Smooth Functions

(1) Definition:

① For $h: \mathbb{R}^n \longrightarrow \mathbb{R}^k$. it's clear what it means to say h is smooth.

② For $h: X \longrightarrow \mathbb{R}^k$. We should see it in chart:

Def: h is smooth at $x \in X$. if $\exists (U_x, f) \in A_x$ st. $h \circ f^{-1}: \tilde{U}_x \longrightarrow \mathbb{R}^k$ is smooth at $f(x)$.

Remark: It's indepdent with the choice of Charts: Since for (U_1, f_1). (U_2, f_2).

$\tilde{h}_1 = h_1 \circ f_1^{-1}$. $\tilde{h}_2 = h_2 \circ f_2^{-1}$

$\Rightarrow \tilde{h}_1 = \tilde{h}_2 \circ \phi_{21}$. Note: ϕ_{21} is smooth.

③ Genelization: $H: X \xrightarrow{\text{conti}} Y$. X, Y are smooth manifolds. $\dim X = n$. $\dim Y = k$.

Def: H is smooth at $x \in X$. if $\exists (U_x, f) \in A_x$. $(V, g) \in A_Y$. st. $H(U_x) \subseteq V$.

$g \circ H \circ f^{-1}: \tilde{U}_x \rightarrow \tilde{V}$ is smooth at $f(x)$.

Remark: i) Smooth Func is automatically conti: since $h = g \circ H \circ f^{-1}$ smooth. so conti

$\Rightarrow H = g^{-1} \circ h \circ f$ is conti.

ii) If H is conti. It's easy to find (U, f). (V, g). St. $H(U) \subseteq V$:

Firstly find $(U_x, f) \in A_X$. St. $x \in U_x$.

and $(V_{H(x)}, g) \in A_Y$. St. $H(x) \in V_{H(x)}$.

restrict f on: $H^{-1}(V_{H(x)}) \cap U_x$, open set.

Then: $(H^{-1}(V_{H(x)}) \cap U_x, f|_{\Box})$. $(V_{H(x)}, g)$.

iii) It's indepdt with choice of charts:

For (U_1, f_1). (V_1, g_1) and (U_2, f_2). (V_2, g_2).

$$g_1 \circ H \circ f_1 = \psi_{12} \circ (g_2 \circ H \circ f_2) \circ \phi_{21}$$

Since ψ_{12}, ϕ_{21} are smooth.

<u>Def.</u> For $H: X \longrightarrow Y$. X. Y are smooth manifolds with boundary. H is smooth at $x \in X$. if $\exists (U, f)$. (V, g) of A_X. A_Y. $x \in U$. $H(U) \subseteq V$. And $\exists \hat{U}. \hat{V} \underset{\text{open}}{\subseteq} \mathbb{R}^n. \mathbb{R}^k$. [resp]. $\hat{F}: \hat{U} \longrightarrow \hat{V}$ smooth. St. $\breve{U} \subseteq \hat{U}$. $\breve{V} \subseteq \hat{V}$. $\hat{F}|_{\breve{U}} = g \circ H \circ f^{-1}$.

<u>Remark</u>: It's indept with charts and extension $\hat{F}$. since we can calculate derivates at $\{x_1 = 0\}$. from $x_1 \to 0^+$ or 0^-.

<u>Lemma.</u> X, Y, Z are smooth manifolds. $H: X \to Y$. $G: Y \longrightarrow Z$. smooth. Func's. Then $G \circ H$ is smooth as well.

Pf. Find $(U, f) \in A_X$. st. $x \in U$.

$(V, g) \in A_Y$. st. $H(x) \in V$. $\Rightarrow$ restrict on open nbd.

$(W, h) \in A_Z$ st. $G(H(x)) \in W$.

For $h \circ G \circ H \circ f^{-1} = h \circ G \circ g^{-1} \circ g \circ H \circ f^{-1}$. smooth.

Remark: There's a category $\begin{cases} \text{morphism: smooth Funcs} \\ \text{objects: smooth manifolds} \end{cases}$

Lemma. i) Z is submanifold of X. $\iota : Z \longrightarrow X$ is inclusion.

