Bounded Linear Functions

(1) Main Theorems:

① Uniform Boundedness Principle:

i) Baire Lemma:

The complete metric space X is Baire space.

i.e. (X_n) is seq of closed sets. s.t. $\operatorname{int} X_n = \emptyset$.

Then $\operatorname{int} \bigcup X_n = \emptyset$.

Cor. (X_n) seq of closed sets. $\bigcup\limits_{1}^{\infty} X_n = X$. Then

$\exists n_0$. s.t. $\operatorname{int} X_{n_0} \neq \emptyset$.

Pf: $\Leftrightarrow$ $O_n = X_n^c$ dense. Then $G = \bigcap O_n$ dense.

By contradiction: suppose $\exists W \cap G = \emptyset$. W open.

Choose seq $(B(x_n, r_n))$. $x_n \in B(x_{n-1}, r_{n-1}) \cap O_{n}$.

$r_n < \frac{r_{n-1}}{2}$. $\overline{B(x_0, r_0)} \subseteq W$. $\overline{B(x_n, r_n)} \subseteq B(x_{n-1}, r_{n-1}) \cap O_n$

$\{x_n\}$ is Cauchy $\rightarrow$ $l \in G \cap W$.

ii) Let E. F. n.v.s. $\mathcal{L}(E.F)$ is space of BLO's:

$T: E \rightarrow F$. with norm $\|T\|_{\mathcal{L}(E.F)} = \sup\limits_{\substack{x \in E \\ \|x\| \leq 1}} \|Tx\|$.

prop. F is Banach $\Leftrightarrow$ $\mathcal{L}(E.F)$ is Banach.

Pf: ($\Rightarrow$) $(M_n) \subseteq \mathcal{L}(E,F)$. Cauchy $\Rightarrow \forall x.\ (M_n x) \subseteq F$ Cauchy

Denote $M_n x \to \eta_x$. Check $M: x \mapsto \eta_x$ BLO.

$M_n \to M$ in $\mathcal{L}(E,F)$.

($\Leftarrow$) $(\eta_n) \subseteq F$. Cauchy. Fix $x_0 \in E$. $\|x_0\| = 1$. $x_0^*(x_0) = 1$

Set $A^\eta(x) = x_0^*(x)\eta$. $\eta \in F$. $A^\eta: E \to F$ BLO.

$\Rightarrow (A^{\eta_n})$ Cauchy in $\mathcal{L}(E,F)$. check $\eta_n \to \eta = A^\eta(x_0)$

Rmk: $Y \xrightarrow{\phi} \mathcal{L}(X,Y)$. $\phi(\eta) = A^\eta$. ϕ is isometric embed.

Thm. (UBP).

E Banach. F n.v.s. $(T_i)_{i\in I} \subseteq \mathcal{L}(E,F)$. If $(T_i)_{i\in I}$ satisfies

$\sup_{i\in I} \|T_i x\| < \infty$. for any fixed $x \in E$. Then $\exists C$. const.

$\|T_i x\| \le C\|x\|$. $\forall i \in I$. $x \in E$. (i.e. $\sup \|T\| \le C < \infty$)

Pf: $X_n = \{x \in E \mid \|T_i x\| \le n.\ \forall i \in I\}$. $\therefore \bigcup X_n = X$.

Apply Baire Lemma. $\exists n_0$. $\operatorname{int} X_{n_0} \neq \emptyset$. Choose a ball.

Remark: i) It's remarkable since it claims: pointwise estimate $\Rightarrow$ global estimate.

ii) In general. pointwise limit of conti operators need not be conti. Linearity is essential.

Cor. G is Banach space. $B \subset G$. subset. If $\forall f \in G^*$.

$f(B) = \{\langle f, x\rangle, x \in B\}$ is bounded. Then B is bounded

Pf: Let $T_b(f) = \langle f, b\rangle$. for $b \in B$.

Remark: To check B is bounded. in finite dimension case, we can check every $f \in G^*$.

