Linear Space

(1) Hanh-Banach Thm:

① Analytic Form:

· It claims the extension of linear functional defined on a subspace.

Thm. $p: E \to \mathbb{R}$ satisfies i) $p(\lambda x) = \lambda p(x)$, $\forall x \in E$, $\forall \lambda > 0$; ii) $p(x+y) \le p(x) + p(y)$, $\forall x, y \in E$. (Minkowsky)

$G \subseteq E$ (vector space) is ~~subspace~~. g is a linear functional $: G \to \mathbb{R}$, st. $g(x) \le p(x)$, $\forall x \in G$.

Then exists $f: E \to \mathbb{R}$, st. $f|_G = g$, $f(x) \le p(x)$ $\forall x \in E$. i.e. f extend g to whole space.

Pf: The ideal is collect all the functionals extending $g(x)$. By Zorn's Lemma, find the most thorough extension. prove it's $f(x)$.

1°) $P = \{ h : D(h) \subseteq E \to \mathbb{R} \mid$ h is linear, $G \subseteq D(h)$, $D(h)$ is linear subspace, $h(x) \le p(x)$ $\}$.

with order "$\le$". $h_1 \le h_2 \iff D(h_1) \subseteq D(h_2)$

Check $P \ne \emptyset$. every chain has a maximal.

2°) Denote f is the maximal element in P.

prove: $D(f) = E$.

By contradiction: $\exists x_0 \in E / D(f)$.

extend f: $h(x+tx_0) = f(x) + t\alpha$. $x \in D(f)$. $t \in \mathbb{R}$

And find α. satisfies:

$h(x+tx_0) = f(x) + t\alpha \leq p(x+tx_0)$. $\forall t \in \mathbb{R}$.

which is a contradiction, since $h \geq f$.

<u>Cor.</u> $G \subseteq E$. linear subspace. $g: G \to \mathbb{R}$. continuous linear functional. Then there exists $f \in E^*$.

s.t. $f|_G = g$. $\|f\|_{E^*} = \|g\|_{G^*}$

<u>Pf.</u> Dominate g by $p(x) = \|g\|_{G^*} \|x\|$.

Check $p(x)$ satisfies i). ii).

<u>Cor.</u> $\forall x_0 \in E$. exists $f \in E^*$. s.t. $\|f\| = \|x_0\|$.

$\langle f, x_0 \rangle = \|x_0\|^2$.

<u>Pf:</u> Def $g(x)$ on $\mathbb{R}x_0$: $g(tx_0) = t\|x_0\|^2$.

$\exists f$. $\|f\|_{E^*} = \|g\|_{\mathbb{R}x_0} = \|x_0\|$. $f|_{\mathbb{R}x_0} = g$.

since g is conti. linear.

<u>Remark</u>: f isn't unique.

<u>Cor.</u> $\forall x \in E$. $\|x\| = \sup\limits_{\substack{\|f\|=1 \\ f \in E^*}} \|f(x)\| = \max\limits_{\substack{\|f\|=1 \\ f \in E^*}} \|f(x)\|$

<u>Pf:</u> $\|f(x)\| \leq \|f\| \|x\|$.

The converse is from above!

Remark: James Thm:

For E is reflective Banach space

$\Leftrightarrow \|f\|_{E^*}$ can be attained.

② Geometric Form:

It claims convex sets can be separated by linear functional.

Def: i) $H \subseteq E$ is an affine hyperplane if H is form: $\{f=\alpha\}$. for some LF $f \not\equiv 0$

ii) H is a half space if $H = [f<\alpha]$ or $[f>\alpha]$. for some LF $f \not\equiv 0$.

prop. $H=[f=\alpha]$ is closed $\Leftrightarrow$ f is conti. (f is BLF).

Pf: ($\Rightarrow$) H^c is open. $\forall x_0 \in H^c$. $\exists B(x_0, r) \subseteq H^c$.

prove: $f<\alpha$ when $x \in B(x_0, r)$

Otherwise, $\exists x_1 \in B(x_0, r)$. s.t. $f(x_1) > \alpha$.

Then $\exists x_2 \in \overline{x_0 x_1}$. s.t. $f(x_2) = \alpha$. contradict!

$\therefore f(x_0 + rz) < \alpha$. $z \in B(0,1)$. $\therefore \|f\| \le \frac{1}{r}(\alpha - f(x_0))$

Rmk: For a LO: $A: X \longrightarrow Y$. between n.v.s. Then:

$N(A)$ closed $\not\Rightarrow$ A is bdd. e.g. $X = span\{e_n\}_{n \in \mathbb{Z}} = Y$.

$Ae_n = ne_n$. $\|e_n\|=1$. $\forall n$. $N(A) = \{0\}$.

But if $\dim Y < \infty$. It holds. ($\hat{A}: X/N(A) \rightarrow Y$. conti)

Def. i) $[f=\alpha]$ separates A, B. if $f(A) \le \alpha \le f(B)$.

ii) separate strictly. if $\exists \varepsilon > 0$. $f(A) + \varepsilon \le \alpha \le f(B) - \varepsilon$

i) Thm. (First geometric Form)

$A, B \subseteq E$. n.v.s. nonempty, convex, disjoint

If A is open. Then exists a closed affine hyperplane $[f=\alpha]$ separate A, B

Lemma. $0 \in C \subseteq E$. open, convex. $\forall x \in E$. Set gauge of C is $p(x) = \inf\{\alpha \mid \alpha^{-1}x \in C\}$

Then $p(x)$ is Minkowsky. $\exists M > 0$.

s.t. $0 \le p(x) \le M\|x\|$. and $C = \{p < 1\}$.

