $(W_t)_{0<t<\mu(\Omega_d)}$ is what we need.

If $\Omega_d = \varnothing$. Then since $\Omega_a = (a_n)_{n\in\mathbb{Z}^+}$.

Let $W_A = \bigcup_{k\in A} \{a_k\}$. $(W_A)_{A\subseteq\mathbb{N}}$ is what we need.

2°) $O_i = \{f \in L^\infty(\Omega) \mid \|f - \chi_{W_i}\|_\infty < \frac{1}{2}\}$ is what we need.

Since $\|\chi_{W_i} - \chi_{W_j}\|_\infty = 1$. if $i \neq j$.

(3) l^p sequence spaces:

Def: i) $x \in \mathbb{R}^{\mathbb{N}}$. $\|x\|_p = (\sum_1^\infty |x_k|^p)^{\frac{1}{p}}$. $\|x\|_\infty = \sup|x_k|$

Denote $l^p = \{\|x\|_p < \infty\}$. $1 \le p \le \infty$.

ii) Denote: $c = \{x \in \mathbb{R}^{\mathbb{N}} \mid \lim_{k\to\infty} x_k \text{ exists}\}$.

$c_0 = \{x \in \mathbb{R}^{\mathbb{N}} \mid \lim_{k\to\infty} x_k = 0\}$.

Then $(c_0, \|\cdot\|_\infty) \underset{\text{close}}{\subseteq} (c, \|\cdot\|_\infty) \underset{\text{close}}{\subseteq} l^\infty$.

Hölder Inequality in discrete form:

$|\sum_1^\infty x_k \eta_k| \le \|x\|_{l^p} \|\eta\|_{l^{p'}}$ for $x \in l^p$. $\eta \in l^{p'}$.

① Properties:

i) l^p is Banach space. $\forall 1 \le p \le \infty$.

Pf: $l^p \simeq L^p(\Omega)$. when $\Omega = \mathbb{N}$. μ is counting measure.

ii) l^p is reflexive, even uniformly convex. $\forall 1 < p < \infty$.

iii) l^p $(1 < p < \infty)$. c. c_0 are separable.

Pf: Check: $D = \{ (x_k) \mid x_k \in \mathbb{Q},\ x_k \equiv 0.\ \forall k \ge N.\ N \in \mathbb{Z}^+ \}$.

is dense in c_0

So $D + \lambda(1,1,\cdots 1,\cdots)$. $\lambda \in \mathbb{Q}$ dense in c.

Remark: l^∞ isn't separable.

If $A \subseteq l^\infty$. countable. $A = (a^k)$.

Let $b_k = \begin{cases} a_k^k + 1. & a_k^k \le 1 \\ 0. & a_k^k > 1 \end{cases}$ $\therefore b = (b_k) \in l^\infty$.

But $\| b - a^k \|_\infty \ge 1$. $\therefore b \notin \bar{A}$.

iv) $l^p \le l^q$. for $1 \le p \le q \le \infty$.

Pf: $\| x \|_{l^2}^2 = (\sum_1^\infty |x_k|^2) \le \sup |x_k|^{2-p} \sum |x_k|^p$

$= \sup |x_k|^{2-p} \| x \|_{l^p}^p \le \| x \|_{l^p}^2$.

$\| x \|_{l^\infty} \le \| x \|_{l^p}$. $l^p \subseteq l^2$.

Remark: It's totally reversed in $L^p(\Omega)$.

Because $x_k \to 0$. its order will

increase when $p \uparrow$. Then it's easy

to converge.

② Representation:

Thm. $1 \le p < \infty$. $\forall \phi \in (l^p)^*$. Then exists a unique $u \in l^{p'}$. s.t. $\langle \phi, x \rangle = \sum_1^k u_k x_k$. $\forall x \in l^p$. Besides $\|\phi\|_{(l^p)^*} = \|u\|_{l^{p'}}$

Pf: Only consider ϕ on $\{e_k\}_{k \in \mathbb{Z}^+}$.

$e_k = (0, 0 \cdots 0, 1, 0 \cdots 0)$, $e_k^k = 1$. $e_n^k = 0$. $\forall n \ne k$.