② Open mapping Thm:

E, F are Banach Space. $T \in \mathcal{L}(E, F)$. surjective.

Then $\exists c > 0$. s.t. $B_F(0, c) \subseteq T(B_E(0,1))$

Pf: 1°) Prove: $\exists c > 0$. s.t. $\overline{T(B_E(0,1))} \supseteq B_F(0, 2c)$

$X_n = n\overline{T(B_E(0,1))}$. $\cup X_n = F$. by surjection.

Apply Baire Thm. $\exists B_F(y_0, 4c) \in \operatorname{int} X_{n_0}$

2°) Prove: $T(B_E(0,1)) \supseteq B_F(0, c)$

$\Leftarrow\,\forall y \in F$. $\|y\|_F < c$. $\exists x \in E$. $\|x\|_E < 1$. $Tx = y$.

From $\overline{T(B_E(0,\frac{1}{2}))} \supseteq B_F(0,c)$. we have:

$\forall \varepsilon > 0$. $\exists z \in E$. s.t. $\|Tz - y\|_F < \varepsilon$. $\|z\| < \frac{1}{2}$

Let $\varepsilon = \frac{c}{2}$. Then $\|Tz_1 - y\|_F < \frac{c}{2}$. $\|z_1\| < \frac{1}{2}$

$\therefore Tz_1 - y \in B_F(0, \frac{c}{2}) \subseteq \overline{T(B_E(0,\frac{1}{4}))}$. Apply again...

Then $x_n = \sum_1^n z_k \to x$ is what we need.

Cor. Under the assumption above. T is open mapping.

Pf: $\forall y_0 \in T(U)$. $\exists x_0 \in E$. $Tx_0 = y_0$. $\therefore \exists B(x_0, r) \subseteq U$.

$\therefore T(B(x_0, r)) = T(x_0 + B(0,r)) \subseteq T(U)$.

i.e. $T(x_0) + T(B(0,r)) \subseteq T(U)$. Apply Thm: $\exists c$.

$T(B(0,r)) \supseteq B(0, rc)$ $\therefore B(y_0, rc) \subseteq T(U)$.

Cor. With addition, T is bijective. Then $T^{-1} \in \mathcal{L}(F, E)$.

Pf: $\forall y \in B_F(0,c)$. $\exists x \in B_E(0,1)$. $Tx = y$

$\therefore \forall z \in F$. $z = \frac{z}{\|z\|} \cdot \frac{c}{2} \cdot \frac{2\|z\|}{c}$. $\exists x_0 \in B_E(0,1)$.

s.t. $Tx_0 = \frac{z}{\|z\|} \cdot \frac{c}{2}$ $\therefore z = \frac{2\|z\|}{c} \cdot Tx_0$.

$\therefore \|T^{-1}z\| = \|\frac{2\|z\|}{c}\cdot x_0\| \le \frac{2}{c}\|z\|$. $T^{-1}\in\mathcal{L}(F,E)$

Cor. E. v.s. with norm $\|\cdot\|_1$, $\|\cdot\|_2$. If $(E,\|\cdot\|_1)$ and $(E,\|\cdot\|_2)$ are both Banach. and exist $c>0$. st. $\|x\|_2 \le c\|x\|_1$, $\forall x\in E$. Then $\|\cdot\|_1 \sim \|\cdot\|_2$.

Pf: $(E,\|\cdot\|_1) \xrightarrow{I} (E,\|\cdot\|_2)$. $\therefore I^{-1}$ is conti.

③ Closed Graph Thm:

Thm. E. F are Banach space. $T: E\longrightarrow F$. linear. Denote: $G(T) = \{(x, Tx) \mid x\in E\}$. Then: $T\in\mathcal{L}(E,F) \Leftrightarrow G(T)$ is closed.