Pf: 1°) $p(x+y) \le p(x) + p(y)$. $p(\alpha x) = |\alpha| p(x)$

$$\frac{x+y}{p(x)+p(y)} = \frac{x}{p(x)} \cdot \frac{p(x)}{p(x)+p(y)} + \frac{y}{p(y)} \cdot \frac{p(y)}{p(x)+p(y)} \in C.$$

2°) $\exists B(0,r) \subseteq C \Rightarrow \|p\| \le \frac{1}{r}$

3°) $\forall x \in C$. $\exists B(x, \varepsilon) \subseteq C$.

$\therefore p(x) < \frac{1}{1+\varepsilon/\|x\|} < 1$

Lemma. (From one point)

$C \subseteq E$. convex, open. If $\exists x_0 \in E/C$. Then $\exists [f = f(x_0)]$ separates $\{x_0\}$ and C where $C \neq \emptyset$. $f \in E^*$.

Pf: By translation suppose $0 \in C$.

$p(x)$ is gauge of C.

Begain from $G = \mathbb{R}x_0$: $g(tx_0) = t$.

$\therefore g(tx_0) \le p(tx_0)$. extend g to f.

$\Rightarrow$ Pf: For general case. Consider $A-B$ and $\{0\}$.

$A-B = \bigcup_{x \in B} A-x$ is open. check $A-B$ is convex.

Apply the Lemma. on $\{0\}$. $A-B$

Rmk: $f(B) \leq \gamma \underset{\wedge}{\leq} f(A)$ can be achieved. $f(A)$ is interval.

ii) Thm. (Second geometric Form)

$A \cdot B \subseteq E$. nonempty. convex. disjoint. If A is closed B is cpt. Then exists $[f=\alpha]$ separates $A \cdot B$. strictly. $f \in E^*$.

Pf: $A-B$ is convex. and closed (convex is trivial)

If $\exists \{Z_\alpha\}_{\alpha \in X}$ net $\longrightarrow Z$. prove: $Z \in A-B$.

Since $Z_\alpha = x_\alpha - y_\alpha$. $x_\alpha \in A$. $y_\alpha \in B$.

$\exists Y$. s.t. $\varphi: X \longrightarrow Y$. $y_{\varphi(\alpha)}$ converges to $y \in B$.

$\therefore x_{\varphi(\alpha)} = Z_{\varphi(\alpha)} + y_{\varphi(\alpha)} \longrightarrow Z + y \in A$. Since A is close

$\therefore Z \in A-B$. i.e. $A-B$ is closed.

Since $0 \notin A-B$ $\therefore \exists B(0,r) \cap (A-B) = \emptyset$.

Apply i) to $B(0,r)$ and $A-B$.

Remark: If $A \cdot B$ are only closed. $A-B$ may not be closed.

Then the conclusion may not hold!

Cor. $F \subseteq E$. linear subspace. $\bar{F} \neq E$. Then $\exists f \in E^*$. $f \not\equiv 0$.

s.t. $f(x) = 0$. $\forall x \in F$. i.e. $F \subseteq \ker f$.

Pf: $\exists \{x_0\} \subseteq E / \bar{F}$. separate $\{x_0\}$ and $\bar{F}$.

$\therefore \exists f \in E^*$. α. $\langle f, x \rangle \leq \alpha \leq \langle f, x_0 \rangle$.

Note that F is linear subspace $\therefore \forall \lambda \in \mathbb{R}$.

$\langle f, \lambda x \rangle = \lambda \langle f, x \rangle \leq \alpha$. $\therefore \langle f, x \rangle = 0$.

Cor. BLF f vanishes on $F \Rightarrow$ vanishes on E. Then $F \underset{\text{dense}}{\subseteq} E$

(3) Norm Vector Space:

i) Linear Span: $S = \{x_i\}_1^n$. $L(S)$ is the smallest linear space containing S. i.e. the intersection of L.S. containing S.

prop. $L(S) = \{\alpha \mid \alpha = \sum_1^n \lambda_i x_i,\ \lambda_i \in \mathbb{K}\}$.

pf: 1°) RHS is a linear space containing S.

2°) $\forall$ L.S containing S will contain RHS.

$\therefore L(S) = \{\alpha \mid \alpha = \sum_1^n \lambda_i x_i,\ \lambda_i \in \mathbb{K}\}$.

ii) Convex set: $S = \{x_i\}_1^n$. Denote $Conv(S)$ is the convex set generated by S. i.e. $x, y \in Conv(S) \Rightarrow ax + (1-a)y \in Conv(S)$.

prop. $Conv(\{x_i\}_1^n) = \{x \mid x = \sum_1^n a_i x_i,\ \sum_1^n a_i = 1\}$.

pf: By induction. for $\sum_1^{k+1} a_i = 1$.

$$\sum_1^{k+1} a_i x_i = a_{k+1} x_{k+1} + (1-a_{k+1}) \sum_1^k \frac{a_i}{1-a_{k+1}} x_i$$

since x_{k+1}, $\sum_1^k \frac{a_i}{1-a_{k+1}} x_i \in Conv(\{x_i\}_1^n)$

$\therefore \sum_1^{k+1} a_i x_i \in Conv(S)$.

① Banach Space:

• n.v.s. $(X, \|\cdot\|)$ is complete called Banach Space.

e.g. $\forall X$. n.v.s. X^* is Banach. ($\mathbb{K}$ is complete)

Next, we will complete a n.v.s X with initial norm $\|\cdot\|$.

i) Step one:

· Denote: $Z = \{(x_i) \mid x_i^n \in X.\ \{x_i^n\}_{n\in\mathbb{Z}^+} \text{ is Cauchy Seq}\}$.

