Integration

For $U \underset{\text{open}}{\subseteq} \mathbb{R}^n$. $h \in C^\infty(U)$. if $F: U \xrightarrow{\sim} V$. diffeomorphism.

We have: $\int_U h \, dx_1 \cdots dx_n = \int_V (h \circ F) \, |\mathrm{Det}\, DF| \, dx_1 \cdots dx_n$. ($U$ connected)

Written in n-form: $\int_V F^* \tau = \mathrm{sgn}(|DF|) \int_U \tau$.

For arbitrary manifold X:

Define: $\int_X \tau \in \mathbb{R}^1$ for $\tau \in \Lambda^n(X)$.

Pick $(U, f) \in A_X$. $f: U \cong \tilde{U}$. Then integrate it on $\tilde{U}$.

$\Rightarrow$ problems $\begin{cases} \text{(1) Integral may not be coordinate indept.} \\ \text{(2) Integral may not be convergent.} \end{cases}$

(1) <u>Oritentions</u>:

· Note that for $\alpha \in \Lambda^n(X)$. (U_1, f_1). $(U_2, f_2) \in A_X$.

WLOG. $U_1 = U_2 = U$. Then $\tilde{\alpha}_1 = \phi_{21}^* \tilde{\alpha}_2$. $\phi_{21}: \tilde{U}_1 \cong \tilde{U}_2$

$\therefore \int_{\tilde{U}_1} \tilde{\alpha}_1 = \mathrm{sgn}(|D\phi_{12}|) \int_{\tilde{U}_2} \tilde{\alpha}_2$

It's not be cordnate indept.

<u>Def</u>: $\omega \in \Lambda^n(X)$ is called volum form if $\omega \neq 0$.

If X has an volum form. Then say it's orientable.

Remark: i) by conti. It said $w>0$ or $w<0$

ii) If we have a volum form: w.

$h \in C^\infty(X)$. $h \neq 0$. Then so hw does.

prop. X is orientable n-dim manifold. $Z = h^{-1}(y) \subset X$

for $h \in C^\infty(X)$. level set at regular value y.

Then Z is orientable.

Pf: Suppose w is volum form on X.

Fix $z \in Z$. $\therefore w|_z \neq 0 \in \Lambda^n T_z^* X$.

1°) $Dh|_z : T_z X \longrightarrow T_z \mathbb{R} \cong \mathbb{R}^1$. Surjective.

$\ker Dh|_z = T_z Z$. Choose $\vec{n} \in T_z X$. st. $Dh|_z \cdot \vec{n} = 1$.

Find basis $(e_k)_1^m$ for $T_z Z$.

$\therefore (e_k)_1^m \cup \{\vec{n}\}$ is basis of $T_z X$.

2°) Define: $w'|_z : (v_1, \cdots v_{m}) \longmapsto w|_z(v_1, \cdots v_{m}, n)$

$\forall v_k \in T_z Z$. Check: $w'|_z \in \Lambda^m T_z^* Z$.

It's indept with choice of $\vec{n}$:

since $w|_z(e_1, \cdots e_m, e_k) = 0$. $\forall 1 \leq k \leq m$.

$\therefore$ It makes no difference to consider: $\vec{n} + \ker Dh|_z$.

3°) $w'|_z \neq 0$. since $\{e_k\}_1^m \cup \{\vec{n}\}$ is basis of $T_z X$.

$\therefore w|_z(e_1, \cdots e_m, \vec{n}) = w'|_z(e_1, \cdots e_m) \neq 0$.

4°) w' is smooth $\in \Lambda^m T^* Z$.

WLOG. $y=0$. $\forall z \in Z$. Choose $(U, f) \in A_z$. $z \in U$.

s.t. $\tilde{h} = h \circ f^{-1} = x_n$ in $\tilde{U}$. by IFT. ($h \circ f^{-1} = \pi$).

$\therefore f(Z \cap U) = f \circ h^{-1}\{0\} \cap \tilde{U} = \mathbb{R}^{n-1} \cap \tilde{U} = \{x_n = 0\} \cap \tilde{U}$.

Then choose $\vec{n} = (0 \cdots \partial/\partial x_n) \in T_{\tilde{z}}\tilde{U}$. $D\tilde{h}|_{\tilde{z}} \cdot \vec{n} = 1$. $\forall z \in Z$.

Written in coordinate: $\tilde{w} = \tilde{g}\, dx_1 \wedge \cdots dx_n$. $\tilde{g} \in C^\infty(\tilde{U})$.

$\Rightarrow \tilde{w}' = \tilde{g}|_{\{x_n=0\}}\, dx_1 \wedge \cdots dx_{n-1}$. smooth. $\square$

<u>Cor.</u> For $h: X \longrightarrow Y$. smooth between orientable manifolds X, Y. Then $Z = h^{-1}(y) \subset X$ is orientable for regular value $y \in Y$.