Then ι is smooth.

ii) $H : X \longrightarrow Y$ smooth. between two manifolds.

$Z \subseteq Y$. submanifold. If $H(X) \subseteq Z$. Then

$H : X \longrightarrow Z$ is smooth.

Pf. i) Find $(U \cap Z, g)$. (U, f).

$$f \circ \iota \circ g^{-1} : \tilde{U} \cap \mathbb{R}^k \xrightarrow{id} \tilde{U} \cap \mathbb{R}^k.$$

ii) $\exists\ (U, f). (V, g). \in A_X, A_Y$.

By $g \circ H \circ f^{-1}$ is smooth.

Since $H(U) \subseteq Z$. It equals with:

$(V \cap Z, \tilde{g})$ and (U, f). $\tilde{g}$ is induced by $g|_{V \cap Z}$.

i.e. $g \circ H \circ f^{-1} = \tilde{g} \circ H \circ f^{-1}$ smooth.

Remark: From i) conclude:

$H : X \longrightarrow Y$ is smooth. Z is submanifold of X. Then $H|_Z$ is smooth.

(2) Rank:

<u>Def.</u> $F: X \to Y$ smooth func between X, Y two smooth manifolds. The rank of F at x is : $D\tilde{F}|_{f(x)}$. where $\tilde{F} = g \circ F \circ f^{-1}$. $(U, f) \in A_X$, $(V, g) \in A_Y$. $F(U) \subseteq V$.

<u>Remark:</u> It's indept with choices of charts:

$$\text{For } \begin{cases} \tilde{F}_1 = g_1 \circ F \circ f_1^{-1}. \text{ from } (U_1, f_1), (V_1, g_1) \\ \tilde{F}_2 = g_2 \circ F \circ f_2^{-1}. \text{ from } (U_2, f_2), (V_2, g_2) \end{cases}$$

$$\Rightarrow D\tilde{F}_2|_{f_2(x)} = D\psi_{21}|_{g_1(F(x))} \circ D\tilde{F}_1|_{f_1(x)} \circ D\phi_{12}|_{f_2(x)}$$

$\Rightarrow$ We can define regular points or critical points for $F: X \to Y$. (Note that x is regular point of $F \Leftrightarrow f(x)$ is regular point of $\tilde{F}$. for some chart)

<u>prop.</u> $k \le n$. $F: X \to Y$.

If y is regular value of F. Then the level set $Z_y = F^{-1}(y)$ is k-codim submanifold of X.

<u>Pf.</u> For $(U, f) \in A_X$. $(V, g) \in A_Y$. $\tilde{F} = g \circ F \circ f^{-1}$.

$x \in Z_y$. $f(x)$, $g(y)$ is regular point / value of $\tilde{F}$.

$\therefore \tilde{F}^{-1}(g(y))$ is k-codim submanifold of $\mathbb{R}^n$.

$\exists (W, h)$. $f(x) \in W$. coordinate chart.

i.e. $h \circ \tilde{F}^{-1}(g(y)) = \mathbb{R}^{n-k} \cap \tilde{W}$.

$\therefore (f(W), h \circ f)$ is chart of Z_y at x.

(3) Special Kinds of Smooth Func's:

① Def: For: $F: X \to Y$. Smooth

i) It's submersion if $\forall x \in X$. $r(D\tilde{F}|_{f(x)})$ $= \dim Y$.

ii) It's immersion if $\forall x \in X$. $r(D\tilde{F}|_{f(x)}) =$ $\dim X$.

Where $\tilde{F}$ is F in chart.

Remark: Immersion or submersion do nothing with surjection or injection of F.

iii) It's diffeomorphism if F is bijection and F^{-1} is smooth as well

Remark: F is both immersion and submersion since $D\tilde{F}^{-1} = (D\tilde{F})^{-1}$ exists.

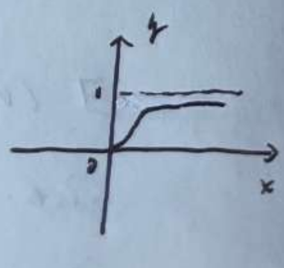
Lemma. For $\dim X = \dim Y$ equals rank of F.