$Y = \{(x_i) \mid x_i^n \to 0\ (n\to\infty).\ x_i^n \in X\}$.

$X_0 = \{(x_k) \mid x_k^n = x.\ \forall n.\ \text{for some } x\in X\}$.

Then: $Z.\ Y.\ X_0$ are linear space.

We claim: $Z/Y = \overline{X_0}$

ii) Step two:

· Define: $[(x_i)] \in Z/Y$ with norm $\|[(x_i)]\| = \lim\limits_{n\to\infty} \|x_i^n\|$.

check it's well-defined.

iii) Step three:

prove: $(Z/Y.\ \|\cdot\|)$ is a Banach Space

For Cauchy Seq $\{[(x_j)]\}_{j\in\mathbb{Z}^+}$ in Z/Y.

Check $[(x_i)] - [(x_j)] = [(x_i - x_j)]$ holds.

Since $\forall\ \varepsilon = \frac{1}{2^{p+1}}.\ \exists N_p.$ st. $i.j > N_p > N_{p-1}$

$\|[(x_i)] - [(x_j)]\| = \|[(x_i - x_j)]\| < \frac{1}{2^{p+1}}$

i.e. $\lim\limits_n \|x_i^n - x_j^n\| \le \frac{1}{2^{p+1}}$. $\therefore \exists n_p.$ st. $\|x_i^{n_p} - x_j^{n_p}\| < \frac{1}{2^p}$

Denote: $(x) = (x_{N_1}^{n_1}.\ x_{N_2}^{n_2} \cdots .\ x_{N_p}^{n_p}. \cdots)$. i.e. $x^k = x_{N_k}^{n_k}$.

$\Rightarrow (x)$ is a Cauchy Seq

Besides, $\lim\limits_i \|[(x_i)] - [(x)]\| = 0$

since $\|x_i^k - x^k\| \le \|x_i^k - x_i^{n_k}\| + \|x_i^{n_k} - x^k\|$

$\therefore [(x_i)] \longrightarrow [(x)]$ in $(Z/Y.\ \|\cdot\|)$

iv) Step Four:

Prove: $Z/Y = \overline{X_0}$

Since we have proved Z/Y is closed.

1') $X_0 \subseteq Z/Y \Rightarrow \overline{X_0} \subseteq Z/Y$.

2') $\forall [(x_i)] \in Z/Y$. $\exists \{[(\eta_n)]\}_{n\in Z^+}$. s.t.

$\eta_n^k = x_i^n$. $\forall k$. $\therefore [(\eta_n)] \to [(x_i)]$

$\therefore \overline{X_0} \supseteq Z/Y$.

③ Finite dimensional n.v.s:

Suppose X is a linear space. $\dim X = N$. $\{z_k\}_1^N$ is the Basis of X. For $x = \sum_1^N \alpha_i(x) z_i \in X$. we can define a norm $\|x\| = \sum_1^N |\alpha_i(x)|$

i) Lemma. $B(0,1) = \{x \mid \|x\| \le 1\}$ is cpt in X

Pf: $\forall (x_n) \in B(0,1)$. $\sum_1^N |\alpha_i(x_n)| \le 1$.

$\therefore \exists \{n_k\} \subseteq Z$. s.t. $\alpha_i(x_{n_k})$ converges $\forall 1 \le i \le N$.

$\therefore (x_{n_k})$ converges in $B(0,1)$

Lemma. All norms in X are equivalent with $\|\cdot\|$.

where $\|x\| = \sum_1^N |\alpha_i(x)|$.

Pf. $\|x\| = \| \sum_1^N \alpha_i(x) z_i \| \le \sum |\alpha_i(x)| \|z_i\|$

$\le \sup \|z_i\| (\sum_1^N |\alpha_i(x)|)$

Conversely. if $\forall n$. $\exists x_n$. s.t.

$\sum_1^N |\alpha_i(x_n)| \ge n \|x_n\|$. let $y_n = \frac{x_n}{\sum_1^N |\alpha_i(x_n)|}$

$\therefore \frac{1}{n} \geq \|\eta_n\| \cdot \sum_1^N |\alpha_i(\eta_n)| = 1$. Set $n \to \infty$. $\eta_n \to 0$.

but. $\exists (\eta_{n_k}) \subseteq (\eta_n)$. $\alpha_i(\eta_{n_k})$ converges. $\forall 1 \leq i \leq N$.

$\Rightarrow \sum_1^N |\alpha_k(\eta_{n_k})| = 1 \not\to 0$. Contradiction!

<u>Cor.</u> $\forall$ norm $\|\cdot\|$ in X. $\dim X < \infty$. $B_X = \{\|x\| \leq 1\}$ is cpt.

<u>Thm.</u> $\forall$ norm. $\|\cdot\|$. $(X, \|\cdot\|)$ is Banach.

<u>Pf:</u> $\|x_p - x_q\| \leq \varepsilon \Rightarrow \sum_1^N |\alpha_i(x_p) - \alpha_i(x_q)| \leq C\varepsilon$.

So $(\alpha_i(x_p))_p$ converges. $\forall 1 \leq i \leq N$, $\to (\alpha_i)_1^N$.

$\therefore x_p \to \sum_1^N \alpha_i z_i = \tilde{x} \in X$.

<u>Prop.</u> E is n.v.s. $X \subseteq E$. $\dim X < \infty$. subspace $\Rightarrow \bar{X} = X$.

