# Weak Topology.

## (1) Coarsest Topology:

· Firstly, we will find the coarsest topo $\tau$ on $X$, associated with $(Y_i, \varphi_i)_{i\in I}$, where $X \xrightarrow{\varphi_i} Y_i$, s.t. $\varphi_i$ is conti. $\forall i \in I$.

It's easy to see: $(\tau, X)$ is generated by:

$\{\varphi_i^{-1}(w) \mid w \underset{open}{\subseteq} Y_i,\ \text{for } i\in I\}$. Denote $(U_\lambda)_{\lambda\in\Lambda}$.

Secondly, consider $\cap_{finite}$, $\cup_{arbitrary}$ operates on $(U_\lambda)_{\lambda\in\Lambda}$.

$\tau_1 = \{\bigcap_{\lambda\in I} U_\lambda \mid I \subseteq \Lambda,\ |I| \text{ is finite}\}$.

$\tau = \{\bigcup_{\alpha\in A} \tilde{U}_\alpha \mid \tilde{U}_\alpha \in \tau_1\}$. Claim: $(\tau, X)$ is a topo.

<u>Lemma</u> $\tau$ is closed under $\cap_{finite}$ operation.

<u>Pf</u>: Note that $(\bigcup_{\alpha\in A}\tilde{U}_\alpha) \cap (\bigcup_{\beta\in B}\tilde{U}_\beta) =$

$\bigcup_{\alpha\in A}(\tilde{U}_\alpha \cap (\bigcup_{\beta\in B}\tilde{U}_\beta)) = \bigcup_{\alpha\in A}\bigcup_{\beta\in B}(\tilde{U}_\alpha\cap\tilde{U}_\beta)$.

$\tilde{U}_\alpha\cap\tilde{U}_\beta$ is finite intersection of $(U_\lambda)_{\lambda\in\Lambda}$.

$\therefore (\bigcup_{\alpha\in A}\tilde{U}_\alpha)\cap(\bigcup_{\beta\in B}\tilde{U}_\beta) \in \tau$. □

<u>Remark</u>: Reverse the order of operation $\cap_{finite}$, $\cup_{arbitrary}$, $\tau$ may not be closed.

<u>prop.</u> $(x_n)$ in $(Z, X)$. $x_n \to x \Leftrightarrow \varphi_i(x_n) \to \varphi_i(x)$. $\forall i \in I$.

<u>Pf.</u> $(\Rightarrow)$ By conti of $\varphi_i$

$(\Leftarrow)$ $\forall U_x$ of $x$. has the form: $U_x = \bigcap_i^n \varphi_i^{-1}(w_i)$

$\exists N_i$ st. $n > N_i$. $\varphi_i(x_n) \in w_i$. Let $N = \max_{1 \le i \le n} N_i$

<u>prop.</u> $Z$ topo space. $Z \xrightarrow{\varphi} X$. Then $\varphi$ is conti

$\Leftrightarrow$ $Z \xrightarrow{\varphi_i \circ \varphi} Y_i$ conti. $\forall i \in I$.

<u>Pf:</u> $(\Leftarrow)$ $\forall W \subseteq_{open} X$. $U = \bigcup_{arbitrary} \bigcap_{finite} \varphi_i^{-1}(w_i)$

$\therefore \varphi^{-1}(W) = \bigcup_A \bigcap_F (\varphi_i \circ \varphi)^{-1}(w_i)$ open

(2) <u>Weak Topo $\sigma(E, E^*)$</u>:

- For Banach space $E$. $f \in E^*$. Denote $\varphi_f : E \to \mathbb{R}$.
  $x \mapsto \langle f, x \rangle$

Define $\sigma(E, E^*)$ is the coarsest topo on $E$.

associated with $(\varphi_f, \mathbb{R})_{f \in E^*}$.

<u>prop.</u> $\sigma(E, E^*)$ is Hausdorff

<u>Pf.</u> Apply Hahn-Banach Thm on $\{x_1\}, \{x_2\}$.

<u>prop.</u> For $x_0 \in E$. $V_k^\varepsilon(x_0) = \{x \in E \mid |\langle f_i, x - x_0 \rangle| < \varepsilon. \forall 1 \le i \le k\}$

is a basis of neighbour of $x_0$ in $\sigma(E, E^*)$

<u>Pf.</u> Check for $\forall W = \bigcap_{finite} \varphi_i^{-1}(w_i)$

<u>Remark:</u> It forms a convex basis.

① Convergence of seq:

Thm. $(x_n)$ seq $\subseteq E$

i) $x_n \rightharpoonup x \iff \langle f, x_n\rangle \longrightarrow \langle f, x\rangle.\ \forall f \in E^*$. (Replace with: $\|x_n\| \le C < \infty.\ \forall n.$ $\forall f \in D \underset{\text{dense}}{\subseteq} E^*$ ✓)

ii) $x_n \to x \Rightarrow x_n \rightharpoonup x$

iii) $x_n \rightharpoonup x \Rightarrow \|x_n\| \le C < \infty.\ \forall n.\ \underline{\lim} \|x_n\| \geq \|x\|$.

iv) $x_n \rightharpoonup x.\ f_n \to f. \Rightarrow \langle f_n, x_n\rangle \longrightarrow \langle f, x\rangle$.

Pf. i) By definition. Note $|\langle f, x_n\rangle| \le |\langle \ell, x_n\rangle| + \|\ell - f\| \|x_n\|$

ii) $|\langle f. x_n - x\rangle| \le \|f\| \|x_n - x\|$.

iii) For $\forall$ fixed $f$. $\langle f, x_n\rangle \to \langle f, x\rangle$.

$\therefore \langle f. x_n\rangle$ is bounded. $\forall n$. By UBP ✓.

Besides. $|\langle f, x_n\rangle| \le \|f\| \|x_n\|$. Take $\underline{\lim}$

$\therefore |\langle \frac{f}{\|f\|}. x\rangle| \le \underline{\lim} \|x_n\|$.

iv) $|\langle f_n, x_n\rangle - \langle f, x\rangle| \le |\langle f_n - f. x_n\rangle| + |\langle f. x - x_n\rangle|$.