Pf: Only prove $(\Leftarrow)$:

Denote: $\|x\|_1 = \|x\|_E + \|Tx\|_F$. Check $(E,\|\cdot\|_1)$ is Banach

Since $G(T)$ is closed. By Cor. above. $\|\cdot\|_1 \sim \|\cdot\|_E$

Cor. $A: D(A)\subset E\longrightarrow F$. bijection. $G(A)$ is closed. Then: A^{-1} is bdd on F. (i.e. $A^{-1}\in\mathcal{L}(F,E)$)

Pf: $(D(A), \|\cdot\|_1)$ is Banach space.

Rmk: A isn't necessary bdd. e.g. $A: C^1[0,\pi]\cap\{f(0)=0\}\subset C[0,\pi]\cap\{f(0)=0\}\longrightarrow C[0,\pi]$. $A(f) = f'$

(2) Transpose of BLO's:

For X. Y Banach spaces. $M: X\longrightarrow Y$. BLO. Then we can define transpose of M: $M^*: Y^*\longrightarrow X^*$. st. $\langle M^*, l\rangle = l\circ M$. for $\forall l\in Y^*$

Claim: $M^*\in\mathcal{L}(Y^*, X^*)$.

i) M^* is linear.

ii) $\|M^*\| = \|M\|$

1°) $\|\langle M^*, l\rangle\| = \sup\limits_{\|x\|\le 1} |\langle M^*, l\rangle(x)| = \sup |l(Mx)|$

$= \sup |l(\frac{Mx}{\|M\|})| \, \|M\| \le \|l\| \|M\|. \Rightarrow \|M^*\| \le \|M\|$

2°) $\|M\| = \sup_x \|M(x)\| = \sup_x \sup_l \|\langle l, Mx\rangle\|$

$= \sup_{x,l} |\langle M^* l, x\rangle| \le \|M^* l\| \|x\| \le \|M^*\|.$

iii) For $M_1^*, M_2^* \in \mathcal{L}(Y^*, X^*)$. $(\alpha M_1^* + M_2^*) = (\alpha M_1 + M_2)^*$

iv) For E, F, G Banach spaces. $T \in \mathcal{L}(E, F)$. $S \in \mathcal{L}(F, G)$

Then: $(S \circ T)^* = T^* \circ S^*$ (Cotravariant)

$E \xrightarrow{T} F \xrightarrow{S} G. \Rightarrow G^* \xrightarrow{S^*} F^* \xrightarrow{T^*} E^*$

v) For $T \in \mathcal{L}(E, F)$. bijection $\Rightarrow T^*$ is bijection.

vi) $T \in \mathcal{L}(E, F)$. between Banach spaces. $R(T)$ closed $\Rightarrow R(T^*)$ closed.

e.g. X is Hilbert Space. $M \in \mathcal{L}(X, X)$.

M^* is transpose of M. To obtain adjoint in $(\cdot,\cdot)$:

By Riesz's Thm: $\exists \eta \in X$. Corresponds $l_\eta \in X^*$.

st. $\langle l_\eta, x\rangle = (\eta, x)$. $\forall x \in X$.

Define: $\tilde{M}: X \to X$. $\tilde{M}(\eta) = M^*(l_\eta)$.

$\Rightarrow (Mx, \eta) = (x, \tilde{M}\eta)$.

Pf: $(Mx, \eta) = \langle l_\eta, Mx\rangle = \langle M^* l_\eta, x\rangle$

$= (\tilde{M}(\eta), x)$

(3) BLF's of Completion:

. $M \in \mathcal{L}(X, Y)$. X, Y n.v.s. Suppose $\bar{X}, \bar{Y}$ are completion of X, Y. Then we can define $M_0 : \bar{X} \to \bar{Y}$. st. $M_0 \in \mathcal{L}(\bar{X}, \bar{Y})$. and it satisfies:

$M_0([(x_n)]) = [(Mx_n)]$.

Check M_0 is well-def $\begin{cases} (Mx_n) \text{ is Cauchy} \\ \text{indept with } (x_n). \end{cases}$. linear.

and bounded. by $\| M_0([(x_n)]) \| = \| [(Mx_n)] \|$

$= \lim\limits_{n} \| Mx_n \| \leq \underline{\lim}_{n} \|M\| \|x_n\| = \|M\| \, \| [(x_n)] \|$.