<u>Pf:</u> $\forall (x_n) \subseteq X \to x$ in E. $\overset{(*)}{\Rightarrow} (x_n)$ is Cauchy.

So $(\alpha_i(x_n))_n$ are Cauchy. $x_n = \sum_1^N \alpha_i(x_n) z_i$.

$\Rightarrow \exists (x_{n_k}) \to x = \sum_1^N \alpha_i z_i \in X$.

<u>Rmk:</u> Remove "finite". it may not hold.

i) F n.v.s. $\dim F = \aleph_0 \Rightarrow F$ is not Banach.

<u>Pf:</u> $(f_n)_{n \in \mathbb{Z}}$ is basis of F. $F_n = \mathrm{span}(f_i)_1^n$.

$F = \bigcup F_n$. union of CLS. Apply Baire Thm.

if F is Banach. which runs into contradict!

$\Rightarrow F \subseteq E$. Banach. $\dim F = \aleph_0$. Then F isn't CLS.

ii) Convergent Net $\rightleftarrows$ Cauchy. generally. (*).

<u>Prop.</u> $\dim X < \infty$. F is Banach space. $T: X \to F$. linear.

$\Rightarrow T$ is BLO.

Pf. $\|Tx\| \le \sum_1^n |\tau_i(x)| \, \|Tz_i\| \le \max \|Tz_i\| \, C \sum_1^n |\tau_i(x)|$

$\le C\|x\|$. $\therefore T$ is BLO.

prop. (About Dual Space)

X is Banach Space st. X^* is finite dimensional

Then X is also finite dimensional. $\dim X = \dim X^*$.

Pf: Lemma. $\dim X < \infty \Rightarrow \dim X^* < \infty$

Pf. $X = \mathrm{span}\{e_i\}_1^n$. then $X^* = \mathrm{span}\{f_i\}_1^n$

where $f_i(x) = x_i$. $x = \sum_1^n x_i e_i$

$\therefore \dim X = \dim X^*$.

$\Rightarrow \dim X^* = \dim X^{**}$. and $X \simeq J(X) \subseteq X^{**}$.

$\therefore \dim X < \infty$ $\therefore \dim X = \dim X^*$. J is canonical injection.

Rmk: $\dim X = \infty \Rightarrow \dim X^* > \dim X$. may happen.

⑧ Infinite dimensional n.v.s:

i) prop. X is n.v.s. $\dim X = \infty$. Then $B(0,1)$ isn't cpt in X.

Cor. $B(0,1)$ is cpt in $X \Leftrightarrow \dim X < \infty$.

Riesz's Lemma: X is an n.v.s. $Y \subseteq X$ closed linear subspace.

$Y \neq X$. Then $\forall \varepsilon > 0$, $\exists u$. st. $\|u\| = 1$. and

$\mathrm{dist}(u, Y) \ge 1 - \varepsilon$.

Pf: $\exists v \in X/Y$. denote $d = \mathrm{dist}(v, Y)$

$\exists m \in Y$. st. $d \le \mathrm{dist}(v, m) \le \frac{d}{1-\varepsilon}$

Then $u = \frac{v-m}{\|v-m\|}$ is what we need.

Cor. If X is reflective. then $\varepsilon = 0$. also holds

Pf: Def f on $Y + \mathbb{R}x$. $x \in X/Y$.

$f(t) = 0$. $\forall t \in Y$. $f(kx) = k\, dist(x, Y)$

By Hahn-Banach Thm, extend f to $\tilde{f}$ on X. $\tilde{f} \neq 0$

Let $g = \frac{\tilde{f}}{\|\tilde{f}\|}$. $\therefore \|g\| = 1$. By James Thm.

$\exists\, x_0$. st. $|g(x_0)| = 1$. $\therefore |g(x_0)| = |g(x_0 - y)| \leq \|x_0 - y\|$.

for $\forall y \in Y$. $\therefore dist(x_0, Y) \geq 1$.

Return to the pf:

Since exist (E_n) seq of subspaces of X,

st. $E_n \subsetneqq E_{n+1}$. $\exists\, (u_n)$ seq of elements.

st. $u_n \in E_n$. $\|u_n\| = 1$. $dist(u_n, E_{n-1}) \geq \frac{1}{2}$

$\therefore \|u_n - u_m\| \geq \frac{1}{2}$. for $n > m$. which is divergent!

ii) Closed Linear Span:

· Def: $A = \{x_\theta\}_{\theta \in I}$. CLS of A is the smallest linear closed set containing A. i.e $\bigcap\limits_{\substack{A \subseteq F_\beta \\ L.S}} F_\beta$.

prop. $CLS(A) = \{ \sum_1^N \alpha_i x_{\theta_i} \mid N \geq 1.\ \alpha_i \in \mathbb{C}.\ \theta_i \in I \}$

Thm. $z \in A$. iff $\forall\, l \in X^*$. st. $l(x_\theta) = 0$. $\forall \theta \in I$.

$\Rightarrow l(z) = 0$.

Pf: $(\Rightarrow)$ $z = \lim\limits_{n \to \infty} z_n = \lim\limits_{N} \sum_1^{m_N} a_{N_j} x_{\theta_{N_j}}$

Then by l is conti $\therefore l(z) = 0$

($\Leftarrow$) Suppose $z \notin A$. Then def f on $A + \mathbb{R}z$

$f(A) = \{0\}$. $f(kz) = k\, \mathrm{dist}(z, A) \neq 0$.

By Hahn-Banach Thm extend to $\tilde{f}$ on X.