<u>Pf:</u> Fix z: $Dh|_z : T_z X \longrightarrow T_{h(z)} Y$. $\dim X = n$. $\dim Y = k$

$\ker Dh|_z = T_z Z$. Choose basis of $T_{h(z)}Y$: $(n_i)_1^k$.

With basis of $T_z Z$: $(e_i)_1^{n-k}$.

$\Rightarrow (e_i)_1^{n-k} \cup (n_i)_1^k$ is basis of $T_z X$.

Suppose $w \in \Lambda^n TX$. $w \neq 0$. Define:

$w'|_z : (v_1 \cdots v_{n-k}) \longmapsto w|_z (v_1 \cdots v_{n-k}, n_1 \cdots n_k)$.

Check $w'|_z \in \Lambda^{n-k} T_z^* Z$. $w' \neq 0$. smooth.

<u>Remark:</u> i) Not every submanifold of orientable manifold is orientable.

There exists nonorientable manifolds:

<u>e.g.</u> Klein bottle. Mobiüs band.

ii) For application: $X = \mathbb{R}^{n+1}$ is orientable ($\bigwedge^{n} dx_k$)

$\therefore S^n, T^n$ are orientable. ($T^n \cong T^1 \times \cdots T^1$)

Def. $w \in \Lambda^n(X)$ is a volum form. For $(U, f) \in A_X$ is said oriented. if: w_0 is standard volum form of $\tilde{U}$. when write w in chart: $\tilde{w} = h w_0$. $h > 0$. $\forall x \in \tilde{U}$.

Remark: It's not hard to find oriented chart: for U is connected. If $h < 0$. Then: Choose $F: (x_1 \cdots x_n) \longmapsto (-x_1, \cdots x_n)$.
$\therefore (U, F \circ f)$ is oriented.

Def: An oriention of X is an equivalent class of volum forms on X. i.e. $w_1 \in [w] \Leftrightarrow$ $\exists g \in C^\infty(X)$. $g > 0$. st. $gw = w_1$.

If we fix an oriention on X. Then say X is oriented.

Remark: A manifold has two orientions or no oriention.

prop. X is oriented manifold. For (U_1, f_1). $(U_2, f_2) \in A_X$. oriented charts. Then $\forall \tilde{x} \in f_2(U_1 \cap U_2)$. we have: $D\phi_{12}|_{\tilde{x}} > 0$.

Pf: Pick $w \in [w]$. from orientation.

$\tilde{w}_1 = h_1 w_0 \in \Lambda^n(\tilde{U}_1)$, $\tilde{w}_2 = h_2 w_0 \in \Lambda^n(\tilde{U}_2)$.

$h_1, h_2 > 0$. By pull-back along ϕ_{12}:

$h_2|_{f_2(x)} = Det(D\phi_{12}|_{f_2(x)}) h_1|_{f_1(x)} \Rightarrow |D\phi_{12}|_{f_2(x)}| > 0$.

Remark: This solves problem (1). We only consider in oriented charts of oriented manifold.

(2) Integration:

① Bump Forms:

Def: $0 \leq p \leq n$. $\tau \in \Lambda^p(X)$. We call τ a bump form if $\exists (U, f) \in A_X$. some cpt set $W \subset U$. s.t. $\alpha \equiv 0$ outside W

Remark: For (U, f) is oriented. We can compose it with reflection: $F: (x_1, \cdots x_n) \longmapsto (-x_1, \cdots x_n)$

prop. (U_1, f_1). $(U_2, f_2) \in A_X$. two oriented charts s.t. $\exists W_1, W_2$ cpt set. α vanishes outside W_1, W_2. Then $\int_{\tilde{U}_1} \tilde{\alpha}_1 = \int_{\tilde{U}_2} \tilde{\alpha}_2$.

Pf: $U = U_1 \cap U_2$. $W = W_1 \cap W_2$. $\alpha \equiv 0$ on W^c.

$\therefore \int_{\tilde{U}_1} \tilde{\alpha}_1 = \int_{f_1(U)} \tilde{\alpha}_1 = \int_{f_2(U)} \tilde{\tau}_2 = \int_{\tilde{U}_2} \tilde{\tau}_2$.

since $\tilde{\alpha}_2 = \phi_{12}^* \tilde{\tau}_1$

Remark: So for any bump form $\alpha \in \Lambda^n(X)$.

We have a well-def $\int_X \alpha$ by

using charts.

② For arbitrary n-forms:

Def: A partition of Unity on smooth manifold

X is set of func $\varphi_\bullet = \{\varphi_i\}_{i\in I} \subset C^\infty(X)$.

st. i) φ_i is bump form. $\forall i \in I$.

ii) $\forall x \in X$. There're only finite $i \in I$. st. $\varphi_i(x) \neq 0$.

iii) $\sum_{i\in I} \varphi_i(x) = 1$. $\forall x \in X$.