② Finite dimension:

Thm. $\dim E < \infty$. Then $\sigma(E. E^*)$ t.v.s $\iff E$. n.v.s.

moreover $x_n \rightharpoonup x \iff x_n \to x$.

Pf: Check strongly open set is weakly open.

Find $V = \bigcap_1^k \varphi_i^{-1}(a_i - \varepsilon. a_i + \varepsilon) \subseteq B(x_0, r) \subseteq U$. (strongly open)

Let $a_i = \langle \varphi_i, x_0\rangle$. suppose $(e_i)_1^k$. $(\varphi_i)_1^k$ basis of $E. E^*$.

Then $\forall x \in E.\ x = \sum_1^k \langle \varphi_i, x\rangle e_i$. By equivalence of norm.

$\therefore \|x - x_0\| \le C \sum_1^k |\langle \varphi_i. x - x_0\rangle| \le kC\varepsilon$. choose $\varepsilon = \frac{r}{kC}$

**Prop.** In infinite-dimensional vector space $E$

$\sigma(E, E^*) \subsetneqq E$. n.v.s.

**Pf:** $S = \{\|x\|=1\}$ isn't closed in $\sigma(E, E^*)$.

Actually $\overline{S}^{\sigma(E,E^*)} = \overline{B_E(0,1)}$. (closure in $E$)

1°) $\overline{B_E(0,1)}$ is closed in $\sigma(E, E^*)$

since $\overline{B_E(0,1)} = \bigcap_{f \in E^*} \{x \in E \mid |\langle f, x\rangle| \le 1\}$.

2°) $\{\|x\|<1\} = B_E(0,1) \subseteq \overline{S}^{\sigma(E,E^*)}$

i.e. $\forall x_0$. $\|x_0\|<1$. $\forall V_k^\varepsilon(x_0)$ of $x_0$. $V_k^\varepsilon(x_0) \cap S \neq \emptyset$.

$\exists \eta_0 \in E$. $\eta_0 \neq 0$. st. $\langle f_i, \eta_0\rangle = 0$. $\forall 1 \le i \le k$.

Otherwise. $E \xrightarrow{\varphi} \mathbb{R}^k$. $\varphi = (\langle f_i, x\rangle)_{1\le i\le k}$ surjection

since $\ker\varphi = \{0\}$. $\therefore E \cong \mathbb{R}^k$. contradict with $\dim E = \infty$.

By conti. find $t_0 \in \mathbb{R}$. st. $\|x_0 + t_0\eta_0\| = 1$.

Since $x_0 + t\eta_0 \in V_k^\varepsilon(x_0) \cap S$. We're done.

**Cor.** $E$ is Banach space. $\overline{\mathrm{span}\{f_i\}_1^k} \subsetneqq E$.

Then $\exists x_0 \in E$. st. $\langle f_i, x_0\rangle = 0$. $\forall 1\le i \le k$. $x_0 \neq 0$.

**Remark:** i) The infinite dimensional space equipped with weak topo can never be metrizable.

ii) In infinite dimension, there also exists

$x_n \rightharpoonup x \not\Rightarrow x_n \to x$.

Note that two metric spaces $(X, d_1)$ $(Y, d_2)$ with same convergent sequences has same topologies.

Pf: let $(X_n) =: x_n = x$. $\forall n$. $x \in X$. $\therefore X = Y$.

Denote: $B_i(x, \frac{1}{n}) = \{\eta \mid d_i(x, \eta) < \frac{1}{n}\}$. $i = 1, 2$.

If $\exists x \in (X, d_1)$. St. $\exists U$ neighbor of $X$.

There's no open set $V$ in $(X, d_2)$. $x \in V \subseteq U$.

Then $B_2(x, \frac{1}{n}) \nsubseteq U$. $\forall n$. Choose $x_n \in B_2(x, \frac{1}{n}) / U$

$(X_n) \to X$ in $(X, d_2)$. But not in $(X, d_1)$

Cor. If the metric spaces are norm space

(with $\|\cdot\|_1, \|\cdot\|_2$) Then $\|\cdot\|_1 \sim \|\cdot\|_2$.

Pf: $(X, \|\cdot\|_1) \xrightarrow{T} (X, \|\cdot\|_2)$ $T, T^{-1}$ are conti

$x \mapsto Tx = X$

③ Convex sets and

linear operators:

Thm. $C \subseteq E$. convex set then $C$ is weakly closed

$\Leftrightarrow C$ is strongly closed.

Pf: $(\Leftarrow)$ Prove: $C^c$ is weakly open.

$\forall x_0 \in C^c$. Apply Hahn-Banach on $\{x_0\}$. $C$.

Obtain a neighbor $V = [f < \alpha]$ of $x_0$.

Cor. If $C$ is closed. Then $C = \bigcap_{i \in I} H_i$

the intersection of all closed half planes $\supseteq C$

Pf: $C \subseteq H_i$ $\therefore C \subseteq \cap H_i$.

If $\exists x_0 \in \cap H_i$. $x_0 \notin C$. Apply Hahn-Banach again.

Cor (Mazur)

$(x_n) \rightharpoonup x \Rightarrow \exists (\eta_n) \subseteq \operatorname{conv}(\bigcup_i \{x_i\})$ (finite sum)

s.t. $\eta_n \to x$.

Pf: $x \in \overline{\operatorname{conv}(\bigcup_i \{x_i\})}^{\sigma(E,E^*)} = \overline{\operatorname{conv}(\bigcup_i \{x_i\})}$

Remark: Variant Form:

$\exists z_n \in \operatorname{conv}(\bigcup_n \{x_i\})$. $p_n \in \operatorname{conv}(\bigcup_i^n \{x_i\})$

$z_n \to x$. $p_n \to x$.