$\therefore \tilde{f} \in X^*$. which is a contradiction.

(4) Dual of n.v.s:

① Linear Function:

Def: l is LF. in X metrizable space.

l is conti $\Leftrightarrow \forall (x_n) \subseteq X \to x$. $l(x_n) \to l(x)$

l is bound $\Leftrightarrow \|l\| = \sup_{\|x\|=1} |l(x)| < \infty$

prop: l is a LF. then l is conti $\Leftrightarrow l$ is bound.

Pf: ($\Leftarrow$) $|l(x_n) - l(x)| \leq \|l\| \, \|x_n - x\|$

$\therefore (x_n) \to x \Rightarrow l(x_n) \to l(x)$.

($\Rightarrow$) If $\exists (x_n)$. st. $\|x_n\| = 1$. $\forall n \in \mathbb{Z}^+$.

$|l(x_n)| \geq n$. Since $|l(\frac{x_n}{\sqrt{n}})| \geq \sqrt{n}$.

$\frac{x_n}{\sqrt{n}} \to 0$. But $|l(\frac{x_n}{\sqrt{n}})| \to \infty$.

② Dual of X:

i) Def: $X^* = \{ f \mid f: X \to \mathbb{R}.\ \text{conti. LF} \}$.

$N_l = \{ z \in X \mid l(z) = 0 \}$. kernel of l.

Thm. For $l \in X^*$. N_l is closed linear space

If $l \neq 0$. Then $\exists x \in X$. for any $y \in X$.

$\exists \alpha \in \mathbb{K}$. $m \in N_l$. $y = \alpha x + m$.

Pf: $\exists x$. st. $l(x) \neq 0$. $\therefore \exists \alpha$. st. $l(y) = \alpha l(x)$.

Let $m = y - \alpha x$. $\in N_l$.

Cr. $X = Z \oplus N_l$. l is LF. $Z = \{\alpha x \mid \alpha \in \mathbb{K}\}$. $l(x) \neq 0$.

So codim $N_l = 1$. moreover. if $N_l = N_g$. Then $l = cg$.

ii) Dual of $C[a,b]$:

· Suppose X is cpt. separable measure space

For $(C(X), \|\cdot\|_{C(X)})$. $\|f\|_{C(X)} = \sup_{x \in X} |f(x)|$. and sign Borel

measure space $(M_X, \|\cdot\|_{M_X})$. $\|M\|_{M_X} = M^+(X) + M^-(X)$.

Thm. (Riesz Representation)

$C^*(X) = M_X$. i.e. $\forall l \in C^*(X)$. $l: C(X) \to \mathbb{R}$.

$\exists \nu \in M_X$. st. $\langle l, g \rangle = \int_X g \, d\nu$. $\forall g \in C(X)$

Besides. $\|\nu\| = \|l\|$.

Pf: Only consider the case $X = [a,b]$.

i°) $M_X \cong BV_X$

$\forall \nu \in M_X$. Def: $l(t) = \nu\{a\} + \nu(a,t]$.

since $\nu = \nu^+ - \nu^-$. difference of increasing

$\therefore l(t) \in BV_X$.

$\forall l \in BV_X$. Def: $\nu(c,d] = l(d) - l(c)$

$\nu[a,d] = l(d) - l(a)$

By Carathéodory, extend ν to σ-algebra M.

where M is generated by $\{(c,d]\} \cup \{[a,d]\}$.

Def: $\|\cdot\|$ in BV_X is $\|e\| = T_e[a,b]$. $\therefore \|\ell\| = \|\nu\|$.

2°) Extend ℓ on $C[a,b]$ to $B[a,b]$.

Since $C[a,b] \subseteq B[a,b]$. ℓ is BLF

By Hahn-Banach Thm. $\exists L$. $L|_{C[a,b]} = \ell$.

and $\|L\| = \|\ell\|$.

Next. we will consider L rather than ℓ.

3°) By Approximation. For $\chi_{[a,t]} \in B[a,b]$.

Def: $L(\chi_{[a,t]}) = \nu[a,t] \triangleq \ell(t)$.

prove: $\ell(t) \in BV_{[a,b]}$.

For $\{t_i\}$ partition of $[a,b]$. $s_i = \operatorname{sgn}(\ell(t_{i+1}) - \ell(t_i))$

$\sum |\ell(t_i) - \ell(t_{i-1})| = \sum s_{i-1}(\ell(t_i) - \ell(t_{i-1}))$

$= \sum s_{i-1} L(\chi_{(t_{i-1}, t_i]}) = L(\sum s_i \chi_{(t_{i-1}, t_i]}) \le \|L\|$.

Since $k(x) = \sum s_i \chi_{(t_{i-1}, t_i]} \le 1$. $\therefore \ell \in BV_X \Rightarrow \nu \in M_X$.

4°) Check: $L(f) = \int_X f \, d\nu$. for $\forall f \in C[a,b]$.

$$L(f) = \lim_{\|T\| \to 0} L\left(f(a)\chi_{[a,t_1]} + \sum_1^{n-1} f(t_i)\chi_{(t_i, t_{i+1}]}\right)$$

$$= \lim_{\|T\| \to 0} \left[f(a)\ell(a) + \sum f(t_i)(\ell(t_{i+1}) - \ell(t_i)) \right]$$

$$= \int_X f \, d\ell = \int_X f \, d\nu. = \ell(f). \; \forall f \in C(X).$$

5°) $\|\ell\| = \|\nu\|$.

$|\langle \ell, f \rangle| = |\int_a^b f \, d\nu| \le \|f\| \, \|\nu\|$, $\forall f \in C(X)$.