Remark: Given any chart A_X. There exists a POU.

subordinate it for any manifold X.

prop. X is cpt manifold $\Leftrightarrow$ There exists a POU

$\varphi_\bullet = \{\varphi_i, i \in I\}$ on X. where I is finite.

pf: ($\Leftarrow$) $X = \bigcup_1^n \operatorname{supp}(\varphi_i) = \bigcup_1^n W_i$. W_i's are cpt.

($\Rightarrow$) For any $x \in X$. Find bump Func. ψ_x.

st. $\exists$ nbd V_x of x. $\psi_x \equiv 1$ in V_x.

$\exists$ cpt $W \subseteq X$. $\psi_x \equiv 0$ outside W.

$X \subseteq \bigcup V_x \Rightarrow \exists (V_{x_i})_1^n$ cover X.

Set $\varphi_i = \psi_{x_i} / \sum_1^n \psi_{x_k} \in C^\infty(X)$.

Remark: since $\alpha = \sum_i^n \varphi_i \alpha$. Then we denote:

$$\int_X^{\varphi_\cdot} \alpha = \sum_i^n \int_X \varphi_i \alpha.$$

prop. X is cpt. oriented manifold. $\varphi_\cdot = \{\varphi_i\}_1^r$. $\hat{\varphi}_\cdot = \{\hat{\varphi}_i\}_1^s$

are two finite POU of X. Then for any $\alpha \in \Lambda^n(X)$.

we have: $\int_X^{\varphi_\cdot} \alpha = \int_X^{\hat{\varphi}_\cdot} \alpha$.

Pf: 1°) For bump form β. $\varphi_\cdot$ is finite POU.

Then $\int_X \beta = \int_X^{\varphi_\cdot} \beta$:

Pf: For each $\varphi_i \beta$. it's bump form.

Written in charts:

$$\int_X \beta = \int_{\tilde{U}} \tilde{\beta} = \int_{\tilde{U}} \sum_1^n \tilde{\varphi}_i \tilde{\beta} = \sum_1^n \int_{\tilde{U}} \tilde{\varphi}_i \tilde{\beta} = \int_X^{\varphi_\cdot} \beta.$$

2°) It follows from the claim:

$$\int_X^{\varphi_\cdot} \alpha = \sum_1^n \int_X \alpha \varphi_i = \sum_1^n \int_X^{\hat{\varphi}_\cdot} \varphi_i \alpha = \sum_i^n \sum_j^n \int_X \hat{\varphi}_j \varphi_i \alpha.$$

$$= \sum_1^n \int_X^{\varphi_\cdot} \hat{\varphi}_i \alpha = \int_X^{\hat{\varphi}_\cdot} \alpha.$$

Remark: Then we can define a well-def

integration: $\int_X \alpha = \int_X^{\varphi_\cdot} \alpha$ on every

cpt. oriented manifold X.

(3) Stokes' Thm:

① For cpt oriented manifold X:

· Note that for $w \in \Lambda^n(X)$, a volum form. Fix orientation $[w]$. Then $\int_X w > 0$.

$\therefore \int_X : \Lambda^n(X) \longrightarrow \mathbb{R}^1$ defines a surjective linear map. (Note: $\lambda w \in \Lambda^n(X)$, $\forall \lambda \in \mathbb{R}^1$).

We claim: $d\alpha \in \ker \int_X$, $\forall \alpha \in \Lambda^{n-1}(X)$.

i) For $U \underset{\text{op}}{\subseteq} \mathbb{R}^n$:

Lemma. For any $\alpha \in \Lambda^{n-1}(U)$ vanishes outside a cpt set $W \subseteq U$. Then $\int_U d\alpha = 0$.

Pf: Suppose $\alpha = \alpha_1 dx_2 \wedge \cdots \wedge dx_n$.

$\exists [-r,r]^n \subseteq U$. s.t. $W \subseteq [-r,r]^n$.

$$\therefore \int_U d\alpha = \int_{[-r,r]^n} \frac{\partial \alpha_1}{\partial x_1} dx_1 \wedge \cdots dx_n.$$

$$= \int_{[-r,r]^{n-1}} (\alpha_1|_{x_1=r} - \alpha_1|_{x_1=-r}) = 0.$$

Since $\alpha_1 \equiv 0$ outside $[-r,r]^n$

ii) X is cpt. oriented manifold:

Thm. $\int_X d\alpha = 0$. for $\forall \alpha \in \Lambda^{n-1}(X)$. ($\dim X = n$).

Pf: $\exists$ POU. $\varphi_\cdot = \{\varphi_i\}_1^n$.

