Operators of Finite Rank

Def: X, Y n.v.s. $A: X \longrightarrow Y$ is called finite rank operator if $\mathrm{Im} A$ has finite rank.

e.g. $\forall x^* \in X$. $\forall \eta \in Y$. Define:

$$\eta \otimes x^*: X \longrightarrow Y$$
$$x \longmapsto x^*(x)\eta$$

Consider $(\eta \otimes x^*)^*: Y^* \longrightarrow X^*$. for $\forall \eta^*. x$

$$((\eta \otimes x^*)^*(\eta^*), x) = (\eta^*, (\eta \otimes x^*)(x))$$

$$= \eta^*(\eta)\, x^*(x) = ((x^* \otimes \eta)(\eta^*), x)$$

$$\Rightarrow (\eta \otimes x^*)^* = x^* \otimes \eta$$

Rmk: Define sum of such operator:

$$\sum_1^n \eta_i \otimes x_i^* : (\sum_1^n \eta_i \otimes x_i^*)(x) =$$

$$\sum_1^n x_i^*(x)\eta_i \in Y. \text{ for } \forall x \in X.$$

prop. Any finite rank. bdd operator $T: X \xrightarrow{\text{linear}} Y$. has form: $\sum_1^n \eta_i \otimes x_i^*$. where $n = \dim \mathrm{Im} T$. $\{\eta_i\}_1^n$. $\{x_i^*\}$ are l.i sets.

Pf: Choose η_i: $\mathrm{span}\{\eta_i\}_1^n = \mathrm{Im} T$.

$$\forall x \in X. \quad T(x) = \sum_1^n \tau_i(x)\eta_i$$

1°) $\alpha_i(x)$ is linear:

check: $T(x+\alpha z) = T(x) + \alpha T(z)$

with $\{\eta_i\}$ are l.i.

2°) $\alpha_i(x)$ is bdd:

$$\|T\|\|x\| \geq \|T(x)\| = \|\sum_1^n \alpha_i(x)\eta_i\|$$

$$\geq c\sum_1^n |\alpha_i(x)| \, \|\eta_i\|$$

$$\geq C \sum_1^n |\alpha_i(x)|$$

by equi of norm in finite rank LS.

$\Rightarrow \alpha_i(\cdot) \in X^*$ $\therefore$ $T = \sum_1^n \eta_i \otimes \alpha_i$.

3°) check $\{\alpha_i\}$ are l.i.

if $\alpha_1 = \sum_2^n k_i \alpha_i \Rightarrow \operatorname{Im} T = \operatorname{span}\{\eta_i + k_i \eta_1\}_2^n$

then $\dim \operatorname{Im} T < n$. contradict.

<u>Cor.</u> For BLO with finite rank: T.

$\dim \operatorname{Im} T = \dim \operatorname{Im} T^*$.

<u>Rmk</u>: Its expression isn't unique.

<u>Thm.</u> X n.v.s. $F(X) = \{$BLO with finite rank from X to $X\}$.

For $K \in F(X) \Rightarrow \operatorname{Im}(I-K)$ is closed. $\dim(\ker(I-K))$ $= \dim \ker(I-K^*)$.

<u>pf:</u> $y \in \operatorname{Im}(I-K)$. $K = \sum_1^n x_i \otimes x_i^*$. $\dim \operatorname{Im} K = n$.

$\Leftrightarrow y = x - \sum_1^n x_i^*(x) x_i$.

1°) $y = x - \sum_i^n x_i^*(x) x_i \iff \begin{pmatrix} x_1^*(y) \\ \vdots \\ x_n^*(y) \end{pmatrix} = (I-M) \begin{pmatrix} \tau_1 \\ \vdots \\ \tau_n \end{pmatrix}$

has solution, where $M = (x_i^*(x_j))_{n\times n}$.

Pf: $(\Rightarrow)$ $\begin{pmatrix} x_1^*(y) \\ \vdots \\ x_n^*(y) \end{pmatrix} = (I-M) \begin{pmatrix} x_1^*(x) \\ \vdots \\ x_n^*(x) \end{pmatrix}$. $\therefore \tau^T = \begin{pmatrix} x_1^*(x) \\ \vdots \\ x_n^*(x) \end{pmatrix}$.

$(\Leftarrow)$ Intuitively, set $x = y + \sum_i^n \tau_i x_i$, check $\tau_i = x_i^*(x)$

$$\begin{pmatrix} x_1^*(x) \\ \vdots \\ x_n^*(x) \end{pmatrix} = \begin{pmatrix} x_1^*(y) \\ \vdots \\ x_n^*(y) \end{pmatrix} + M\tau^T = (I-M)\tau^T + M\tau^T$$

$$= \tau^T.$$

2°) For $|I-M| \neq 0$. $\Rightarrow \mathrm{Im}(I-K) = X$.

3°) For $|I-M| = 0$. Note that $y \in \mathrm{Im}(I-K) \iff$

$\begin{pmatrix} x_1^*(y) \\ \vdots \\ x_n^*(y) \end{pmatrix} \in C(I-M) \iff \beta^T \begin{pmatrix} x_1^*(y) \\ \vdots \\ x_n^*(y) \end{pmatrix} = 0$, $\forall \beta \in C^\perp(I-M)$

$\therefore \mathrm{Im}(I-K) = \{ \sum_i^n \lambda_i x_i^* \mid \vec{\lambda}^T (I-M) = 0 \}^\perp$, closed.

4°) Check: $\ker(I - \sum_i^n x_i \otimes x_i^*) = \{ \sum_i \tau_i x_i \mid (I-M)\tau = 0 \}$

$\xrightarrow{f} \{ (\tau_1, \dots, \tau_n) \mid (I-M)\tau = 0 \} = \ker(I-M)$

f is bijection.

$\Rightarrow \dim\ker(I-K) = \dim\ker(I-M) = \dim\ker(I-M^T)$

$= \dim\ker(I-K^*)$