$\therefore \|\ell\| \le \|\nu\|$

Conversely. since $L(\chi_{[a,t]}) = \ell(t)$.

$\|\ell\| = \sup_{\|T\|\to 0} \sum |\ell(t_i) - \ell(t_{i-1})| \leq \sup |L(x)| \leq \|L\|$.

$\therefore \|\ell\| = \|v\| \leq \|L\| = \|\ell\|$.

③ Extension of BLF's:

prop. For $\{x_i\}_1^N \subseteq X$. n.v.s. $\{\alpha_k\}_1^N \subseteq \mathbb{R}$. $\exists L : X \to \mathbb{R}$.

L is BLF. s.t. $L(x_i) = \alpha_i$. $\forall 1 \leq i \leq N$.

Pf: Consider the subspace $Y = \text{span}\{x_i\}_1^N$. Let $l(x_i) = \alpha_i$

$l : Y \to \mathbb{R}$. $|l(\sum_1^N \beta_i x_i)| \leq \max|\alpha_i| (\sum |\beta_i|) \leq c\|x\|$.

$\therefore l$ is BLF. By Hahn-Banach Thm. ✓.

Cor. $Y \subseteq X$. $\dim Y < \infty$. Then exists closed linear space M.

s.t. $X = Y \oplus M$. (Finite dimension admits complement)

Pf: $\{x_i\}_1^N$ is Basis of Y. Let $L_i(x_j) = \delta_{ij}$.

$\therefore M = \bigcap_1^N N_{L_i}$ is Closed. Check $X = Y \oplus M$.

④ Norm in subspace:

$Y \subseteq X$. linear subspace. $l \in X^*$. Then $\|l\|_{Y^*} = \sup_{\substack{\|y\|=1 \\ y \in Y}} |l(y)|$

$= \inf_{m \in Y^\perp} \|l - m\|_{X^*} = \text{dist}(l, Y^\perp)$

Pf: $|l(y)| = |l(y) - m(y)| \leq \|l - m\|_{X^*}$

For the converse: $l|_Y : Y \to \mathbb{R}$ is BLF

$\|l|_Y(z)\|_{Y^*} \leq \|l|_Y\|_{Y^*}\|z\| = p(z)$. By Hahn-Banach Thm.

$\exists L$ on X. s.t. $\|L\|_{X^*} \leq \|l\|_{Y^*}$ Let $m_0 := l - L$.

$\therefore \exists m_0$. s.t. $\|l - m_0\|_{X^*} = \|L\|_{X^*} \leq \|l\|_{Y^*}$

(5) Bidual and Orthogonality:

① E is n.v.s. Bidual E^{**} is dual of E^*. with norm:

$$\|\xi\|_{E^{**}} = \sup_{\substack{f \in E^* \\ \|f\| \le 1}} |\langle \xi, f \rangle|.$$

<u>Def</u>: Canonical injection: $J: E \longrightarrow E^{**}$. satisfies
$x \longmapsto Jx$.

$\langle Jx, f \rangle = \langle f, x \rangle, \ \forall f \in E^*.$

$\Rightarrow J$ is linear. isometry. $\|Jx\|_{E^{**}} = \|x\|_E$.

② <u>Def</u>: For $M \subseteq E$. linear subspace.

Set $M^\perp = \{ f \in E^* \mid \langle f, x \rangle = 0, \ \forall x \in M \}$.

For $N \subseteq E^*$. linear subspace. $\Rightarrow$ Both $M^\perp, N^\perp$ are closed!

Set $N^\perp = \{ x \in E \mid \langle f, x \rangle = 0, \ \forall f \in N \}$.

<u>Remark</u>: Note that $N^\perp \subseteq E$ rather than E^{**}

we may have following proposition.

<u>prop</u>. i) $(M^\perp)^\perp = \overline{M}$

ii) $(N^\perp)^\perp \supseteq \overline{N}$

<u>Pf</u>: $\because (M^\perp)^\perp \supseteq M$, $(N^\perp)^\perp \supseteq N$.

$\therefore (M^\perp)^\perp \supseteq \overline{M}$, $(N^\perp)^\perp \supseteq \overline{N}$

If $\exists x_0 \in (M^\perp)^\perp$, $x_0 \notin \overline{M}$. By Hahn-Banach.

$\therefore \langle f, x \rangle < \alpha < \langle f, x_0 \rangle$. $\forall x \in \overline{M}$. $\langle f, x \rangle = 0$.

$\therefore \langle f, x_0 \rangle > 0$. which is a contradiction!

<u>Remark</u>: $(N^\perp)^\perp = \overline{N}$ in $\sigma(E^*, E)$

If E is reflexive. Then $(N^\perp)^\perp = \overline{N}$

(6) Conjugate Convex Functions:

· Denote: i) $\varphi: E \longrightarrow (-\infty, +\infty]$. $D(\varphi) = \{x \in E \mid \varphi(x) < +\infty\}$.

ii) Epigraph of φ is: $epi(\varphi) = \{(x, \lambda) \in E \times \mathbb{R},\ \varphi(x) \le \lambda\}$.

i.e. $epi(\varphi) = \bigcup_{\lambda \in \mathbb{R}} \{\varphi \le \lambda\} \times \{\lambda\}$.

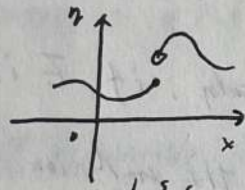
Next, we suppose E is topo space.