(4) Example of BLF: integral operator

① S_j is metric space. Consider measure space $(S_j, \mathcal{B}_{S_j}, \mu_j)$. $\mu_j(S_j) < \infty$. $j = 1, 2$. $\mathcal{B}_{S_j}$ is Borel of S_j.

If: $A : L^2(\mu_1) \to L^2(\mu_2)$. $A(f) = \int_{S_1} k(s,t) f(t) \, d\mu_1$

where $k : S_1 \times S_2 \to \mathbb{C}$.

Find condition st. A is BLF:

Note that $|A(f)|^2 \leq \|k\|^2_{L^2(\mu_1)} \|f\|^2_{L^2(\mu_1)}$

$\therefore \| A(f) \|^2_{L^2(\mu_2)} \leq \| k \|^2_{L^2(\mu_1 \times \mu_2)} \|f\|^2_{L^2(\mu_1)}$

If $k \in L^2(\mu_1 \times \mu_2)$. Then A is BLO. call it integral operator.

② Representation of norm:

$\| Af \|_{L^2} = \sup\limits_{\substack{h \in L^2(\mu_2) \\ \|h\|_{L^2} = 1}} | \langle Af, h \rangle | = \sup\limits_{\square} \left| \int_{S_2} A(f)(s) \, h(s) \, d\mu_2 \right|$

$$|\langle Af, h\rangle| = \left|\int_{S_1\times S_2} k(s,t) f(t) h(s)\, dt\, ds\right|$$

$$\le \frac{1}{2}\int_{S_1\times S_2} \gamma |k| f^2 + \frac{1}{\gamma}|k| h^2 \le \frac{1}{2} C_1 \gamma \|f\|_{L^2}^2 + \frac{1}{2} C_2 \frac{1}{\gamma}\|h\|_{L^2}^2$$

If $\|h\|_{L^2} \le 1$. Then choose γ is optimum.

$\|Af\| \le C\|f\|_{L^2}$. $C = \sqrt{C_1 C_2} = \sqrt{\sup_{t\in S_1}\int |k|\,d\mu_2 \sup_{t\in S_2}\int |k|\,d\mu_1}$

To guarantee A is BLO. We have different condition. $C<\infty$.

(5) Complementary Subspaces and Invertibility:

Thm (property of closed subspaces)

E is Banach space. $G, L \subseteq E$. closed subspaces st. $G+L$ is also closed. Then $\exists C>0$. Such that $\forall z \in G+L$. $\exists x \in G$. $y \in L$. st. $z = x+y$. $\|x\|+\|y\| \le C\|z\|$.

Pf: $T: G\times L \to G+L$, $(x,y) \mapsto x+y$. Conti. Linear and surjective from $G, L, G+L$ are closed.

Apply open mapping thm. on T.

Cor. $\exists C>0$. st. $\text{dist}(x, G\cap L) \le C(\text{dist}(x,G) + \text{dist}(x,L))$ $\forall x \in E$. under the condition above.

Pf: Choose $a\in G$. $b\in L$. st. $\|a-x\| \overset{\varepsilon}{\sim} \text{dist}(x,G)$ and $\|b-x\| \overset{\varepsilon}{\sim} \text{dist}(x,L)$.

$a-b \in G+L$. Apply Thm. $\exists a'\in G$. $b'\in L$. st.

$a-b = a'+b'$. $C\|a-b\| \ge \|a'\|+\|b'\|$

$\therefore a-a' = b'+b \in G\cap L$. $\text{dist}(x, G\cap L) \le \|x-(a-a')\|$.

<u>Cor.</u> E Banach space. $G, L \subseteq E$. CLS. If $\exists\, c > 0$ s.t.

$d(x, G \cap L) \leq c\, d(x, L)$. Then $G + L$ is closed.

