# Existence of Projection

In Hilbert space, we know for closed convex set A.

$\forall x \in H$. $\exists y \in A$. st. $d(x, y) = d(x, A)$. We call such

point $y$ by projection of $x$ on $A$ : $P_A x$.

In general n.v.s. such point may not exist. even

for closed linear subspace.

<u>Claim:</u> In n.v.s $E$. Projection exists $\Leftrightarrow$ $E$ is

reflexive.

<u>Pf</u>. ($\Leftarrow$) $E = E^{**}$. So $E$ is Banach space.

$\exists y_n \in A$. $\|x - y_n\| \to d(x, A)$

$\therefore (y_n)$ is bdd in $E$. $\exists y_{n_k} \rightharpoonup y$.

Since $A$ is convex. $\therefore \overline{A}^{\sigma(E, E^*)} = \overline{A}$.

i.e. $y \in A$. $\|x - y\| \le \underline{\lim} \|x - y_n\| = d(x, A)$

($\Rightarrow$) If $E$ isn't reflexive. By James Thm.

$\exists f \in E^*$. $\|f\| = \sup\limits_{\|x\|=1} |f(x)|$ can't be achieved.

<u>Claim:</u> $\forall x \notin N(f)$. $d(x, N(f))$ can't be attained.

If not: $\exists y \in N(f)$. $d(x, y) = d(x, N(f))$

Set : $p = (x - y) / \|x - y\|$. WLOG. Let $\|f\| = 1$.

Then: Note: $d(x, N(f)) = \sup\limits_{\substack{\|l\|=1 \\ l \in N(f)^\perp}} |l(x)|$

$l \in N(f)^\perp \iff N(l) \supseteq N(f)$

i.e. $N(l) = N(f)$. Since $\operatorname{codim} N(f) = 1$. $l \neq 0$

$\therefore cl = f$. Since $\|cl\| = 1 = \|l\|$ $\therefore c = 1$.

$\therefore d(x, N(f)) = |f(x)|$

$\Rightarrow |f(p)| = \frac{|f(x-y)|}{\|x-y\|} = \frac{|f(x)|}{d(x, N(f))} = 1$. Contradict!

Rmk: For truely counterexample for $E$ isn't reflexive:

i) $T: C[0,1] \to \mathbb{R}^1$. $T(f) = \int_0^{\frac{1}{2}} f(t) - \int_{\frac{1}{2}}^1 f(t)$

Note for $C(K)$. $K$ cpt. $|K| \geq S^1$. Then:

$C(K)$ is not reflexive.

Besides. $\|T\| = 1$. Note: $g = \begin{cases} 1, & t \in [0, \frac{1}{2}] \\ -1, & t \in [\frac{1}{2}, 1] \end{cases}$

$g \notin C[0,1]$. $\exists\, g_n \to g$. $\|T\| \leq 1$. $|T(g)| = 1$

ii) $f: C_0 \subseteq l_\infty \to \mathbb{R}^1$. $C_0$ isn't reflexive.

$f(x) = \sum \frac{x_n}{2^n}$. $\|x\| = \sup\limits_n |x_n|$

$\|f\| = 1$. But it can't attain.

Lemma. For $C(K)$. $K$ is cpt. $|K| \geq S^1$. Then it's not reflexive.

Pf: Suppose $\{x_n\} \subseteq K \to x$. $x_n$ all distinctive.

By Urysohn. $\exists\, f_n: K \to [0,1]$ s.t. $\begin{cases} f_n|_{\{x_1, \cdots, x_n\}} = 1 \\ f_n|_{\{x_{n+1}, \cdots\} \cup \{x\}} = 0 \end{cases}$

Since $\|f_n\|_{C(K)} = 1$. By weakly cpt of reflexive. (Assume)

Then $\exists\, (f_{n_k}) \subseteq (f_n) \rightharpoonup f$. $ev_{x_k}(f_{n_j}) \xrightarrow{j \to \infty} ev_{x_k}(f)$. $(n_j > k)$

$\therefore f(x_k) = 1$. $\forall k$. But $ev_x(f_{n_k}) \to ev_x(f) = 0$. $f$ conti. Contradict!