Set $u_k = \phi(e_k)$. check $\|u\| = \|\phi\|$.

(let $x = (x_1, \cdots, x_n, 0 \cdots)$, $x_k = |u_k|^{p'-2} u_k$)

Thm. $\forall \phi \in (c_0)^*$. $\exists$ unique $u \in l^1$. s.t. $\langle \phi, x \rangle = \sum_1^\infty u_k x_k$. $\forall x \in c_0$. Besides $\|u\|_{l^1} = \|\phi\|_{(c_0)^*}$.

Remark: Similar method as above. The point is: $\{(x_1, x_2 \cdots x_N, 0 \cdots) \mid x_k \in \mathbb{Q}.\ N \in \mathbb{Z}^+\}$ is dense in l^p. c_0. $1 \le p < \infty$. But not l^∞.

Thm. $\forall \phi \in (c)^*$. Then exists $(u, \lambda) \in l^1 \times \mathbb{K}$.
s.t. $\langle \phi, x \rangle = \sum_1^\infty u_k x_k + \lambda \lim_k x_k$. $\forall x \in c$.
Besides. $\|u\|_{l^1} + |\lambda| = \|\phi\|_{(c)^*}$.

Pf: Let $x = y + a e$. $a = \lim x_k$. $e = \sum_1^\infty e_k$.
Then $y \in c_0$. Consider $\phi(e) = \lambda + \sum_1^\infty u_k$.
which is reduced to c_0 case.
check it's isometry by $x =: \begin{cases} x_k = \mathrm{sgn}(u_k). & k \le N \\ x_k = \mathrm{sgn}(\lambda). & k > N. \end{cases}$

Cor. l^1. l^∞. c. c_0 aren't reflexive

(4) Convolution and Regularization:

① Young Inequality:

$\frac{1}{r}+1=\frac{1}{p}+\frac{1}{q}$. where $1\le p,q,r\le\infty$. And $f\in L^p(\mathbb{R}^n)$. $g\in L^q(\mathbb{R}^n)$. Then we have:

$f*g\in L^r(\mathbb{R}^n)$. $\|f*g\|_r\le\|f\|_p\|g\|_q$.

i) $f(x-y)g(y)$ is integrable on y for a.e. x.

Pf: $\int|f(x-y)g(y)|=\int|f|^{\alpha}|g|^{\beta}|f(x-y)^{1-\alpha}g(y)^{1-\beta}|$

$\le\|f^{\alpha}\|_{\lambda_1}\|g^{\beta}\|_{\lambda_2}\|f(x-y)^{1-\alpha}g(y)^{1-\beta}\|_{\lambda_3}$. $\sum_1^3\frac{1}{\lambda_i}=1$

Let $\begin{cases}\lambda_1\alpha=p & (1-\alpha)\lambda_3=p\\ \lambda_2\beta=q & (1-\beta)\lambda_3=q\end{cases}$

$\therefore\begin{cases}\alpha=p/q'\\ \beta=q/p'\end{cases}$ Note that $\frac{1}{p'}+\frac{1}{q'}+\frac{1}{r}=1$

Then $|f(x-y)g(y)|\in L^1$. (by Fubini Thm on the last term)

ii) $|f*g|\le\int|f(x-y)g(y)|\le\|f\|_p^{\alpha}\|g\|_q^{\beta}$.

$(\int|f(x-y)|^p|g(y)|^q)^{\frac{1}{r}}$. choose α,β as above.

$\therefore\int|f*g|^r\le\|f\|_p^{\alpha r}\|g\|_q^{\beta r}\|f\|_p^p\|g\|_q^q$.

$\therefore\|f*g\|_r\le\|f\|_p\|g\|_q$. $\square$

Cor. when $r=\infty$. Then $f*g\in L^{\infty}(\mathbb{R}^n)\cap C(\mathbb{R}^n)$,

if $1<p<\infty$. then, $f*g\to0$ $(|x|\to\infty)$

Pf: Exists $(f_n), (g_n) \subseteq C_c(\mathbb{R}^n)$, s.t.