<u>Rmk:</u> It's converse of Thm. above.

<u>Lemma</u> $A : D(A) \subset X \longrightarrow Y$. injective CLO. X, Y are Banach. Then: $R(A)$ is closed $\iff \exists\, c > 0$ s.t. $\|x\| \leq c\|Ax\|$. $\forall x \in D(A)$.

<u>Pf:</u> $(\Leftarrow)$ $Ax_n \to y \Rightarrow (x_n)$ Cauchy $\Rightarrow x_n \to x$ in X.

$\Rightarrow$ By closed graph. $y = Ax$.

$(\Rightarrow)$ $(R(A), \|\cdot\|_Y)$ is CLS of Y. So Banach

$A : D(A) \xrightarrow[CLO]{\sim} (R(A), \|\cdot\|_Y)$. $\therefore A^{-1}$ is bdd.

<u>Cor.</u> $A : D(A) \subset X \to Y$. CLO. Then: $R(A)$ is closed

$\iff \exists\, c > 0$. s.t. $d(x, N(A)) \leq c\|Ax\|$. $\forall x \in D(A)$.

<u>Pf.</u>
$$\begin{tikzcd} X \arrow[r, "A"] \arrow[dr, "\pi"'] & Y \\ & X/\ker A \arrow[u, "\tilde{A}"'] \end{tikzcd}$$

Note that $R(\tilde{A}) = R(A)$.

check: 1°) $N(A)$ is closed

2°) $\tilde{A}$ is CLO.

1°) follows from A is CLO. 2°) is trivial

Then $X/\ker A$ is Banach. Apply Lemma.

<u>Pf of cor:</u> $\pi : E \longrightarrow E/L$. $T : G \longrightarrow E/L$. $Tx = \pi x$

$\therefore N(T) = G \cap L$. T is CLO.

$\Rightarrow R(T) = \pi(G)$ closed. $\pi^{-1}(\pi(G)) = G + L$ closed.

<u>Rmk:</u> i) $l : E \to F$. linear bijection among Banach spaces. l isn't necessary to be bdd:

$E = F$. $\dim E = \infty$. where $(e_\alpha)_{\alpha \in A}$ is set of Hamel Basis. set: $l(e_\alpha) = n_\alpha e_\alpha$. $n_\alpha \uparrow \infty$

ii) Remove "Banach". We can't apply Close Graph Thm.

e.g. X. Banach space with Hamel Basis $(e_\gamma)_{\gamma\in A}$.

set $\|x\|_Y = \sum_1^n |a_i|$. for $x = \sum_1^n a_i e_i \in X$.

Then $Y = (X, \|\cdot\|_Y)$ isn't Banach. since $\sum_1^\infty \frac{e_k}{2^k} \notin Y$. $X \xrightarrow{I} Y$. $I = id$ is CLO. I^{-1} is bdd.

$\Rightarrow I$ isn't BLO (Otherwise $X \simeq Y$. complete)

① Complement: Basis:

Def: i) Hamel Basis of vector space E is the maximal l.i. set $(e_\alpha)_{\gamma\in A}$. st. $\forall$ finite set of $(e_\alpha)_{\gamma\in A}$ is l.i. and $\forall x \in E$. x is finitely span by $(e_\alpha)_{\gamma\in A}$.

ii) Schauder Basis of vector space E is a l.i. set $(e_\alpha)_{\gamma\in A}$. st. $\forall x \in E$. $\exists!(\alpha_n)$ st. $x = \sum_{n=1}^\infty \alpha_n e_n$. uniquely.

Rmk: i) By Zorn's Lemma. Hamel Basis always exists. But not for Schauder basis.

ii) o.n.b in Hilbert space is Schauder basis. but not Hamel basis.

iii) Schauder basis can be uncountable.

② Complement of Banach space:

Def: G is CLS of E. $L \subseteq E$ subspace. L is complement (topological) of G if: i) $L = \overline{L}$ ii) $G \cap L = \{0\}$. $G + L = E$

$\Rightarrow \pi_G : G + L \to G$. canonical Proj. is surjective BLO.

