Cotangent Space

(1) Covectors:

<u>Denote</u>: $R_x(X) \subseteq C^\infty(X)$. $R_x(X) = \{ f \in C^\infty(X) \mid$ rank of f is zero at $x \}$. Where $C^\infty(X) = C^\infty(X, \mathbb{R}^1)$.

<u>Def</u>: The cotangent space to X at x is:

$$T_x^* X = C^\infty(X) / R_x(X)$$

. element in $T_x^* X$ is called covector.

① $X \underset{\text{open}}{\subseteq} \mathbb{R}^n$ case:

<u>prop.</u> $\dim(T_x^* X) = n$

<u>Pf</u>:
$$C^\infty(X) \xrightarrow{F} \mathbb{R}^n$$
$$h \longmapsto Dh|_x = \left(\frac{\partial h}{\partial x_1}, \cdots, \frac{\partial h}{\partial x_n}\right)\Big|_x$$

$\therefore \ker F = R_x X$. Besides. $\forall (a_1 \cdots a_n) \in \mathbb{R}^n$.

$\exists\, h = \sum_1^n a_k x_k$. s.t. $F(h) = \vec{a}$. F is surjection.

$$\therefore C^\infty(X) / R_x X \xrightarrow{\sim} \mathbb{R}^n.$$

<u>Remark</u>. Note: $C^\infty(X)$ is a infinite-dimension vector space.

② For general n-dim manifold X:

. For $h \in C^\infty(X)$, and $x \in X$. Find $(U_x, f) \in A_x$.

Then $\tilde{h} = h \circ f^{-1} : \tilde{U} \longrightarrow \mathbb{R}^1$. We can compute the rank of h at x (i.e. $D\tilde{h}|_{f(x)}$).

Fix (U, f):

$\nabla_f : C^\infty(X) \longrightarrow \mathbb{R}^n$. $\nabla f(h) = D(h \circ f^{-1})|_{f(x)}$

$\ker(\nabla f) = R_x X$. ($\because$ $R_x X$ is subspace of $C^\infty(X)$).

<u>Prop.</u> $T_x^* X \xrightarrow{\sim} \mathbb{R}^n$.

<u>Pf</u>: We only need to prove: ∇f is surjection.

$\tilde{h} \in C^\infty(\tilde{U})$. $\tilde{h} = \sum_1^n a_k x_k$. for $\vec{a} = (a_1 \cdots a_n) \in \mathbb{R}^n$.

Choose ψ is Bump $\begin{cases} \psi \equiv 1 & \text{in nbd of } x. \\ \psi \equiv 0 & \text{outside } U \end{cases}$

Let $h = \begin{cases} \psi \cdot \tilde{h}(f(x)) . & x \in U \\ 0 & . \ x \in U^c \end{cases}$ $\therefore \nabla f(h) = a$. $h \in C^\infty(X)$.

<u>Remark</u>: By extension (using bump Func's).

we can prove there're lots of smooth vector field on X.

i.e. for $(U, f) \in \mathcal{A}_X$. Choose $\tilde{h} : \tilde{U} \to TX$.

s.t. $f^{-1}(\tilde{h}) = q$. (determines vector field)

Let $h = \begin{cases} \tilde{h}(f)\psi . & x \in U \\ 0 & . \ x \notin U \end{cases}$

$\therefore \Delta f \circ h \circ f^{-1} = Dq|_x$. locally in $x \in V \subseteq U$.

where $V = f^{-1}(B(x_0, r))$. $\psi(V) = \{1\}$.

③ Physicist's Definition:

. For $h \in C^\infty(X)$. Denote an element in T_x^*X by $dh|_x$. i.e the equivalent class of h.

Since for $(U_1, f_1), (U_2, f_2) \in A_x$. $\begin{cases} \tilde{h}_1 = h \circ f_1^{-1} \\ \tilde{h}_2 = h \circ f_2^{-1} \end{cases}$

$\therefore \tilde{h}_1 = \tilde{h}_2 \circ \phi_{21}$. We obtain:

$\nabla f_1 (dh|x) = \nabla f_2 (dh|x) \cdot D\phi_{21}|_{f_1(x)}$.

written in row vector: $\nabla f_1 = (D\phi_{21}|_{f_1(x)})^T \cdot \nabla f_2$.

prop. T_x^*X is the set collecting Func such ε.

Then. $T_x^*X \cong T_x^*X$.

Pf. Define: $ev_f : T_x^*X \longrightarrow \mathbb{R}^n$
$\varepsilon \longmapsto \varepsilon f$

There exists canonical linear isomorphism.

