Quotient Space.

- E is Banach space. $M \subseteq E$. closed subspace

For quotient space E/M. $\pi: E \longrightarrow E/M$. $(\|\pi\| = 1)$

equipped with a norm: $\|[x]\|_{E/M} = \|\pi x\|_{E/M} = \inf\limits_{\substack{\eta \in [x] \\ \subseteq E}} \|\eta\|$

<u>Remark:</u> Require M is closed is for:

$\|[x]\| = 0 \iff [x] = 0$.

<u>Pf:</u> $(\Rightarrow)$. i.e. $\exists (z_n)$. $\|z_n\| \to 0$. $(z_n \to 0)$

$(z_n) \subseteq [x]$. $\therefore$ $z_n - x \in M$.

$z_n - x \longrightarrow -x \in M$. Since M is closed

$\therefore$ $[x] = 0$. in E/M.

(1) <u>Representation of norm:</u>

$$\|[x]\| = \inf_{\substack{\eta \in [x] \\ \subseteq E}} \|\eta\| = \inf_{m \in M} \|x - m\| = \mathrm{dist}(x, M)$$

$$= \sup_{\substack{\|l\| = 1 \\ l(M) = \{0\}}} |l(x)| \quad \text{(we have proved it before)}$$

<u>Thm.</u> $(E/M, \|\cdot\|_{E/M})$ is Banach space.

<u>Pf.</u> $(\pi(x_k))$ Cauchy $\Rightarrow$ $(x_k - m_k)$ Cauchy, $m_k \in M$.

$\therefore$ $\pi(x_k - m_k) \longrightarrow \pi(a)$. where $x_k - m_k \to a$

Since π is BLF on E.

(2) Dual space :

prop. i) Let $\pi^*: (E/M)^* \to E^*$ adjoint of π. Then

$R(\pi^*) = M^\perp$. i.e. $(E/M)^* \overset{\pi^*}{\cong} M^\perp$. with

$\|\pi^*(\xi)\|_{E^*} = \|\xi\|_{(E/M)^*}$. $\forall \xi \in (E/M)^*$

ii) $T: E^* \to M^*$. $T(f) = f|_M$. $\forall f \in E^*$. Then let

$\tilde{T} \circ \pi = T$. $E^*/M^\perp \overset{\tilde{T}}{\cong} M^*$. isometry. $(E^* \xrightarrow{\pi} E^*/M^\perp)$

Pf: i) $(E/M)^* \subseteq M^\perp$ is easy to check.

Conversely. for $f \in M^\perp$. define ξ satisfies :

$\xi(\eta) = \langle f, x \rangle$. where $\eta = \pi(x)$.

check ξ is well-def. linear. bounded.

$\therefore \xi \in (E/M)^*$. $\pi^*(\xi) = f$. check isometry.

ii) Lemma. $T \in L(F, G)$. F, G are Banach space.

$F/N(T) \overset{\tilde{T}}{\cong} R(T)$. isomorphism. $\|T\| = \|\tilde{T}\|$

Pf: $\forall \eta \in F/N(T)$. $\tilde{T}\eta = Tx$. where $\eta = \pi(x)$

check $\tilde{T}$ is well-def. it's bijective. isometry.

$\Rightarrow N(T) = M^\perp$. $R(T) = M^*$ (By Hahn-Banach)

Then check $\|Tx\| = \|x\|$. i.e. $\tilde{T}$ is isometry.

1°) $\|\eta\| \le \|\tilde{T}\eta\| \iff \|\pi(x)\| \le \|\tilde{T} \circ \pi(x)\|$.

$\iff \|\pi(x)\| \le \|Tx\|$. $\iff dist(x, M^\perp) \le \|Tx\|$.

It's from : $\exists \tilde{x} \in E^*$. $\tilde{x}|_M = x$. $\therefore x - \tilde{x} \in M^\perp$.

$\therefore dist(x, M^\perp) \le \|x - (x - \tilde{x})\| = \|Tx\|$.

2°) $\|\eta\| \ge \|\tilde{T}\eta\|$. is from $\|Tx\| \le \|x\|$

(3) Properties of inherits:

① **prop.** E is reflexive Banach space. Then E/M is reflexive.