Prop. i) Finite-dimension subspace admits a complement.

ii) Closed subspace with finite codimension admits a complement.

iii) Closed subspace of Hilbert space admits a complement.

Pf: ii) Same as i). Denote it by G. Let $N \subseteq E^*$ st. $N^\perp = G$. $\dim N = p < \infty$. prove $N^+ \oplus G = E$.

Remark: For a Banach space E, which isn't Hilbert, There exists $G \subseteq E$, closed linear subspace. st. G admits no complement.

Def: For $T \in \mathcal{L}(E,F)$.

right inverse: $S \in \mathcal{L}(F,E)$. st. $T \circ S = I_F$

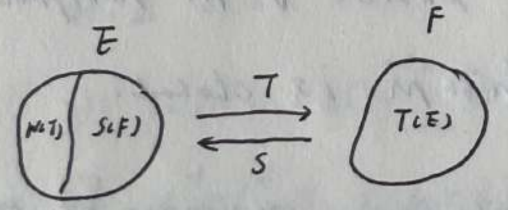
left inverse: $S \in \mathcal{L}(F,E)$. st. $S \circ T = I_E$.

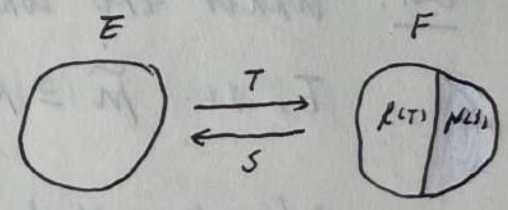
Thm. i) For $T \in \mathcal{L}(E,F)$ surjective. Then

T admits a right inverse $\Leftrightarrow$ $N(T)$ admit a complement.

ii) For $T \in \mathcal{L}(E,F)$ injective. Then

T admits a left inverse $\Leftrightarrow$ $R(T)$ admits a complement.

i) 

ii) 

$T \circ S = I_F$ $\qquad$ $S \circ T = I_E$

prop. E is Banach. $M \subseteq E$. closed linear subspace.

For $X \subseteq E$. finite dimension Then:

i) $M+X$ is closed.

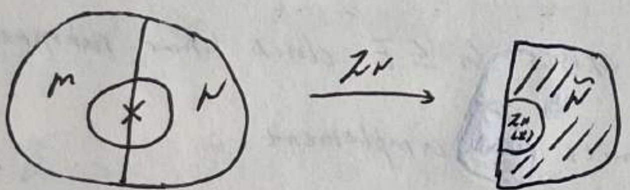
ii) $M+X$ admits a complement $\Leftrightarrow$ M does.

Pf: i) suppose $M \cap X = \{0\}$. (let $\tilde{X}$ = complement of $M \cap X$ in X)

For $x_n + y_n \in M+X \to u \in E$. $x_n \in M$. $y_n \in X$

check (y_n) is bounded. $\Rightarrow$ (y_n) admits convergent subseq.

ii) ($\Leftarrow$) $E = M \oplus N$. $\exists$ π_M. π_N. canonical projection.



$\dim \pi_N(X) < \infty$.
admits a complement
$\tilde{N}$ in N.
check: $(M+X) \oplus \tilde{N} = E$

($\Rightarrow$) $(W + \tilde{X}) \oplus M = E$. where W is the complement of $M+X$ in E

Remark: subspace with finite codimension needn't be closed: $\forall$ L.F. f on E. $N(T)$ has codimension $\dim(E/N(T)) = 1$. when f isn't conti. $N(T) = [f=0]$ isn't closed.

Cor. Under the condition above. $\forall$ $\tilde{M}$ subspace of E. s.t. $\tilde{M} \supseteq M$. Then $\tilde{M}$ is closed.

Pf: let X be the algebraic complement of M in $\tilde{M}$.

then $\tilde{M} = M + X$. $\dim X \leq \dim E/M < \infty$.