(2) Third Definition of tangent vectors:

Claim: $T_x^*X = (T_xX)^*$

① $X \underset{open}{\subseteq} \mathbb{R}^n$:

Since $T_xX \cong \mathbb{R}^n$. We can identify $\vec{v} \in T_xX$ with vector in $\mathbb{R}^n$.

consider the operation: take partial derivative

at x in the direction $\vec{v}$:

$\partial_{x,\vec{v}}: C^\infty(X) \longrightarrow \mathbb{R}^1$. $\partial_{x,\vec{v}}(h) = Dh|_x \cdot \vec{v}$

It's easy to see $\partial_{x,\vec{v}}$ is linear.

Actually, replace $\vec{v}$ by $\sigma \in T_xX$: $D\sigma|_0 = \vec{v}$.

$\therefore$ $\partial_{x,\sigma}(h) = D(h\circ\sigma)|_0 = Dh|_x \cdot D\sigma|_0$. vanish on R_xX.

$\Rightarrow$ $\partial_{x,\sigma}: T_x^*X \longrightarrow \mathbb{R}^1$ is well-def. $\partial_{x,\sigma} \in (T_x^*X)^*$.

$dh|_x \longmapsto \partial_{x,v}(h)$

Remark: $\partial_{x,v}$ is simply: $\begin{array}{l} \mathbb{R}^n \longrightarrow \mathbb{R}^1 \\ u \longmapsto u \cdot v \end{array}$

Conversely, $\forall$ BLO $\delta: T_x^*X \to \mathbb{R}^1$. since $T_x^*X \cong \mathbb{R}^n$, and $(\mathbb{R}^n)^* = \mathbb{R}^n$. by Riesz Thm: $\delta(dh|_x) = (Dh|_x, \vec{v})_2 = Dh|_x \cdot \vec{v}$.

for some $\vec{v}$. $\therefore$ $\delta = \partial_{x,\vec{v}}$. actually.

$\therefore$ $(T_x^*X)^* = T_xX$.

② For X is manifold:

Define $\partial_{\sigma,x}: C^\infty(X) \longrightarrow \mathbb{R}^1$:

Fix $(U,f) \in A_x$. $\partial_{\sigma,x}(h) = D(h\circ\sigma)|_0 = \nabla f(h) \cdot \Delta f(\sigma)$

Since $h\circ\sigma = (h\circ f^{-1})\circ(f\circ\sigma)$. And $\partial_{\sigma,x}$ is well-def.

and chart-indept.

<u>prop.</u> $\exists$ linear isomorphism: $T_x X \cong (T_x^* X)^*$.

<u>Pf:</u>

$$\begin{tikzcd} T_x X \arrow[r, "F"] \arrow[d, "\Delta f"', "S"] & (T_x^* X)^* \arrow[d, "S"] \\ \mathbb{R}^n \arrow[r, "\cong"] & (\mathbb{R}^n)^* \end{tikzcd}$$

$[\sigma] \mapsto \partial_{\sigma.x}$, $\Delta f(\sigma) \mapsto \partial_{v.x}$.

$v = \Delta f(\sigma)$.

Besides, $\partial_{v.x}(u) = u \cdot \Delta f(\sigma)$.

for $u \in \mathbb{R}^n$.

<u>Remark:</u> i) Since $\dim T_x X = n < \infty$. $\therefore (T_x X)^* \cong T_x^* X$.

Explicitly: For $dh|_x \in T_x^* X$. define:

$\widehat{dh|_x} : T_x X \to \mathbb{R}^1$. $\widehat{dh|_x}([\sigma]) = \partial_{\sigma.x}(h)$.

$\therefore \widehat{dh|_x} \in (T_x X)^*$

ii) Note that: $\begin{cases} \Delta f : T_x X \xrightarrow{\sim} \mathbb{R}^n \\ \nabla f : T_x^* X \xrightarrow{\sim} \mathbb{R}^n \end{cases}$

$\therefore \Delta f$ is the dual BLO of ∇f^{-1}.

③ <u>Derivation at x:</u>

<u>Def:</u> For X is manifold. A derivation at x is BLO:

$\partial : C^\infty(X) \to \mathbb{R}^1$. st. $\partial(h_1 h_2) = h_1 \partial(h_2) + h_2 \partial(h_1)$.

for $\forall h_1, h_2 \in C^\infty(X)$. Denote the set by $Der_x(X)$.

<u>prop.</u> BLO $\partial : C^\infty(X) \to \mathbb{R}^1$ is a derivation at x

$\iff \partial$ vanishes on $R_x X$.

Def: (Algebraist 's).

A tangent vector to x on X is a derivation at x.