Pf: $x \in \overline{\operatorname{conv}(\bigcup_n \{x_i\})}^{\sigma(E,E^*)} = \overline{\operatorname{conv}(\bigcup_n \{x_i\})}$, $\forall n$

$\exists (\eta_k^n) \to x$. $\forall n$. Let $(z_n) = (\eta_n^n) \to x$

For the latter, since $z_n \in \operatorname{conv}(\bigcup_i^n \{x_i\}) \Rightarrow z_n \in \operatorname{conv}\{\bigcup_i^{n_k} \{x_i\}\}$

Cor. $\varphi : E \to (-\infty, +\infty]$ convex, l.s.c in strong topo.

Then $\varphi$ is l.s.c in $\sigma(E, E^*)$

Pf: $[\varphi \le \lambda]$ is convex, closed in strong topo.

Remark: $\varphi$ convex, conti in strong topo $\Rightarrow \varphi$ l.s.c in $\sigma(E,E^*)$.

Note that u.s.c won't share it. Since $[\varphi \ge \lambda]$ may not be convex.

Thm. $E, F$ Banach space. $T: E \to F$, linear. Then.

$T$ is conti in strong topo $\Leftrightarrow T: \sigma(E,E^*) \to \sigma(F,F^*)$ conti.

Pf: $(\Rightarrow)$ $\forall f \in F^*$. $\langle f, Tx \rangle = \varphi_f$ is BLF. $\therefore \varphi_f \in E^*$

($\Leftarrow$). $G(T)$ is weakly closed in $\sigma(E,E^*)\times\sigma(F,F^*)$

so strongly closed. by closed graph Thm. ✓.

Remark: Denote $S$ = strong topo. $W$ = weak topo.

The continuity equals:

$S\to S$. $W\to W$. $S\to W$ ($G(T)$ closed in $W$)

But few LF conti on $W\to S$.

## (3) Weak Topo $\sigma(E^*, E)$:

· we're going to the third topo: $\sigma(E^*, E)$

Def: For every $x\in E$. $\varphi_x : E^*\to\mathbb{R}$. $\varphi_x(f) = \langle f, x\rangle$.

$\sigma(E^*, E)$ is the coarsest topo on $E^*$ associated with $(\mathbb{R}, \varphi_x)_{x\in E}$.

Remark: i) Note that $E\subseteq E^{**}$. Then $\sigma(E^*, E)$ is coarser than $\sigma(E^*, E^{**})$.

ii) The motivation on weak topo is: coarser topo $\Rightarrow$ more cpt sets. which plays important role in existence mechanism.

① Prop: i) $\sigma(E^*, E)$ is Hausdorff

ii) $f_0\in E^*$. $V_\varepsilon^k(f_0) = \{f \mid |\langle f-f_0, x_k\rangle| < \varepsilon, 1\le i\le k\}$ forms its basis neighbourhood. (it's convex)

Pf: i) $f_1\neq f_2\in E^*\Rightarrow \exists x_0$. $\langle f_1, x_0\rangle\neq\langle f_2, x_0\rangle$

$\therefore$ WLOG. $\langle f_1, x_0\rangle<\alpha<\langle f_2, x_0\rangle$.

① Converge of seq:

prop. $(f_n) \subseteq E^*$. Then

i) $f_n \xrightarrow{*} f \iff \langle f_n, x\rangle \to \langle f, x\rangle, \forall x \in E$.

(Replace with $\|f_n\| \le C$. $\forall x \in D \underset{\text{dense}}{\subseteq} E$, it holds)

ii) $f_n \to f$ or $f_n \longrightarrow f \Rightarrow f_n \xrightarrow{*} f$.

iii) $f_n \xrightarrow{*} f \Rightarrow \|f_n\| \le C < \infty . \forall n. \ \|f\| \le \underline{\lim} \|f_n\|$.

iv) $f_n \xrightarrow{*} f$. $x_n \to x \Rightarrow \langle f_n, x_n\rangle \to \langle f, x\rangle$.

Remark: When $\dim E < \infty$. Then $E = \sigma(E, E^*)$

$= \sigma(E^*, E)\ (= \sigma(E^*, E^{**}))$

③ Conti LF on $\sigma(E^*, E)$:

prop. $\varphi: E^* \to \mathbb{R}$. linear. conti on $\sigma(E^*, E)$. Then

there exists some $x_0 \in E$. st. $\varphi(f) = \langle f, x_0\rangle . \forall f \in E^*$.

Lemma. $X$ t.v.s, $\varphi, \{\varphi_i\}_1^k$ LF's on $X$. st. for $v \in X$.

$\varphi_i(v) = 0$. $\forall 1 \le i \le k \Rightarrow \varphi(v) = 0$. Then.

$\exists \lambda_i$. st. $\varphi = \sum_1^k \lambda_i \varphi_i$.

Pf: Def: $F(v) = (\varphi(v), \varphi_1(v) \dots \varphi_k(v)) : X \to \mathbb{R}^{k+1}$

Apply Hahn-Banach in $\{(1, 0 \cdots 0)\}$ and $R(F)$.

$\Rightarrow$ Return to the pf:

By conti. $f \in V_\varepsilon^n(0) \Rightarrow |\varphi(f)| < \varepsilon$.

In particularly, $\langle f, x_k\rangle = 0$. $\forall 1 \le k \le n \Rightarrow \varphi(f) = 0$. Apply Lemma.

Remark: It characterizes the conti linear functions

in $\sigma(E^*, E) \longrightarrow \mathbb{R}$.

Cor. $H$ is hyperplane in $E^*$, closed in $\sigma(E^*, E)$.

Then $\exists\, x_0 \neq 0 \in E$, $\alpha \in \mathbb{R}$. $H = \{f \in E^* \mid \langle f, x_0 \rangle = \alpha\}$.

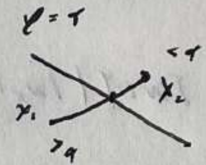
Pf: $H = \{f \in E^* \mid \varphi(f) = \alpha\}$. $\varphi$ is merely linear.