Remark: Topological complement E of F is different from algebraic complement, which requires: $E \oplus F = X$. Besides, E is closed. But the latter only require: $E \oplus F = X$.

Cor. E is Banach Space. $M \subseteq E$ closed linear subspace of finite co-dimension. $D \subseteq E$ dense subspace. Then there exists a complement X of M. st. $X \subseteq D$.

Pf: By induction on $d = \dim(E/M)$. $d=0$ ✓.

For $d=n$. choose $x_1 \in D$. $x_1 \notin M$.

otherwise $D \subset M$. $\Rightarrow \bar{D} = E \subset M \subset E$. contradict!

$\therefore$ $M_1 = M + \mathbb{R}x_1$ has codimension $n-1$. By hypothesis ✓.

Remark: It characterizes the complement of FCLS.

Prop. E is Banach. $G, L \subseteq E$. closed subspace. If $\exists X_1, X_2 \subseteq E$. subspace. st. $\dim X_1, \dim X_2 < \infty$. $G+L+X_1 = E$. $G \cap L \subseteq X_2$. Then G and L admits a complement.

(6) Orthogonallity Revisit:

prop. $G, L \subseteq E$. closed subspaces. Then

i) $G \cap L = (G^{\perp} + L^{\perp})^{\perp}$

ii) $G^{\perp} \cap L^{\perp} = (G+L)^{\perp}$

Pf: Note that $N_1 \subseteq N_2 \Rightarrow N_2^{\perp} \subseteq N_1^{\perp}$.

Thm. $G, L \subseteq E$. closed subspaces. Then the following properties are equivalent:

i) $G+L \underset{\text{closed}}{\subseteq} E$ ii) $G^{\perp}+L^{\perp} \underset{\text{closed}}{\subseteq} E^*$

iii) $G+L = (G^{\perp} \cap L^{\perp})^{\perp}$ iv) $G^{\perp}+L^{\perp} = (G \cap L)^{\perp}$

(7) Unbounded Linear Operators and its Adjoints:

① **Def:** E, F banach spaces. An unbounded linear operator from E to F is: $A: D(A) \subseteq E \to F$.

$D(A) = \{ u \in E \mid \|Au\|_F < \infty \}$. domain of A.

e.g. $A = \frac{d}{dx}\big|_{x=1/2}$ on $C^{\infty}[0,1]$.

Remark: i) A is CLO $\Rightarrow N(A)$ is closed. But consider $R(A)$: $R(A)$ may not be closed since $u_n \to u$, u may $\in E/D(A)$. e.g. $T: C[0,1] \longrightarrow C[0,1]$. $Tf = \int_0^x f\,dt$.

ii) We may assume A is closed LO and $D(A)$ is dense. When $\exists c>0$. st. $\forall u \in D(A)$. $\|Au\| \le c\|u\|$. Then A can extend to E.

Def: adjoint of A is: $A^*: D(A^*) \subseteq F^* \to E^*$.

$D(A^*) = \{ f \in F^* \mid . \exists c>0. \text{ st. } |\langle f, Au\rangle| \le c\|u\|, \forall u \in D(A) \}$.

is linear subspace. Usually suppose $D(A)$ is dense. then extend $f \in F^*$ to E.

As the common transpose, we have:

$< f, Au >_{F^*,F} = < A^* f, u >_{E^*,E}$

<u>Prop.</u> $A: D(A) \subseteq E \to F$ densely defined. Then A^* is closed.

<u>Pf:</u> For $(v_n, A^* v_n) \to (v, f)$.

check i) $v \in D(A^*)$ ii) $f = A^* v$.

From $< v_n, Au > = < A^* v_n, u >$ $\forall u \in D(A)$. Let $n \to \infty$.

$\therefore < v, Au > = < A^* v, u > = < f, u >$. $|< v, Au >| \le \|f\| \|u\|$.