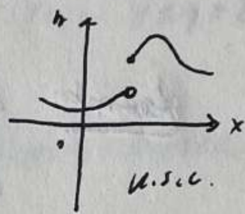
① LSC and USC Functions:

· Def: $\varphi: E \longrightarrow (-\infty, +\infty]$ is l.s.c $\Leftrightarrow \forall \lambda \in \mathbb{R}$, $[\varphi \le \lambda]$ is closed.

$\varphi: E \longrightarrow [-\infty, +\infty)$ is u.s.c $\Leftrightarrow \forall \lambda \in \mathbb{R}$ $[\varphi \ge \lambda]$ is closed.

e.g.





It won't change eruptly over lower / upper semi-part.

Remark: f is l.s.c and u.s.c $\Leftrightarrow$ f is conti

Pf: $(\Rightarrow)$. $[f \le a]$, $[f \ge b]$. are closed

So $[b \le f \le a] = [f \le a] \cap [f \ge b]$ close.

Since $\{(-\infty, a]\} \cup \{[b, +\infty)\} \cup \{[a, b]\}$ generate closed set in $\mathbb{R}^1$.

properties: i) l.s.c / u.s.c Function forms a linear space.

ii) φ is l.s.c $\Leftrightarrow$ $epi(\varphi)$ is closed in $E \times \mathbb{R}$

Pf: $\forall (x, \lambda) \in epi(\varphi)^c$. Then $\varphi(x) > \lambda$.

$\exists \varepsilon > 0$. st. $\varphi(x) > \lambda + \varepsilon$. Besides, $\exists U_x$ of x. st.

$\forall y \in U_x$. $\varphi(y) > \lambda$, Since φ is l.s.c.

Then $(x, \lambda) \in U_x \times (\lambda - \varepsilon, \lambda + \varepsilon) \subseteq epi(\varphi)^c$. Converse is Similar.

iii) φ is l.s.c $\iff \forall x \in E.\ \forall \varepsilon > 0.\ \exists U_x$ s.t.

$\forall \eta \in U_x.\ \varphi(\eta) \geq \varphi(x) - \varepsilon.$

Pf: $(\Rightarrow)$ $x \in \{\varphi > \varphi(x) - \varepsilon\}$ open.

$(\Leftarrow)$ For $x \in [\varphi > \lambda]$. $\exists \varepsilon > 0$. s.t. $\varphi(x) > \lambda + 2\varepsilon$.

Then $U_x \subseteq [\varphi > \lambda]$. Since $\eta \in U_x \Rightarrow \varphi(\eta) > \lambda + \varepsilon$.

Cor. $\forall \{x_n\} \subseteq E$. $x_n \to x$. Then $\underline{\lim}_{n\to\infty} \varphi(x_n) \geq \varphi(x)$.

for φ is l.s.c.

Pf: $\exists U_n$. s.t. $x \in U_n$. $\forall \eta \in U_n$. $\varphi(\eta) > \varphi(x) - \frac{1}{n}$

Then For $x_n \to x$. $\exists n_k$. $\varphi(x_{n_k}) > \varphi(x) - \frac{1}{k}$

Remark: Conversely, if E is metrizable

under the condition. Then φ is l.s.c.

Pf: $\forall \{x_n\} \subseteq [\varphi \leq \lambda]$. s.t. $\varphi(x_n) \leq \lambda$.

$x_n \to x$. Then $\varphi(x) \leq \underline{\lim} \varphi(x_n) \leq \lambda$.

iv) If E is cpt. φ is l.s.c. Then $\inf_E \varphi$ can be achieved.

Pf: Lemma $\inf_E \varphi > -\infty$.

Pf: suppose $\inf_E \varphi = -\infty$. By contradiction:

Let $Y_0 = [0 < \varphi \leq \infty]$. $Y_n = [-n < \varphi \leq -n+1]$ open.

$\therefore E = \bigcup_0^\infty Y_n$. By cpt $E = \bigcup_1^N Y_{n_k}$ $(*)$

$\therefore \varphi(x) > -N$. contradict!

$(*)$. By iii). $\forall x \in Y_n$. $\exists U_x$. neighbour

s.t. $\forall \eta \in U_x$. $\varphi(\eta) \geq \varphi(x) - \varepsilon > n$.

Let $\varepsilon = \frac{\varphi(x) - n}{2}$ $\therefore Y_n = \bigcup_{x \in Y_n} U_x$

$\Rightarrow$ Analogously. suppose φ can't attain $\inf_E \varphi \hat{=} c$

Then $E = [c < \varphi] = (\bigcup_1^{\infty} [c+\frac{1}{n} < \varphi \le c+\frac{1}{n-1}]) \cup [1 < \varphi \le \infty]$. By cpt.

$\therefore E = (\bigcup_1^{N} [c+\frac{1}{n_k} < \varphi \le c+\frac{1}{n_k-1}]) \cup Y_0$. which is same contradiction!

<u>Remark:</u> For u.s.c Function, it also has the dual properties. Note that u.s.c Func can attain suprum on cpt set. That's why Conti Func can attain extremum on cpt set!

② <u>Convex Functions:</u>

<u>Def:</u> $\varphi: E \longrightarrow (-\infty, +\infty]$ is convex if

$\varphi(tx+(1-t)y) \le t\varphi(x) + (1-t)\varphi(y)$. $\forall x, y \in E$. $\forall t \in (0,1)$

<u>properties:</u> i) φ is convex $\Leftrightarrow$ $epi(\varphi)$ is convex.