If F is smooth bijection. Then F is diffeomorphism.

Pf: Apply IFT.

② Bump Function:

Consider $\phi(x) = \begin{cases} 0, & x \le 0 \\ e^{-\frac{1}{x}}, & x > 0 \end{cases}$

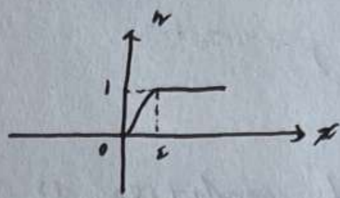


$\phi \in C^\infty(\mathbb{R}^1, \mathbb{R}^1)$. (Smooth). But ϕ isn't analytic

at $x=0$ (It only has Laurent Expansion)

For $\varphi^\varepsilon(x) = \dfrac{\phi(x)}{\phi(x)+\phi(\varepsilon-x)} \in C^\infty$. $\begin{cases} \varphi^\varepsilon = 0 \Leftrightarrow x \le 0 \\ \varphi^\varepsilon = 1 \Leftrightarrow x \ge \varepsilon \end{cases}$

Besides, $0 \le \varphi^\varepsilon \le 1$.



For $\mathbb{R}^n \longrightarrow \mathbb{R}^1$ case:

Consider $\psi^{R,r}(x) = \dfrac{\phi(R-|x-\eta|)}{\phi(R-|x-\eta|)+\phi(|x-\eta|-r)} \in C^\infty(\mathbb{R}^n, \mathbb{R}^1)$, $0<r<R$.

$0 \le \psi \le 1$. Besides: $\psi = \begin{cases} 1 \Leftrightarrow x \in B(\eta, r) \\ 0 \Leftrightarrow x \in B(\eta, R)^c \end{cases}$ (looks like Bump)

i) <u>Extension:</u>

· Bump Func's can be used to extend locally smooth

Functions defined in $U \subseteq X$:

Firstly, for X smooth manifold, $g \in C^\infty(U)$.

We create bump Func on the whole X:

Let $x \in X$, $(U, f) \in A_X$, $x \in U$. Choose $\overline{B(0,R)} \subseteq \tilde{U}$.

which is the largest ball. ($\forall\, 0<r<R$)

<u>Def:</u> $\tilde{\psi}(x) = \begin{cases} (\psi^{R,r} \circ f)(x), & x \in U \\ 0, & x \notin U \end{cases}$ (It's bump-like)

check $\tilde{\psi} \in C^\infty(X)$:

$\because\ \tilde{\varphi} \geq 0 \Leftrightarrow x \in f^{-1}(\overline{B(0,R)})$ opt. in X

$\because\ X$ is Hausdorff $\therefore\ f^{-1}(\overline{B(0,R)})$ is closed, too

$\because\ \tilde{\varphi}$ is smooth in U and $X / f^{-1}(\overline{B(0,R)})$ (open, $\tilde{\varphi} \equiv 0$)

$\therefore\ \tilde{\varphi} \in C^\infty(X)$.

<u>Remark</u>: Hausdorff condition is necessary:

1°) Introduce: topo on disjoint union: $X \sqcup Y$. (X, Y topo)

Generally. $\bigsqcup_{\alpha \in A} X_\alpha = \bigcup_{\alpha \in A} (X_\alpha, \alpha)$

$$\tau\Big(\bigsqcup_{\alpha \in A} X_\alpha\Big) = \Big\{ V \subseteq \bigcup_{\alpha \in A} (X_\alpha, \alpha) \ \Big|\ V \cap X_\alpha \underset{\text{open}}{\subseteq} X_\alpha,\ \forall \alpha \in A \Big\}$$

$$= \Big\{ V \subseteq \bigcup_{\alpha \in A} (X_\alpha, \alpha) \ \Big|\ \sigma_\alpha^{-1}(V) \underset{\text{open}}{\subseteq} X_\alpha,\ \forall \alpha \in A \Big\}.$$

$$\sigma_\alpha : X_\alpha \longrightarrow \bigsqcup_{\alpha \in A} X_\alpha \quad \text{canonical projection.}$$

$$x \longmapsto (x, \alpha)$$

2°) Consider $\mathbb{R}^1 \sqcup \mathbb{R}^1$:

$X = \mathbb{R}^1 \sqcup \mathbb{R}^1 / (x,1) \sim (x,2)$, $\forall x \neq 0$. (has 2 origins)

Then X isn't Hausdorff. Since we can't separate $(0,1)$ and $(0,2)$. $\forall\ U_{(0,1)} \cap U_{(0,2)} \neq \emptyset$. under the equivalent relation.