$\Rightarrow v \in D(A^*)$. Since $D(A)$ dense $\Rightarrow \forall u \in E$ $\checkmark$ $\therefore f = A^* v$.

② <u>Orthogonal Relation between A and A^*</u>

· <u>Def:</u> $I: F^* \times E^* \to E^* \times F^*$. $I[v, f] = [-f, v]$

<u>prop.</u> $I(G(A^*)) = G(A)^\perp$

<u>pf:</u> $[v, f] \in G(A^*) \Leftrightarrow < A^* v, u > = < f, u >$. $\forall u \in D(A)$

$\Leftrightarrow < -f, u > + < v, Au > = 0$. $\forall u \in D(A)$.

i.e. $< [-f, v], [u, Au] > = 0$. $\therefore [-f, v] \in G(A)^\perp$

<u>Cor.</u> $A: D(A) \subseteq E \to F$. densely defined. closed. Then.

$N(A) = R(A^*)^\perp$. $N(A^*) = R(A)^\perp$

$N^\perp(A) = \overline{R(A^*)}$. $N(A^*)^\perp = \overline{R(A)}$

<u>Pf:</u> From: $G = G(A)$. $L = E \times \{0\}$. we have:

$N(A) \times \{0\} = G \cap L$. $E \times R(A) = G + L$

$\{0\} \times N(A^*) = G^\perp \cap L^\perp$. $R(A^*) \times F^* = G^\perp + L^\perp$.

Since $G(A)^\perp = I(G(A^*))$. $L^\perp = \{0\} \times F^*$.

Cor. The following properties are equivalent.

i) $R(A)$ is closed. ii) $R(A^*)$ is closed.

iii) $R(A) = N(A^*)^\perp$. iv) $R(A^*) = N(A)^\perp$

④ Thm. X, Y Banach. $A: D(A) \subseteq X \to Y$ surjective CLO. Then: A is open map.

Pf:
$$\begin{tikzcd} (D(A), \|\cdot\|) \arrow[r, "\bar{A}"] \arrow[dr, "z"'] & Y \\ & D(A)/\ker A \arrow[u, "\tilde{A}"'] \end{tikzcd}$$

$\tilde{A}$ is homeo. z is open quotient map.

③ Characterization of surjective operators:

Thm. $A: D(A) \subseteq E \to F$ closed. densely defined.

i) The following properties are equivalent:

(a) A is surjective (b) $N(A^*) = \{0\}$. $R(A^*)$ is closed.

(c). $\exists$ const. C. st. $\|v\| \leq C \|A^* v\|$. $\forall v \in D(A^*)$

ii) The following properties are equivalent:

(a) A^* is surjective (b) $N(A) = \{0\}$. $R(A)$ is closed

(c) $\exists$ const. C. st. $\|u\| \leq C \|Au\|$. $\forall u \in D(A)$.

Remark: A [resp A^*] is surjective $\Rightarrow$

A^* [resp A] is injective. Converse fails.

(Consider $A: \ell^2 \to \ell^2$. $A(x_n) = (\frac{1}{n} x_n)$.

$A^* = A$. Since ℓ^2 is Hilbert. A is injective.

But A isn't bijective!)

In particular. $\dim E \dim F < \infty$. converse holds.

Pf: i) (b) $\Rightarrow$ (a). $R(A) = N(A^*)^\perp = F$ since $R(A^*)$ is closed

(c) $\Rightarrow$ (b). $A^* v_n \to f \Rightarrow (v_n)$ is Cauchy.

(a) $\Rightarrow$ (c). For $\| \frac{v}{\|A^* v\|} \| \leq C$. $\forall v \in D(A^*)$.

Consider $B^* = \{ \|A^* v\| \leq 1 \}$ is bounded in $D(A^*)$

By UBP. $\forall f_0 \in F$. $f_0 = A e_0$.

$\therefore |\langle f_0, v \rangle| = |\langle A e_0, v \rangle| = |\langle e_0, A^* v \rangle| \leq \|e_0\|$