<u>Pf:</u> ($\Rightarrow$) Check ($\Leftarrow$) $(x_1, \varphi(x_1))$, $(x_2, \varphi(x_2)) \in epi(\varphi)$

ii) φ is convex $\Rightarrow$ $\forall \lambda \in \mathbb{R}$. $[\varphi \le \lambda]$ is convex.

iii) Convex Functions form a linear space.

iv) $(\varphi_i)_{i\in I}$ family of convex Func's. Then $\sup_{i\in I} \varphi_i$ is convex. (By $\sup f+g \le \sup f + \sup g$)

③ <u>Conjugate Functions:</u>

· Suppose E is an n.v.s.

<u>Def:</u> $\varphi: E \longrightarrow (-\infty, +\infty]$. $\varphi \not\equiv +\infty$. Its Conjugate Func.

$\varphi^* = \sup_{x\in E} \{\langle f, x\rangle - \varphi(x)\} : E^* \longrightarrow (-\infty, +\infty]$.

Remark: i) From: $\langle f, x\rangle \le \varphi(x)+\varphi^*(f)$. $\forall x\in E$. $f\in E^*$.

It's called Young's Inequility. (Let $f=|x|^p/p$)

ii) φ^* is convex and l.s.c. since For $\forall$ fixed $x\in E$. $\langle f,x\rangle-\varphi(x)$ is convex. l.s.c. on E^* (conti actually). Then take sup envelope.

prop. $\varphi: E\longrightarrow(-\infty,+\infty]$. convex. l.s.c. Then we have:

$\varphi\not\equiv+\infty \Rightarrow \varphi^*\not\equiv+\infty$.

Pf: Apply Hahn-Banach Thm on $epi(\varphi)$ and $\{(x_0,\lambda_0)\}$ where $x_0\in D(\varphi)$. $\exists\lambda_0$. $\lambda_0<\varphi(x_0)$

Let $f(x)=\varphi[x,0]$. Note that $D(\varphi)\times\{\infty\}\subseteq epi(\varphi)$.

$\varphi^*(f)=\sup\limits_{x\in D(\varphi)}\{f(x)-\varphi(x)\}$, actually!

Def: $\varphi^{**}: E\longrightarrow(-\infty,+\infty]$. $\varphi^{**}(x)=\sup\limits_{f\in E^*}\{\langle f,x\rangle-\varphi^*(f)\}$

$=\sup\limits_{f\in D(\varphi^*)}\{\langle f,x\rangle-\varphi^*(f)\}$. $\forall x\in E$

Thm. (Frenchel-Moreau)

$\varphi: E\longrightarrow(-\infty,+\infty]$ convex. l.s.c. $\varphi\not\equiv+\infty$.

Then $\varphi^{**}=\varphi$.

Pf. 1°) Under $\varphi\ge0$:

Note that $\langle f,x\rangle\le\varphi^*(f)+\varphi(x)$.

$\therefore \varphi(x)\ge\varphi^{**}(x)$

For the converse, by contradiction. Let $\varphi(x_0)>\varphi^{**}(x_0)$.

Apply Hahn-Banach on $epi(\varphi)$ and $(x_0, \varphi^{**}(x_0))$

use the def of φ^*, φ^{**}. contradict with itself.

2°) General Case:

Let $\bar{\varphi}(x) = \varphi(x) - \langle f_0, x \rangle + \varphi^*(f_0) \geq 0$. l.s.c. convex.

where $f_0 \in D(\varphi^*)$. Since $\varphi \not\equiv +\infty \Rightarrow \varphi^* \not\equiv +\infty$.

<u>Thm.</u> (Fenchel-Rockafeller)

$\varphi, \psi: E \longrightarrow (-\infty, +\infty]$. convex. If $\exists x_0 \in D(\varphi) \cap D(\psi)$

s.t. φ is conti at x_0. Then. We obtain:

$$\inf_{x \in E} \{\varphi + \psi\} = \sup_{f \in E^*} \{-\varphi^*(-f) - \psi^*(f)\} = \max_{f \in E^*} \{-\varphi^*(f) - \psi^*(f)\}$$

<u>Lemma.</u> If C is convex in E. n.v.s. Then $\bar{C}$ and

int C are both convex.

<u>Pf:</u> $\forall x, y \in \bar{C}$. $tx + (1-t)y \leftarrow tx_n + (1-t)y_n$.

where $\{x_n\} \to x$. $\{y_n\} \to y$. $\{x_n\}, \{y_n\} \subseteq C$.

$\forall x, y \in$ int C. $\exists B(x, r)$. $B(y, r) \subseteq C$

Then $tx + (1-t)y \in B(tx + (1-t)y, r) = tB(x, r) + (1-t)B(y, r) \subseteq C$.

<u>e.g.</u> Denote $I_K = \begin{cases} \infty, & x \in K^c \\ 0, & x \in K \end{cases}$ $\therefore I_K^* = I_{K^\perp}$ if K is a

linear subspace. We can obtain: for $K \neq \emptyset$. convex.

$$dist(x_0, K) = \inf_{x \in K} \|x - x_0\| = \inf_{x \in E} \{\|x - x_0\| + I_K\} = \max_{\substack{f \in E^* \\ \|f\| \leq 1}} (\langle f, x_0 \rangle - I_K^*(f)).$$

If K is linear subspace Then $dist(x_0, K) = \max_{\substack{f \in K^\perp \\ \|f\| \leq 1}} \langle f, x_0 \rangle$

It's dual with $dist(l, Y) = \sup_{\substack{\|y\| \leq 1 \\ y \in Y^\perp}} |l(y)|$. $l \in E^*$. $Y \underset{l.s}{\subseteq} E^*$.