Consider $f_0 \in H^c$. Find convex neighbour of $f_0$: $V \subseteq H^c$

Convex set $V$ will be separated by $[\varphi = \alpha]$. (*)

WLOG. Suppose $\varphi(f) < \alpha$. $\forall f \in V$.

Prove: $\varphi$ is conti at $0$. $\Rightarrow$ Apply lemma.

Remark: (*) Convex set = 

$\Rightarrow$

It will intersect $\varphi = \alpha$ !

Cor. $\{\varphi = \alpha\}$ closed in $\sigma(E^*, E)$ $\Rightarrow$ $\varphi$ is conti. on $\sigma(E^*, E)$.

Prop. For $T \in \mathcal{L}(E, F)$. $T^* \in \mathcal{L}(F^*, E^*)$. Then

$T^*$ is conti between $\sigma(F^*, F)$ and $\sigma(E^*, E)$

Pf: $\langle T^* f, x \rangle = \langle f, Tx \rangle = \langle J(Tx), f \rangle$

$\therefore\ \varphi_x \circ T^* = \varphi_{T(x)}$. conti

Cor. Continuity equals: $W^* \rightarrow W^*$.

$S \rightarrow S$. $S \rightarrow W^*$ ($W^*$ is weak*)

④ Cpt Ball in $\sigma(E^*, E)$:

Thm.

$B_{E^*} = \{f \in E^* \mid \|f\| \le 1\}$ is cpt in $\sigma(E^*, E)$

Pf: $Y = \mathbb{R}^E$. i.e. map : $E \to \mathbb{R}$. equipped with product topo.

Then $(E^*, \sigma(E^*, E)) \subseteq Y$.

1°) For $\phi : E^* \to Y$. $\phi(f) = (w_x)_{x \in E}$. $w_x = \langle f, x \rangle$.

prove $\phi, \phi^{-1}$ is conti. $\therefore \phi$ is homeomorphism. $(E^* \simeq \phi(E^*))$

(check by $\{\pi_x\}_{x \in E}$. $(\phi(f))_x = \langle f, x \rangle$ conti. inverse is same)

2°) Characterize $\phi(B_{E^*}) \triangleq K$. prove: $K$ is cpt.

$K = \{w \in Y \mid |w_x| \le \|x\|\} \cap \{w \in Y \mid w_x + w_y = w_{x+y}, \lambda w_x = w_{\lambda x}\}$.

$\triangleq K_1 \cap K_2$. prove $K_1$ is cpt. $K_2$ is closed.

$K_1$ is cpt by Tychonoff. since $K_1 = \prod_{x \in E} [-\|x\|, \|x\|]$

$K_2 = [\bigcap_{x,y \in E} \{w_{x+y} = w_x + w_y\}] \cap [\bigcap_{x,y \in E} \{\lambda w_x = w_{\lambda x}\}]$ closed.

## (4) Reflexive Space:

Def: $E$ is Banach space. $J : E \to E^{**}$. canonical injection.

$E$ is said to be reflexive. if $J(E) = E^{**}$.

i.e. $J$ is also surjective.

Remark: i) $E$ is finite dimension $\Rightarrow$ $E$ is reflexive.

ii) It's essential to use "$J$". since there exists $E \xrightarrow{\varphi} E^{**}$. surjective isometry. But $E$ is not reflexive.

Lemma. $X \xrightarrow{\varphi} Y$. $\varphi$ is surjective isometry. If $Y$ is reflexive. Then $X$ is reflexive.

Pf:
$$\begin{tikzcd} X^{**} \arrow[r, "\varphi^{**}"] & Y^{**} \\ X \arrow[u, "J_X"] \arrow[r, "\varphi"] & Y \arrow[u, "J_Y"'] \end{tikzcd}$$
prove: $\varphi^{**}$ is also surjective isometry.

$\Leftrightarrow$ Prove: $\varphi$ is surjective isometry $\Rightarrow$ so does $\varphi^*$

1°) $\varphi^*$ is isometry:

$$\|\varphi^*(l_1 - l_2)\| = \sup_{\substack{x \in X \\ \|x\|=1}} |\langle \varphi^*(l_1 - l_2), x \rangle|$$

$$= \sup_{\substack{\varphi(x) \in Y \\ \|\varphi(x)\|=1}} |\langle l_1 - l_2, \varphi(x) \rangle| = \|l_1 - l_2\|.$$

2°) $\varphi^*$ is surjective:

$\forall f \in X^*$. let $l = f \circ \varphi^{-1}$. $\therefore \varphi^*(l) = f$. $l \in Y^*$.

① Criteria:

Thm. (Kakutani)

$E$ is Banach space. Then $E$ is reflexive

$\Leftrightarrow$ $B_E = \{x \in E \mid \|x\| \leq 1\}$ is cpt in $\sigma(E, E^*)$

Pf: ($\Rightarrow$) $J(B_E) = B_{E^{**}}$ by reflexive. cpt in $\sigma(E^{**}, E^*)$

$\therefore J(B_E)$ cpt in $\sigma(E, E^*)$. check $J^{-1}$ is conti.

($\Leftarrow$) Introduce two Lemmas following:

Lemma (Helly)

$\{f_i\}_1^k \subseteq E^*$. $\{\beta_i\}_1^k \subseteq \mathbb{K}$. The following properties are equivalent:

i) $\forall \varepsilon > 0$. $\exists x_\varepsilon \in E$. s.t. $\|x_\varepsilon\| \le 1$. $|\langle f_i, x_\varepsilon \rangle - y_i| < \varepsilon$. $\forall i = 1 \cdots k$.

ii) $|\sum_1^k \beta_i y_i| \le \|\sum_1^k \beta_i f_i\|$. $\forall \{\beta_i\}_1^k \subseteq \mathbb{R}^1$.