$f_n \to f$ in L^p, $g_n \to g$ in L^q.

Note that $f_n * g_n \in C_c(\mathbb{R}^n)$, $\|f*g - f_n * g_n\|_\infty \to 0$.

Remark: $\overline{C_c(\mathbb{R}^n)} = C_0(\mathbb{R}^n)$, it's the point.

prop. $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$, $h \in L^{r'}(\mathbb{R}^n)$.

where $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, $1 \le p, q, r \le \infty$. Denote

$\check{F}(x) = F(-x)$. Then $\int (f*g)h = \int f(\check{g}*h)$

Pf: $(f*g)h \in L^1(\mathbb{R}^n)$, it's easy to check

② Support:

prop. $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$, $\frac{1}{p} + \frac{1}{q} > 1$.

Then $\operatorname{supp}(f*g) \subseteq \overline{\operatorname{supp} f + \operatorname{supp} g}$.

Pf: $f*g = \int_{x - \operatorname{supp} f \,\cap\, \operatorname{supp} g} f(x-y) g(y)\, dy$ $\therefore$ if $x \notin \operatorname{supp} f + \operatorname{supp} g$

Then $x - \operatorname{supp} f \cap \operatorname{supp} g = \emptyset$ $\therefore f*g = 0$

Remark: If $\operatorname{supp} f$, $\operatorname{supp} g$ are cpt. Then

$\operatorname{supp}(f*g)$ is cpt as well.

Since $\operatorname{supp} f + \operatorname{supp} g$ is cpt, and

$\operatorname{supp}(f*g)$ is closed.

③ Continuity:

prop. $f \in C_c(\mathbb{R}^n)$. $g \in L^1_{loc}(\mathbb{R}^n) \supseteq L^p_{loc}(\mathbb{R}^n)$. $p \geq 1$.

Then $f * g \in C(\mathbb{R}^n)$.

Pf: $f(x-y)g(y)$ is integrable. check to $\forall x_n \to x$.

Since $|f*g(x_n) - f*g(x)| \leq \sup_y |f(x-y) - f(x_n-y)| \, \|g\|_{L^1}$.

prop. $f \in C_c^k(\mathbb{R}^n)$. $g \in L^1_{loc}(\mathbb{R}^n)$. Then $f*g \in C^k(\mathbb{R}^n)$

and $D^\alpha(f*g) = (D^\alpha f * g)$. $\forall \alpha$. $|\alpha| \leq k$,

In particular. $k = \infty$.

Pf: By induction show $\nabla(f*g) = (\nabla f * g)$

④ Mollifiers:

Def: A seq of mollifiers $(\ell_n)_{n\in\mathbb{Z}^+}$ satisfies: $\forall n \in \mathbb{Z}^+$

$\ell_n \in C_c^\infty(\mathbb{R}^n)$. $\operatorname{supp} \ell_n \subset \overline{B(0,\frac{1}{n})}$ $\int \ell_n = 1$. $\ell_n \geq 0$

e.g. let $\ell(x) = \begin{cases} e^{1/(|x|^2-1)} & . \ |x| < 1 \\ 0 & . \ |x| \geq 1. \end{cases}$ $\ell_n(x) = Cn^n \ell(nx)$. $C = \int \ell$.

prop. If $f \in C(\mathbb{R}^n)$. Then $\ell_n * f \xrightarrow{u.c.c} f$.

pf: Compactness is for uniformly conti.

Thm. $f \in L^p(\mathbb{R}^n)$. $\forall 1 \leq p < \infty$. Then $\ell_n * f \to f$ in L^p

Pf: $\exists f_1 \in C_0(\mathbb{R}^n)$. $f_1 \to f$ in L^p.

Then $\ell_n * f_1 \to f_1$ in L^p. $(n\to\infty)$

since $supp(\ell_n * f_1)$ is cpt.

Cor. For $\Omega \underset{open}{\subseteq} \mathbb{R}^n$. $C_c^\infty(\Omega)$ is dense

in $L^p(\Omega)$. $\forall\, 1 \le p < \infty$.