$\therefore d\alpha = \sum_1^n d(\alpha \varphi_i)$. sum of bump forms.

written in charts. Reduce to i).

$\Rightarrow$ Note that $H_{dR}^n(X) = \Lambda^n(X) / \{\text{exact } (n-1)\text{-forms}\}$.

Stokes's Thm said: $\int_X : H_{dR}^n(X) \longrightarrow \mathbb{R}^1$ well-def.

Remark: For X is connected additionally.

Then: $\int_X : H_{dR}^n(X) \cong \mathbb{R}^1$.

② Manifold with boundary:

i) For $U \underset{open}{\subseteq} \mathbb{R}_{\leq 0} \times \mathbb{R}^{n-1}$:

$\partial U = U \cap \{x_1 = 0\}$ is its boundary.

Denote: $\iota : \partial U \hookrightarrow U$. inclusion.

Note that $T_z \partial U = \{x_1 = 0\} \subset \mathbb{R}^n = T_z U$.

For $\forall \alpha = a_1 dx_2 \wedge \cdots dx_n + a_2 dx_1 \wedge dx_3 \cdots + \cdots \in \Lambda^{n-1}(U)$

Pull back α along ι: $\iota^* \alpha = a_1|_{\{x_1=0\}} dx_2 \wedge \cdots dx_n \in \Lambda^{n-1}(\partial U)$

Claim: $\int_U d\alpha = \int_{\partial U} \iota^* \alpha$. $\forall$ suppα is cpt. $\subset U$.

<u>Pf</u>: 1°) Consider the integration on $[-r,0]\times[-r,r]^{n-1}$

for large enough $r\in\mathbb{R}^+$, rather than U.

2°) For $\alpha=\alpha_k\, dx_1\wedge\cdots dx_{k-1}\wedge dx_{k+1}\cdots dx_n$. $\iota^*\alpha=0$.

$$\int_{[-r,0]\times[-r,r]^{n-1}} d\alpha = \int_{[-r,0]\times[-r,r]^{n-2}}\int_{-r}^{r}\left(\frac{\partial\alpha}{\partial x_k}dx_k\right)dx_1\cdots = 0$$

3°) For $\alpha=\alpha_1\, dx_2\wedge dx_3\cdots dx_n$.

$$\int_{[-r,0]\times[-r,r]^{n-1}} d\alpha = \int_{[-r,r]^{n-1}}\left(\int_{-r}^{0}\frac{\partial\alpha_1}{\partial x_1}dx_1\right)dx_2\cdots dx_n$$

$$=\int_{[-r,r]^{n-1}}\alpha_1|_{\{x_1=0\}}\,dx_2\cdots dx_n$$

$$=\int_{\partial U}\iota^*\alpha.$$

ii) <u>For manifold-with-boundary X</u>:

<u>prop</u>. If X is oriented manifold with boundary

Then there's a canonical orientation on

boundary ∂X, which is same as X.

<u>Pf</u>: For $\omega\in\Omega^n(X)$, volum form.

Under chart (U,f), $\partial\tilde{U}=\tilde{U}\cap\{x_1=0\}$.

$\tilde{\omega}=h\, dx_1\cdots\wedge dx_n$, $h\in C^\infty(\tilde{U})$.

Choose $\xi=(\xi_1,\cdots\xi_n):\partial\tilde{U}\to\mathbb{R}^n$, $\xi_1>0$

ξ is a vector field on $\partial\tilde{U}$.

Contract $\tilde{\omega}$ with ξ: $i_\xi \tilde{\omega} = h\xi_1 \, dx_2 \wedge \cdots \wedge dx_n - h\xi_2 \, dx_1 \wedge dx_3 \cdots + \cdots$

$i_\xi \tilde{\omega} \in \Lambda^{n-1}(\tilde{U})$. Pull-back to $\partial \tilde{U}$: $\iota^*(i_\xi \tilde{\omega}) = h\xi_1 \, dx_2 \wedge \cdots \wedge dx_n$

$\therefore$ $\iota^*(i_\xi \tilde{\omega}) \in \Lambda^{n-1}(\partial \tilde{U})$. Volum form. induced by $f|_{\partial X}$

Besides. its orientation is indept with ξ.

For the whole ∂X: Construct ξ by POU.

<u>Remark:</u> For (U, f) is oriented chart. Then

$(\partial U, f|_{\partial U})$ on ∂U is also oriented.

<u>Thm.</u> (Full version)

X is cpt. oriented manifold with boundary.

$\dim X = n$. $\iota : \partial X \hookrightarrow X$ is inclusion. $\forall\, \alpha \in \Lambda^{n-1}(X)$.

$$\int_X d\alpha = \int_{\partial X} \iota^* \alpha .$$

<u>pf:</u> Since we can obtain canonical orientation

on ∂X.

By POU. Then see in charts.

Reduce to i).