Pf: E reflexive $\Rightarrow E^*$ reflexive $\Rightarrow M^\perp$ reflexive $\Rightarrow (E/M)^*$ reflexive ($M^\perp \underset{iso}{\cong} (E/M)^*$) $\Rightarrow E/M$ reflexive.

② **Prop.** E is uniformly convex $\Rightarrow E/M$ is uniformly convex.

Pf: easy to check. see $\pi(x)$ as $x-m$. $m \in M$.

(4) **Dimension:**

prop. i) $\dim M < \infty \Leftrightarrow \operatorname{codim} M^\perp < \infty$. in that case $\dim M = \operatorname{codim} M^\perp$.

ii) $\operatorname{codim} M < \infty \Leftrightarrow \dim M^\perp < \infty$. in that case $\operatorname{codim} M = \dim M^\perp$

pf: Apply $E^*/M^\perp \cong M^*$. $(E/M)^* \cong M^\perp$.

Prop. (Dual statement)

$N \subseteq E^*$. closed subspace. Then $\dim N < \infty \Leftrightarrow \operatorname{codim} N^\perp < \infty$

in that case $\dim N = \operatorname{codim} N^\perp$. Besides, $\dim N^\perp \leq \operatorname{codim} N$.

Remark: $\dim N^\perp < \operatorname{codim} N$ may happen. (when $\operatorname{codim} N < \infty$)

since $\bar{N} \subsetneqq N^{\perp\perp}$ may happen. e.g. Choose $\xi \in E^{**}$.

$\xi \notin E$. $\xi^{-1}\{0\}$ has codimension one. $\langle \xi, f \rangle = 0$. $\forall f \in \xi^{-1}\{0\}$.

Denote $\xi^{-1}\{0\} = N$. since $\xi \notin E$. $\therefore N^\perp = \{0\}$. $N^{\perp\perp} = E^*$.

$\therefore \bar{N} = N \subsetneqq N^{\perp\perp} = E^*$

Pf. ($\Rightarrow$) Argue $N = N^{\perp\perp}$

($\Leftarrow$) Deduce $\dim N < \infty$. reduce to the first one.

(5) Operators:

prop. X, Y are Banach spaces. $A, B \in \mathcal{L}(X, Y)$

Then. $R(A) \subset R(B) \iff \tilde{A} = \hat{B} \circ T$. where

$$\begin{tikzcd} X \arrow[r, "B"] \arrow[dr, "T"'] & (R(B), \|\cdot\|_Y) \arrow[r, hook] & Y \\ & X/\ker B \arrow[u, "\hat{B}"'] & \end{tikzcd}$$

$\exists \tilde{A}$. $\tilde{A}x = Ax$.

Pf: $(\Leftarrow)$ It's trivial. $R(\hat{B}) = R(B)$.

$(\Rightarrow)$ $\|\hat{B}x\|_Y = \|Bx\|_Y \le c\|[x]\| = c\, dist(x, \ker B)$

Define a norm: $|||\hat{B}x||| = \|[x]\|$ in $R(B)$.

$\Rightarrow (R(B), |||\cdot|||) \underset{iso}{\xrightarrow{\hat{B}^{-1}}} X/\ker B$. Banach space.

$$\begin{tikzcd} X \arrow[r, "\tilde{A}"] \arrow[dr, "z_B"'] & (R(B), |||\cdot|||) \\ & X/\ker B \arrow[u, "\hat{B}"', "\sim"] \end{tikzcd}$$

claim: $\tilde{A}$ is bdd.

since: $X \times (R(B), |||\cdot|||) \underset{conti}{\xrightarrow{I}} X \times (R(B), \|\cdot\|_Y)$

$I^{-1}(G(A))$ closed. Apply closed Graph Thm.

Cor. X, Y are Banach space. $B \in K(X, Y)$.

If $A \in \mathcal{L}(X, Y)$. $R(A) \subset R(B)$. Then:

$A \in K(X, Y)$.

Pf: $\tilde{A} = \hat{B} \circ T$. check $\hat{B} \in K(X/\ker B, Y)$

Rmk: Alternative proof:

Consider $X = \bigcup \{x \mid Ax \in \overline{B(B(x_0, n))}\}$

Apply Baire Thm.