Pf: Set $\bar{f}(x) = \begin{cases} f(x). & x \in \Omega \\ 0 & x \in \Omega^c \end{cases}$ $\therefore \bar{f} \in L^p(\mathbb{R}^n)$

Consider exhaustion of $\Omega = \bigcup\limits_1^\infty K_n$. K_n cpt.

Set $g_n = \bar{f} \cdot \chi_{K_n}$. $f_n = \ell_n * g_n$.

Let $K_n = \{ |x| \le n,\ d(x, \Omega^c) \ge \frac{2}{n} \}$. for $supp f_n \subseteq \Omega$.

Check $f_n \in C_c^\infty(\mathbb{R}^n) \to f$. in L^p.

Remark: For $p = \infty$. $C_c^\infty(\Omega)$ is dense in

$L^\infty(\Omega)$ w.r.t $\sigma(L^\infty, L^1)$.

Lemma. $\forall u \in L^\infty(\mathbb{R}^n)$. If $(g_n) \subseteq L^\infty(\mathbb{R}^n)$. st.

$\|g_n\|_{L^\infty} \le 1$. $g_n \to g$. a.e. Then set

$v_n = \ell_n * (g_n u)$. $v = gu$. $v_n \xrightarrow{*} v$. and

$\int_B |v_n - v| \to 0$ on every ball.

Pf: 1°) $\left|\int v_n \varphi - \int v\varphi\right| = \left|\int (\ell_n * (g_n u))\varphi - v\varphi\right|$

since $u \in L^\infty(\mathbb{R}^n)$. $\varphi \in L^1(\mathbb{R}^n)$.

remove the convolution from u to φ

$= \left|\int g_n u (\check{\ell}_n * \varphi) - g u \varphi\right|$

$\le \|u\|_\infty \left(\|g_n\|_\infty \|\check{\ell}_n * \varphi - \varphi\|_{L^1} + \|g_n - g\|_\infty \|\varphi\|_1 \right)$

2°) Let $\varphi = \chi_B \in L^p(\mathbb{R}^n)$. $\forall 1 \le p \le \infty$.

$\Rightarrow$ Then for $u \in L^\infty(\Omega)$, we can find $(u_n) \subseteq C_c^\infty(\Omega)$

s.t. (a) $\|u_n\|_\infty \le \|u\|_\infty$ (b) $u_n \to u$, a.e. on Ω.

(c) $u_n \xrightarrow{*} u$. in $\sigma(L^\infty, L^1)$

Pf: $\Omega = \bigcup K_n$ exhaustion of Ω. Let $\zeta_n = \chi_{K_n}$.

Let $\bar{u} = \begin{cases} 0, & x \in \Omega^c \\ u, & x \in \Omega \end{cases}$. $V_n = \zeta_n * (\bar{u}\chi_{K_n}) \in C_c^\infty(\mathbb{R}^n)$

For $\forall B_n = B(0,n)$. $\exists (V_k^n) \subseteq (V_k)$. $V_k^n \to \bar{u}$ in B_n.

Since $V_k^n \xrightarrow{L^1} \bar{u}$ in B_n. Let $u_n = V_n^n$. Done.

Cor. $u \in L^1_{loc}(\Omega)$. s.t. $\int u f \, d\mu = 0$. $\forall f \in C_c^\infty(\Omega)$

Then $u \equiv 0$. a.e. on Ω.

Pf: prove: $\int u f \, d\mu(x) = 0$. $\forall f \in L^\infty(\Omega)$, supp$f$ is cpt.

Then Let $f = \mathrm{sgn}(u)\chi_{K_n}$ $\therefore u \equiv 0$ $\forall x \in K_n$. $\forall n$.

Note that $\ell_n * f \in C_c^\infty(\Omega)$. $\to f$ in L^1.

$\exists$ $\ell_{n_k} * f \to f$. a.e. By Dominated Converge Thm.