<u>Pf</u>: i) $\Rightarrow$ ii) is trivial. Consider ii) $\Rightarrow$ i).

$\varphi: E \to \mathbb{R}^k$. $\varphi(x) = (\langle f_i, x \rangle)_{1 \le i \le k}$.

i) $\Leftrightarrow$ $y = (y_i) \in \overline{\varphi(B_E)}$. By contradiction:

Apply Hahn-Banach Thm. (Note: $\mathbb{R}^k = (\mathbb{R}^k)^*$)

<u>Lemma.</u> (Goldstine)

i) $J(B_E)$ is dense in $B_{E^{**}}$. w.r.t. $\sigma(E^{**}, E^*)$

ii) $J(E)$ is dense in $E^{**}$ w.r.t $\sigma(E^{**}, E^*)$

<u>Pf</u>: For $\xi \in B_{E^{**}}$. with neighbour $V_k^\varepsilon(\xi)$.

prove $\exists x \in B_E$. $J(x) \in V_k^\varepsilon(\xi)$.

It's from the Lemma. ii) is from i)

<u>Remark</u>: $J(B_E)$ is closed in $B_{E^{**}}$ equipped with strong topo. (By $J$ is conti. isometry)

$\therefore$ $J(B_E)$ won't dense unless $E$ is reflective.

$\Rightarrow$ Return to the pf:

$J$ is conti. $B_E$ is cpt in $\sigma(E, E^*)$ $\Rightarrow$ $J(B_E)$ is cpt in $\sigma(E^{**}, E^*)$.

By Hausdorff. $J(B_E)$ is close in $\sigma(E^{**}, E^*)$. $\therefore J(B_E) = B_{E^{**}}$.

$\therefore$ $J(E) = E^{**}$ (since $\forall R > 0$. $J(B_E(R)) = B_{E^{**}}(R)$)

<u>Thm.</u> i) $\forall W \subset X^*$. l.s. $\dim W < \infty$ $\Rightarrow$ $W$ is closed in $\sigma(X^*, X)$.

ii) $X$ is reflexive $\Leftrightarrow$ $W \underset{l.s}{\subseteq} X^*$. strongly close is $\sigma(X^*, X)$-closed

② Sequential cpt:

Thm. $E$ is Banach space. Then $E$ is reflective

$\Leftrightarrow$ Every bounded seq $(x_n)$ admits a weakly

convergent subseq in $\sigma(E, E^*)$

Pf: ($\Rightarrow$) $M = CLS(\{x_n\})$, which is reflexive and separable

$\therefore M^*$ is separable as well.

$\therefore B_M$ is cpt and metrizable in $\sigma(M, M^*)$

$\therefore B_M$ is cpt sequentially in $\sigma(E, E^*)$

since $\sigma(M, M^*) = \sigma(E, E^*)|_M$.

($\Leftarrow$) It's complicated.

③ Properties:

i) prop. $E$ is reflexive Banach space. $M \subseteq E$ closed

linear subspace. Then $M$ is reflexive.

Pf: By Hahn-Banach Thm. BLF on $M$ can correspond to BLF on $E$ (By extend and restrict)

Note that $B_M = B_E \cap M$. cpt in $\sigma(E, E^*)$

$\sigma(M, M^*)$ is topo subspace of $\sigma(E, E^*)$.

$\therefore \sigma(M, M^*) = \sigma(E, E^*)|_M \therefore B_M$ cpt in $\sigma(M, M^*)$

Remark: cpt in subspace topo $\Leftrightarrow$ cpt in initial topo.

Pf: ($\Rightarrow$) $\forall \{U_i\}_{i\in I}$ covers $K$ in $M$.

Then $\{U_i \cap M\}_{i\in I} \underset{\text{open}}{\subseteq} M$. covers $K$.

$\exists \{U_i \cap M\}_1^n \subseteq \{U_i\}_1^n$ covers $K$.

($\Leftarrow$). $\exists \{U_k\}_1^n$ covers $K$. so $\{U_k \cap M\}_1^n$ covers $K$.

**Cor.** $E$ is Banach space. $E$ is reflexive $\iff$ $E^*$ is reflexive.

**Pf:** $(\Rightarrow)$ Check $\forall \varphi \in E^{***}$. $\exists f \in E^*$. s.t.

$\langle \varphi, \xi \rangle = \langle Jf, \xi \rangle$. $\forall \xi \in E^{**}$. i.e.

$\langle \varphi, \xi \rangle = \langle \xi, f \rangle$. But $\exists x \in E$. $Jx = \xi$.

$\therefore \langle \varphi, Jx \rangle = \langle f, x \rangle$. $\varphi_f(x) = \langle \varphi, Jx \rangle \in E^*$.

$\therefore$ such $f \in E^*$ exists.

$(\Leftarrow)$ $E^*$ reflexive. Then so $E^{**}$.

since $E \cong J(E)$. $J(E)$ closed subspace of $E$ ✓.

**Cor.** $E$ is reflexive Banach. $K \subseteq E$. bounded closed convex set. Then $K$ is cpt in $\sigma(E, E^*)$.

**Pf:** $\exists m \in \mathbb{Z}^+$. $K \subseteq m B_E$. $K$ is also closed in $\sigma(E, E^*)$.

**Cor.** $E$ is reflexive Banach. $A \neq \emptyset \subseteq E$. closed convex subset. $\varphi: A \longrightarrow (-\infty, +\infty]$. convex, l.s.c. s.t. $\varphi \not\equiv +\infty$. $\lim\limits_{\substack{x \in A \\ \|x\| \to \infty}} \varphi(x) = +\infty$. Then $\varphi$ achieve minimum on $A$.

**Pf:** $\tilde{A} = \{x \in A \mid \varphi(x) \le \varphi(a)\}$ is bounded. closed. convex.

$\therefore \tilde{A}$ is cpt in $\sigma(E, E^*)$. So $\varphi$ attain min on $\tilde{A}$.