Remark: It only uses the fact: $C^\infty(X)$ is a ring.

(3) Vector Fields as Derivations:

① $X \underset{\text{open}}{\subseteq} \mathbb{R}^n$:

· For $\tilde{\xi}: X \to TX$. We can define:

$\hat{\xi}: C^\infty(X) \to C^\infty(X)$. $\hat{\xi}(h): x \longmapsto \partial_{x,\tilde{\xi}|_x}(h)$.

i.e. $\hat{\xi} = \sum \tilde{\xi}_i \frac{\partial}{\partial x_i}$ ($\tilde{\xi} = (\tilde{\xi}_1 \cdots \tilde{\xi}_n)$).

Remark: i) $(\frac{\partial}{\partial x_i})_1^n$ is Standard Basis

ii) Since $\partial_{x,\tilde{\xi}|_x} \in \mathrm{Der}_x(X)$

$\therefore \hat{\xi}(h_1 h_2) = h_1 \hat{\xi}(h_2) + h_2 \hat{\xi}(h_1)$.

② For X is arbitrary manifold:

· For $\xi: X \to TX$. vector field. (smooth)

Define: $\hat{\xi}: C^\infty(X) \to C^\infty(X)$.

$\hat{\xi}(h): x \longmapsto \partial_{x,\xi|_x}(h)$

Check: $\hat{\xi}(h)$ is smooth. (See in charts)

For $(U, f) \in A_X$. $\xi(h) \circ f^{-1} : \tilde{x} \longmapsto \partial_{f^{-1}(\tilde{x})} \cdot \xi|_{f^{-1}(\tilde{x})}(h)$.

i.e. $\xi(h)(f^{-1}(\tilde{x})) = D(h \circ \xi|_{f^{-1}(\tilde{x})})|_0$

$= D(h \circ f^{-1} \circ f \circ \xi|_{f^{-1}(\tilde{x})})|_0$

$= D\tilde{h}|_{\tilde{x}} \cdot D\tilde{\xi}|_0 = \nabla f(h|_{f^{-1}(\tilde{x})}) \cdot \Delta f(\xi|_{f^{-1}(\tilde{x})})$

$\therefore$ $\xi(h) \circ f^{-1}$ is smooth. since ξ is smooth.

Def: A derivation on manifold X is LF:

$\mathcal{D} : C^\infty(X) \longrightarrow C^\infty(X)$. s.t. $\mathcal{D}(h_1 h_2) = h_1 \mathcal{D}(h_2) + h_2 \mathcal{D}(h_1)$

for $\forall h_1, h_2 \in C^\infty(X)$. Denote the set by $Der(X)$.

prop. $\forall \mathcal{D} \in Der(X)$ defines a smooth vector field.

pf. 1°) $\mathcal{D}|_x \in Der_x(X) = T_x X$

$\therefore$ $\xi : X \longrightarrow TX$. $\xi(x) = \mathcal{D}|_x$ is vector field.

2°) Check $\mathcal{D}$ is smooth.

For $(U, f) \in A_X$. $\tilde{\xi} = \Delta f \circ \xi|_x \circ f^{-1} : \tilde{U} \rightarrow \mathbb{R}^n$.

Denote $\tilde{\xi} = (\tilde{\xi}_1, \cdots \tilde{\xi}_n)$.

$\therefore$ $\tilde{\xi} : C^\infty(\tilde{U}) \longrightarrow C^\infty(\tilde{U})$. $\tilde{\xi}(h) = \sum_i^n \tilde{\xi}_i \frac{\partial h}{\partial x_i}$. $\forall h \in C^\infty(\tilde{U})$.

3°) Check $\tilde{\xi}_i$ is smooth. $\forall 1 \le i \le n$. (at $\forall \tilde{\eta} = f(\eta) \in \tilde{U}$).

Choose $\phi \in C^\infty(\tilde{U})$. $\phi \equiv 1$ in $\tilde{V}_{\tilde{\eta}}$.

Define: $\varphi_k = \begin{cases} (x_k \phi) \circ f & , x \in U \\ 0 & , x \notin U \end{cases}$ $\therefore$ $\begin{matrix} \varphi_k \in C^\infty(X) \\ \varphi_k \circ f^{-1} \in C^\infty(\tilde{U}) \end{matrix}$

By def: $\mathcal{D}(\varphi_k)|_x = \sum_j \tilde{\xi}_j|_{f(x)} \frac{\partial \tilde{\varphi}_k}{\partial x_j}|_{f(x)} = \tilde{\xi}_k|_{f(x)}$

is smooth. $\forall f(x) \in \tilde{V}_{\tilde{\eta}}$.