Remark: It can be applied to $\forall u \in L^p(\Omega)$. $\forall 1 \le p \le \infty$.

since $u \in L^p(\Omega) \Rightarrow u \in L^p_{loc}(\Omega) \Rightarrow u \in L^1_{loc}(\Omega)$

(5) Strong Compactness in L^p:

Denote: $\tau_h f(x) = f(x+h)$

We will introduce Ascoli Thm in L^p space:

Thm: F is bounded in $L^p(\mathbb{R}^n)$. $1 \le p < \infty$.

and equicont in L^p. i.e. $\|\tau_h f - f\|_{L^p} \to 0$. uniform

with $f \in F$. Then $\overline{F|_\Omega}$ is cpt in $L^p(\Omega)$.

for $\forall \Omega \in M(\mathbb{R}^n)$. $m(\Omega) < \infty$.

Pf: 1°) Approxi $f \in F$ by $\varphi_n * f$

2°) Denote $\mathcal{H} = \{\varphi_n * f \mid f \in F\}$.

Note that:

$\|\varphi_n * f\|_\infty \le \|\varphi_n\|_{L^{p'}} \|f\|_{L^p}$

$|\varphi_n * f(x_1) - \varphi_n * f(x_2)| = |\nabla(\varphi_n * f)(\xi)| |x_1 - x_2|$

$\le \|\nabla \varphi_n\|_{L^{p'}} \|f\|_{L^p} |x_1 - x_2|$.

3°) $\forall \Omega \subseteq \mathbb{R}^n$. $m(\Omega) < \infty$. Then $\forall \varepsilon > 0$. $\exists W$. cpt.

s.t. $W \subseteq \Omega$. $\|f\|_{L^p(\Omega/W)} \le \varepsilon$. $\forall f \in F$.

by approxi of $\varphi_n * f$.

4°) $L^p(\Omega)$ is complete metrizable space

$\therefore$ prove: $\overline{F|_\Omega}$ is totally bounded.

By 2°) and Ascoli. $\overline{\mathcal{H}|_W}$ is cpt

$\therefore \overline{\mathcal{H}|_W} \subseteq \bigcup_{i}^{N(\varepsilon)} B(g_i, \varepsilon)$. totally bounded

use them to cover $\overline{F|_\Omega}$

Remark: We can't conclude F has cpt closure if it satisfies conditions above

Cor. F is bounded in $L^p(\mathbb{R}^n)$. $1 \le p < \infty$. equiconti in L^p. Moreover. $\forall \varepsilon > 0$. $\exists \Omega \in \mathcal{M}(\mathbb{R}^n)$. bounded.
st. $\|f\|_{L^p(\mathbb{R}^n/\Omega)} \le \varepsilon$. $\forall f \in F$.
Then F has cpt closure in $L^p(\mathbb{R}^n)$.

Pf: $\overline{F|_\Omega} \subseteq \bigcup_i^{N(\varepsilon)} B(g_i, \varepsilon) \Rightarrow \overline{F} \subseteq \bigcup_i^{N(\varepsilon)} B(\bar{g}_i, 2\varepsilon)$

Remark: The converse is true:
If $F \subseteq L^p(\mathbb{R}^n)$. $1 \le p < \infty$. cpt. Then $\overline{F} \subseteq \bigcup_i^{\tilde{N}} B(g_i, \varepsilon)$
Convert F to finite elements set!

Cor. $g \in L^p(\mathbb{R}^n)$. $B \subseteq L^q(\mathbb{R}^n)$. bounded set. $\frac{1}{p} + \frac{1}{q} > 1$.
$1 \le p, q < \infty$. Then $g * B|_\Omega$ has cpt closure.
in $L^r(\Omega)$. $\forall \Omega \in \mathcal{M}(\mathbb{R}^n)$. $m(\Omega) < \infty$.

Pf: 1) $g * B|_\Omega$ is bounded
2) $\|\tau_h(g * f) - g * f\|_{L^r} = \|(\tau_h g - g) * f\|_r$
$\le \|\tau_h g - g\|_p \|f\|_q$.
$\|\tau_h g - g\|_p \to 0$ $(h \to 0)$ since $\overline{C_c^\infty(\mathbb{R}^n)} \supseteq L^p(\mathbb{R}^n)$

By the thm above. Done.