**Remark:** e.g. let $\varphi = \|x - a\|$.

ii) **Thm.** $E, F$ are reflexive Banach space. $A: D(A) \subseteq E \to F$ linear. densely defined, closed. Then $D(A^*)$ is dense in $F^*$. Besides, $A^{**} = A$.

Pf: 1°) $D(A^*)$ is dense:

$\Leftrightarrow$ prove: $\forall \varphi \in F^{**}$, s.t. $\langle \varphi, f\rangle = 0$, $\forall f \in D(A^*)$. Then $\varphi \equiv 0$.

By reflective, suppose $\varphi \in F$. $\langle f, \varphi\rangle = 0$. $\forall f \in D(A^*)$

By contradiction. $(0, \varphi)$ & $G(A)$. Separate by Hahn-Banach

$\exists (f, v) \in E^* \times F^*$. $\langle f, u\rangle + \langle v, Au\rangle < \alpha < \langle v, \varphi\rangle$. $\forall u \in D(A)$

$\therefore \langle f, u\rangle + \langle v, Au\rangle = 0$ i.e. $\langle A^* v, u\rangle = -\langle f, u\rangle$. $\therefore v \in D(A^*)$

But let $v = w$. $\langle w, \varphi\rangle > 0$. Contradict!

2°) $A = A^{**}$:

$I(G(A^*)) = G(A)^\perp$. $I(G(A^{**})) = G(A^*)^\perp$.

Check $I^2 = -id$. $I(G(A)) = G(A^*)^\perp$, $I(G(A)^\perp) = I(G(A))^\perp$

$\therefore$ since $G(A^{**})$ is symmetry. $\therefore G(A^{**}) = I^2(G(A^{**})) = G(A)$

(5) <u>Separable Space</u>:

<u>Def</u>: Metric space $E$ is separable if $\exists D$ countable dense subset of $E$.

<u>Remark</u>: Finite dimensional spaces is separable:

$$D = \{ \sum_1^n t_k u_k \mid \{t_k\}_1^n \subseteq \mathbb{Q} \}.$$

① <u>Properties</u>:

i) <u>Prop</u>. Any subset of separable metric space $E$ is separable

<u>Pf</u>: $\{u_n\} \subseteq E$. countable dense. If $F \subseteq E$

Choose a point $a_{m,n}$ from $B(u_n, r_m)$. $r_m \to 0$

Then $(a_{m,n}) \cap F \underset{\text{dense}}{\subseteq} F$.

Remark: $D = \{u_n\} \cap F$ may be null set. The ideal is from: $\forall f \in F$. $U_f$ neighbour. $\exists B(u_m, r)$ s.t. $U_f \cap B(u_m, r) \neq \emptyset$. Since $U_f \cap D \neq \emptyset$.

ii) Thm. $E$ is Banach space. $E^*$ separable $\Rightarrow$ $E$ separable.

Remark: Converse is false: $L^1$ separable $\not\Rightarrow$ $L^\infty$ separable.

Pf: $(f_n) \underset{dense}{\subseteq} E^*$. $\exists x_n \in E$. $\|x_n\| = 1$. $|\langle f, x_n \rangle| \geq \frac{\|f_n\|}{2}$.

Claim: $L_0 = CLS(\{x_n\}_{n \in \mathbb{Z}^+}) = E$

If $\exists \eta \in E$. $\eta \notin L_0$. By Hahn-Banach Thm.

extend $f(L_0) = \{0\}$. $f(\eta) = d(\eta, L_0) \neq 0$.

from $\{L_0 + \alpha\eta \mid \alpha \in \mathbb{R}\}$ to $E$.

WLOG. Let $\|f\| = 1$. $\exists f_n \in (f_n)$. $\|f_n - f\| \leq \varepsilon$.

$\therefore \|f_n\| \geq \|f\| - \|f - f_n\| \geq 1 - \varepsilon > 2\varepsilon$.

But $\|f_n\| \leq 2\langle f_n, x_n \rangle = 2\langle f_n - f, x_n \rangle \leq 2\|f_n - f\| \leq 2\varepsilon$.

Which is a contradiction. $\therefore L_0 = E$.

Let $D = \{\sum_1^N \alpha_k x_{n_k} \mid N \in \mathbb{Z}^+.\ \alpha_k \in \mathbb{Q}.\ (x_{n_k}) \subseteq (x_n)\}$ dense!

Cor. $E$ is Banach space. Then, we obtain:

$E$ reflexive and separable $\Leftrightarrow$ $E^*$ does so.

② Related to Metrizability:

For Banach space $E$. Then,

i) $E$ is separable $\Leftrightarrow$ $B_{E^*}$ is metrizable in $\sigma(E^*, E)$

ii) $E^*$ is separable $\Leftrightarrow$ $B_E$ is metrizable in $\sigma(E, E^*)$

Pf: i) ($\Rightarrow$). Suppose $\langle x_n \rangle = D$. Define a norm $[\cdot]$ on $E^*$.

$[f] = \sum \frac{1}{2^n} |\langle f, x_n \rangle|$ . $[f] \le \|f\|$. $d(f,g) = [f-g]$.

prove: $(B_{E^*}, d) = (B_{E^*}, \sigma(E^*, E))$

("$\le$") For $V_k^{\varepsilon}(f_0)$, only consider $\{y_i\}_1^k$. By dense of $(x_n)$

("$\ge$"). Consider the finite sum $\sum_1^k \frac{1}{2^n} |\langle f - f_0, x_n \rangle|$. of $[f - f_0]$.

($\Leftarrow$) $U_n = \{ f \in B_{E^*} \mid d(f,0) < \frac{1}{n} \}$. $\exists V_n \subseteq U_n$. with form:

$V_n = \{ f \in B_{E^*} \mid |\langle f, x \rangle| < \varepsilon_n,\ x \in \Phi_n \}$. $\Phi_n$ is finite set of $E$.