3°) Consider $\varphi(x) \in C^\infty(\mathbb{R}^1)$. $\begin{cases} \varphi \equiv 1 & \text{in } (-r, r) \\ \varphi \equiv 0 & \text{in } [-m, m]^c. \end{cases}$

Extend φ on X:

$$\tilde{\varphi} = \begin{cases} \varphi, & x \in (\mathbb{R}^1, 1) \\ 0, & x \in (\{0\}, 2) \end{cases} \quad \text{choose } (U, f) = ((\mathbb{R}^1, 1), \mathrm{id})$$

Restrict $\tilde{\psi}$ on $(\mathbb{R}, 2)$: $\begin{cases} \tilde{\psi} = 1 & \text{in } ((-r,0)\cup(0,r), 2) \\ \tilde{\psi} = 0 & \text{in } (\{0\}, 2) \end{cases}$

$\therefore \tilde{\psi}$ isn't conti at 0_2.

Secondly. Let $\hat{g} = \begin{cases} g\tilde{\psi}. & x \in U \\ 0. & x \notin U \end{cases} \in C^\infty(X)$. $\hat{g}|_{f^{-1}(B(0,r))} = g$.

ii) <u>Witney Thm:</u>

<u>Thm.</u> $\forall A \underset{close}{\subseteq} X$. smooth manifold. $\exists f \in C^\infty(X)$. st. $f^{-1}\{0\} = A$.

<u>pf.</u> $U = X/A \underset{open}{\subseteq} X$. $U = \bigcup f_x^{-1}(B(x, r))$, $(U_x, f_x) \in A_x$.

Since X is C_2 $\therefore U = \bigcup\limits_{n \in \mathbb{Z}} f_n^{-1}(B(x_n, r_n))$

By i). $\exists \phi_n \in C^\infty(X)$. $\phi_n \equiv 1$ in $f_n^{-1}(\overline{B(x_n, r_n)})$. $\phi_n \equiv 0$ outside U

Choose ε_n. st. $\varepsilon_n \sup\limits_x \left| \frac{\partial^k \phi_n}{\partial x_{i_1} \cdots \partial x_{i_k}} \right| \le \frac{1}{2^n}$. $\forall k \le n$. (supp ϕ_n is cpt)

Set $g_n(x) = \sum\limits_1^n \varepsilon_n \phi_n \in C^\infty(X)$.

$\because \sum\limits_1^\infty \left| \varepsilon_n \frac{\partial^k \phi_n}{\partial x_{i_1} \cdots \partial x_{i_k}} \right| \le \sum\limits_1^k \square + \sum\limits_{k+1}^\infty \frac{1}{2^n} < \infty$. $\forall k \in \mathbb{Z}^+$.

$\therefore g_n \in C^\infty(X) \xrightarrow{u} g = \sum\limits_1^\infty \varepsilon_n \phi_n \in C^\infty(X)$. what we need.

<u>Cor.</u> $\forall A, B \underset{close}{\subseteq} X$. $\exists \phi: X \to [0,1]$, st. $\phi^{-1}\{0\} = A$. $\phi^{-1}\{1\} = B$. $\phi \in C^\infty(X)$.

<u>pf.</u> Choose $\phi_1, \phi_2 \in C^\infty(X)$. $\phi_1^{-1}\{0\} = A$. $\phi_2^{-1}\{0\} = B$. $\phi = \frac{\phi_1}{\phi_1 + \phi_2} \in C^\infty(X)$.

<u>Remark:</u> i) It's extension of Urysohn Thm.

ii) In particular, Let $X = \mathbb{R}^n$.