Banach Space Over $\mathbb{C}$.

· Suppose E is vector space over $\mathbb{C}$. (Denote $E_{\mathbb{R}}$ over $\mathbb{R}$)

Def: i) linear subspace: $x+y \in M$. for $x, y \in M$

$\lambda x \in M$. $\forall \lambda \in \mathbb{C}$. $\forall x \in M$

ii) linear function: $f: E \to \mathbb{C}$. satisfies:

$$f(x+y) = f(x) + f(y)$$

$$f(\lambda x) = \lambda f(x). \ \forall \lambda \in \mathbb{C}.$$

iii) Norm: $\|\cdot\|: E \to [0, +\infty)$. satisfies:

$\|x\| \geq 0$. $\|x\| = 0 \Leftrightarrow x = 0$. $\|\lambda x\| = |\lambda| \|x\|$. $\forall \lambda \in \mathbb{C}$.

$\|x+y\| \leq \|x\| + \|y\|$.

Remark: i) M is linear subspace of $E_{\mathbb{R}}$ $\not\Rightarrow$ LS of E.

e.g. $x=0$ in $\mathbb{R}^2 \underset{iso}{\cong} \mathbb{C}$.

ii) LF's on $E_{\mathbb{R}}$ is: $f: E \to \mathbb{R}$. but only satisfies $\lambda \in \mathbb{R}$. $\therefore$ $E_{\mathbb{R}}^* \neq E^*$. absolutely.

iii) Norm in $E_{\mathbb{R}}$ $\not\Rightarrow$ Norm in E.

Prop. $I: f \in E^* \xrightarrow{\sim} \mathrm{Re} f \in E_{\mathbb{R}}^*$ is isomorphic isometry.

Pf: 1°) $I(f) \in E_{\mathbb{R}}^*$ (well-def). $\|I(f)\| \leq \|f\|_{E^*}$.

2°) I is injective.

Consider $\mathrm{Re}\langle f, x\rangle = \mathrm{Re}\langle f, ix\rangle = 0$.

3°) I is surjective:

$\forall\, \varphi \in E_{\mathbb{R}}^*$. Let $f(x) = \varphi(x) - i\varphi(ix) \in E^*$. (check)

Let $\lambda = \frac{f}{|f|}$ (rotation). $|f(x)| = \varphi(\frac{x}{\lambda})$.

Obtain: $\|I(f)\| = \|\varphi\|$.

Remark: We can modify the Thm in $E_{\mathbb{R}}$ before (About $E_{\mathbb{R}}^*$) to $E_{\mathbb{C}}$ case. By replacing f with $\mathrm{Re} f$. (For $\mathrm{Im} f$. Let $x = ix$).

Next. Consider H is Hilbert space over $\mathbb{C}$. with $(\cdot,\cdot)$. $H_{\mathbb{R}}$ is over $\mathbb{R}$. equipped with $\mathrm{Re}(\cdot,\cdot)$.

In H. $a(u,v)$ is coercive $\Leftrightarrow$ $\mathrm{Re}\, a(u,u) \geq \alpha |u|^2$. $\exists \alpha > 0$. $\forall u \in H$.

Thm. (Lax-Milgram)

For $T \in \mathcal{L}(H)$ (BLO). St. $|(Tu,u)| \geq \alpha |u|^2$. $\exists \alpha > 0$.

Then T is bijection.

Pf: It's similar. Note that the condition is weaker than coercive.

Remark: Claim: $|(Tu,u)| \geq \alpha |u|^2 \Rightarrow \exists\, \zeta \in \mathbb{C}.\ |\zeta| = 1$. st. $\mathrm{Re}(\zeta Tu, u) \geq \alpha |u|^2$

Pf: $W(T) = \{(Tu,u) \mid |u| = 1.\ u \in H\}$ is numerical range of operator $T \in \mathcal{L}(H)$.

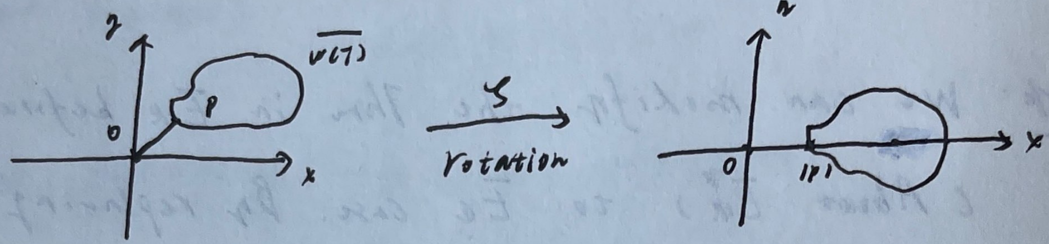
It's convex. by Toeplitz. Hausdorff.

$0 \notin \overline{W(T)}$. Consider $p = P_{\overline{W(T)}}\, 0$.

i.e. $\|p-0\| = \text{dist}(0, \overline{W(T)}) \geq \alpha > 0$. for some α

($|p| = \min\limits_{|u|=1} |(Tu,u)|$.).

Choose $\xi = \frac{\bar{p}}{p}$. $\|\xi\|=1$. $\xi p = |p|$. on $(|p|, 0)$



$\therefore$ $\text{Re}(\xi T u, u) \geq |p| \geq \alpha$

For spectrum of $T \in \mathcal{L}(E)$.

<u>Def</u>: $\rho(T) = \{\lambda \in \mathbb{C} \mid \lambda I - T \text{ is bijection}\}$.

$\sigma(T) = \mathbb{C} / \rho(T)$. $EV(T) = \{\lambda \in \mathbb{C} \mid N(\lambda I - T) \neq \{0\}\}$

<u>prop</u>. $\sigma(T) \neq \emptyset$. cpt. $\sigma(T) \subseteq \{|\lambda| \leq \|T\|\}$.

Besides $r(T) = \max\{|\lambda| \mid \lambda \in \sigma(T)\}$. (may not hold in $E_{\mathbb{R}}$)

<u>prop</u>. In $E = E_{\mathbb{C}}$. we have:

$\varphi(EV(T)) = EV(\varphi(T))$. $\varphi(\sigma(T)) = \sigma(\varphi(T))$

<u>Pf</u>: That's because $\varphi(t) - \mu = \prod_{i}^{k}(t - t_i)$

The polynomials splits in $\mathbb{C}$.