Claim: $D = \bigcup_1^\infty \Phi_n$ is dense. (check by BLF)

ii) ($\Rightarrow$) Analogously, let $[x] = \sum_1^\infty \frac{1}{2^n} |\langle f_n, x \rangle|$ . $(f_n) = D$.

($\Leftarrow$) Analogously, $U_n = \{ x \in B_E \mid d(x,0) < \frac{1}{n} \}$. $\exists V_n \subseteq U_n$. st.

$V_n = \{ x \in E \mid |\langle f, x \rangle| < \varepsilon_n,\ f \in \Phi_n \}$. $D = \bigcup_1^\infty \Phi_n$. $\Phi_n$ finite.

prove: $F = CLS(D) = E^*$. By contradiction:

1°) By Hahn-Banach. $\exists \xi \in E^{**}$ $f_0 \in E^*/F$. st.

$\langle \xi, f_0 \rangle > 1$. $\xi(F) = \{0\}$. $\|\xi\| = 1$. ($\|f_0\| > 1$. afterward)

2°) Let $W = \{ x \in B_E \mid |\langle f_0, x \rangle| < \frac{1}{2} \}$

since $V_n \subseteq U_n$. $(U_n)$ neighbour basis.

$\therefore \exists n_0$. st. $V_{n_0} \subseteq W$.

3°) We can find $x_1 \in B_E$. st. $\begin{cases} |\langle f, x_1 \rangle - \langle \xi, f \rangle| < \varepsilon_{n_0}.\ \forall f \in \Phi_{n_0} \\ |\langle f_0, x_1 \rangle - \langle \xi, f_0 \rangle| < \frac{1}{2} \end{cases}$

since $J(B_E)$ is dense in $B_{E^{**}}$. $\xi \in B_{E^{**}}$.

$\therefore x_1 \in V_{n_0}$. But $|\langle f_0, x_1 \rangle| > \frac{1}{2}$ contradict!

Cor. $E$ is separable Banach space. If $(f_n)$ is bounded seq. Then $\exists (f_{n_k}) \subseteq (f_n)$ weakly convergent in $\sigma(E^*, E)$

③ Characterization:

i) Thm. Every separable Banach space $E$.

exists an isometry $\varphi$. st. $E \xrightarrow{\varphi} l^{\infty}$.

Pf: $B_{E^*}$ is cpt and metrizable in $\sigma(E^*, E)$.

Then $\forall n.\ \exists (t_k^n)_{k=1}^{N_n}$. $B_{E^*} = \bigcup_1^{N_n} B_{E^*}(t_k^n, \frac{1}{n})$

$\therefore D = \bigcup_{n=1} (t_k^n)_1^{N_n}$ is dense in $B_{E^*}$.

Denote $D = \cup \{t_k\}$. $\varphi(x) = (\langle t_1, x\rangle . \langle t_2, x\rangle \cdots \langle t_n, x\rangle \cdots)$

Check $\|\varphi(x)\|_{l^\infty} = \sup_D |\langle t_k, x\rangle| = \|x\|$.

ii) Thm $\dim E = \infty$. Banach space. If one of assumptions

holds in the following:

(a) $E^*$ is separable

(b) $E$ is reflexive

Then $\exists (x_n) \subseteq E$. st. $\|x_n\| = 1$. $x_n \to 0$ in $\sigma(E, E^*)$

Pf: (a) $B_E$ is metrizable in $\sigma(E, E^*)$. By Seq Lemma:

Let $S = \{\|x\| = 1\}$. $\overline{S}^{\sigma(E, E^*)} = B_E$. $0 \in B_E$.

(b) Suppose $\{u_i\}_{i\in I}$ is Basis of $E$.

Choose $\{u_k\}_{k\in\mathbb{Z}^+} \subseteq \{u_i\}_{i\in I}$. $M = CLS(\{u_k\}_{k\in\mathbb{Z}^+})$

$\therefore M$ is reflexive. separable. So $M^*$ does.

Then reduce to (a).

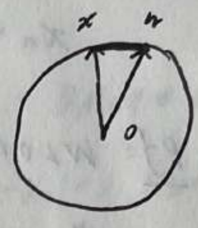
Remark: e.g. Hilbert space $H$. $\{e_n\}_{n\in\mathbb{Z}^+}$ is

its orthonormal basis. $e_n \rightharpoonup 0$.

## (6) Uniformly convex.

**Def:** Banach space $E$ is said to be uniformly convex if $\forall \delta > 0.\ \exists \varepsilon > 0.$ s.t.

$x, \eta \in E.\ \|x\|, \|\eta\| \leq 1.\ \|x - \eta\| > \delta \Rightarrow \left\|\frac{x+\eta}{2}\right\| < 1 - \varepsilon.$



**Remark:** i) It's related with norm

ii) It's a geometric property of unit ball.

**Thm.** Every uniformly convex Banach space $E$ is reflexive.

**Pf:** $\forall \xi \in E^{**}$. WLOG. Let $\|\xi\| = 1$.

prove: $\xi \in J(B_E)$. $\Longleftrightarrow \forall \varepsilon > 0,\ \exists x.$ s.t. $\|J(x) - \xi\| \leq \varepsilon.\ x \in B_E$.

since $J(B_E)$ is closed in $E^{**}$ strong topo.

$\exists f \in E^*,\ \|f\| = 1.\ \langle \xi, f\rangle > \|\xi\| - \frac{\delta}{2} = 1 - \frac{\delta}{2}.\ (\langle \xi, f\rangle \overset{\delta/2}{\approx} \|\xi\|)$

$V = \{\eta \in E^{**} \mid |\langle \eta - \xi, f\rangle| < \frac{\delta}{2}\}.\ V \cap J(B_E) \neq \emptyset.$

since $V$ is neighbor of $\xi$. $J(B_E)$ is dense in $\sigma(E^{**}, E^*)$

Claim: $\exists x \in B_E.\ J(x) \in V.$ which is what we need.

(The ideal is find $x \in B_E$. s.t. $J(x) \overset{\varepsilon}{\approx} \xi$ in norm $\|\cdot\|$.

first. Let $\langle \xi, f\rangle \overset{\delta/2}{\approx} \|\xi\|$, then consider $\langle \eta, f\rangle \overset{\delta/2}{\approx} \langle \xi, f\rangle$)

By contradiction: $\xi \in (Jx + \varepsilon B_{E^{**}})^c \triangleq W$. neighbor of $\xi$.

$\therefore V \cap W \neq \emptyset$ in $\sigma(E^{**}, E^*)$. By dense of $J(B_E)$: $V \cap W \cap J(B_E) \neq \emptyset$.

Find another $\eta \in B_E$. $J(\eta) \in V \cap W \cap J(B_E)$.

Apply uniform convex on $x, \eta$. Come into contradiction.

prop. $E$ is uniformly convex Banach space. Then

$(x_n) \rightharpoonup x$ in $\sigma(E,E^*)$. $\overline{\lim}\|x_n\| \leq \|x\|$

$\Leftrightarrow$ $(x_n) \to x$ in $E$ strong.

Cr. Under the assumption:

$x_n \rightharpoonup x$. $\|x_n\| \to \|x\|$ $\Leftrightarrow$ $x_n \to x$.

pf: WLOG. Let $x \neq 0$. Denote $\lambda_n = \max\{\|x_n\|, \|x\|\}$.

$y_n = \frac{x_n}{\lambda_n}$. $y = \frac{x}{\|x\|}$. $\lambda_n \to \|x\|$. $\therefore \frac{y_n + y}{2} \rightharpoonup y$.

$\therefore \underline{\lim}\|\frac{y_n+y}{2}\| \geq \|y\| \geq \frac{\|y\|+\|y_n\|}{2} \geq \frac{\|y+y_n\|}{2}$.

$\therefore \underline{\lim}\|\frac{y_n+y}{2}\| \geq \|y\| \geq \overline{\lim}\frac{\|y_k+y\|}{2}$. $\lim\|\frac{y_n+y}{2}\| = 1$

$\therefore \|y_n - y\| \to 0$. i.e. $x_n \to x$.

(7) Application of weak Topo:

- $X = C[-1,1]$. $f \in X$. $\|f\|_X = \sup_{x\in[-1,1]} |f(x)|$. By Riesz

Representation: $X^* = M[-1,1]$. for $\mu_n \in M[-1,1]$.

$d\mu_n = f_n\, dt$. Denote: Dirac measure $\delta_0 = \begin{cases} 0, & 0 \notin A \\ 1, & 0 \in A \end{cases}$.

If $\mu_n \xrightarrow{*} \delta_0$. Then $\forall g \in X$:

$\int_{-1}^{1} g\, d\mu_n = \int_{-1}^{1} g(t) f_n(t)\, dt \to g(0)$.

Next, we find the necessary and sufficient conditions of $(f_n)$, s.t. $\mu_n \xrightarrow{*} \delta_0$. ($d\mu_n = f_n\, dt$.)

Thm.

$\int_{-1}^{1} g(t) f_n(t)\,dt \longrightarrow g(0)$ for $\forall g(t) \in X$ $\iff$

i) $\int_{-1}^{1} f_n\,dt \longrightarrow 1$

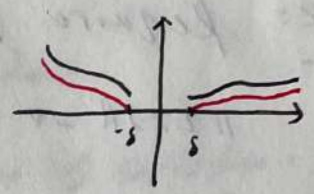
ii) $\forall g(t) \in C^\infty[-1,1]$, $0 \notin \operatorname{supp} g(t)$ then $\int_{-1}^{1} f_n g\,dt \longrightarrow 0$

iii) $\exists C_0 < \infty$, $\int_{-1}^{1} |f_n|\,dt \le C_0$.

<u>Pf:</u> $(\Rightarrow)$ i) $g = 1$. ii) $g(0) = 0$. iii) $\mu_n \xrightarrow{*} \delta_0$ $\therefore \|\mu_n\| \le C_0$.

$(\Leftarrow)$ For $g \in X$. $\forall \varepsilon > 0$. $\exists \delta < 0$. s.t. $|x_1 - x_2| < \delta \Rightarrow |g(x_1) - g(x_2)| < \varepsilon$.

Let $g_\varepsilon = \begin{cases} 0, & x \in [-\delta, \delta] \\ g(x), & x \in [-1,1]/[-\delta,\delta] \end{cases}$




If $g_\varepsilon(-\delta) = a_1$, $g_\varepsilon(\delta) = a_2$. $|a_1|, |a_2| < \varepsilon$.

Let $\tilde{g}_\varepsilon = \begin{cases} g_\varepsilon - a_1, & x \in [-1, -\delta] \\ g_\varepsilon - a_2, & x \in [\delta, 1] \\ 0, & x \in [-\delta, \delta] \end{cases}$ $\therefore \tilde{g}_\varepsilon \in C[-1,1] = X$, $|\tilde{g}_\varepsilon - g_\varepsilon| < \varepsilon$

Consider $\tilde{g}_\varepsilon * \phi_n \triangleq k_n(t)$. Suppose $\operatorname{supp} \phi_n = [-\delta_n, \delta_n]$.

Let n is big enough. s.t. $\delta > \delta_n$. $\therefore k_n(t) \equiv 0$. $|t| < \delta - \delta_n$

Besides, $\left| \int_{-1}^{1} f_n(t) \tilde{g}_\varepsilon(t)\,dt - \int f_n(t) k_N(t)\,dt \right|$

$$\le \sup |\tilde{g}_\varepsilon - k_N| \int_{-1}^{1} |f_n| \le C_0 \varepsilon$$

Apply ii) on $k_N(t)$.

Note that $k_N \overset{C_0\varepsilon}{\sim} \tilde{g}_\varepsilon \overset{\varepsilon}{\sim} g_\varepsilon \overset{\varepsilon}{\sim} g$ in $\sigma(X^*, X)$

By approximation!