Submanifolds

(1) <u>Pre</u>:

· For $h: \mathbb{R}^n \longrightarrow \mathbb{R}^k$. Fix $\vec{x} \in \mathbb{R}^n$. $Dh|_x : \mathbb{R}^n \longrightarrow \mathbb{R}^k$.

where $Dh|_x \in M^{k\times n}(\mathbb{R})$. $Dh|_x = \left(\frac{\partial h_i}{\partial x_j}|_x\right)_{k\times n}$.

<u>Thm</u>. (IFT).

$U \underset{open}{\subseteq} \mathbb{R}^n$. $F: U \longrightarrow \mathbb{R}^n$ smooth. For $x \in U$.

If $DF|_x : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is isomorphism (i.e. $|DF|_x| \neq 0$)

Then $\exists V$. st. $x \in V \underset{open}{\subseteq} U \subseteq \mathbb{R}^n$.

$F: V \longrightarrow F(V)$ is diffeomorphism.

<u>Cor</u>. $F: U \rightarrow W$ smooth bijection. If $|DF|_x| \neq 0$.

$\forall x \in U$. Then $F^{-1}: W \longrightarrow U$ is smooth.

(2) <u>Definitions</u>:

① <u>Def</u>: i) An affine subspace of $\mathbb{R}^n$ is a translation of a linear subspace. i.e. $A = \{x + v \mid x \in W\}$.

W is subspace of $\mathbb{R}^n$. $v \in \mathbb{R}^n$. fix.

<u>e.g.</u> Standard affine subspace:

$\{(x_1 \cdots x_m, 0, 0 \cdots 0) \mid x_k \in \mathbb{R}^1\}$. $m \in [0, n]$.

i.e. $W = \mathbb{R}^m$. $v = \vec{0}$.

ii) For X is smooth manifold. $Z \subseteq X$.

Z is m-dimension submanifold of X if

$\forall z \in Z$. $\exists (U, f)$. $z \in U$. coordinate chart.

and $\exists$ m-dimension affine subspace $A \subseteq \mathbb{R}^n$.

$$f: U \xrightarrow{\sim} \tilde{U} \subseteq \mathbb{R}^n. \quad f(U \cap Z) = A \cap \tilde{U}.$$

Remark: We can replace A by standard affine

subspace $\{(x_1 \cdots x_k, 0, 0 \cdots 0) \mid x_i \in \mathbb{R}^1\}$.

Pf: $\exists$ $\tau: \mathbb{R}^n \to \mathbb{R}^n$. τ just exchanges

the position of component.

replace (U, f) by $(U, \tau \circ f)$.

e.g. i) The interior of n-dimension manifold with

boundary is n-dimension manifold. For $\forall (U, f)$

$\in A$. $\tilde{U} \underset{\text{open}}{\subseteq} \mathbb{R}^n$.

ii) The boundary ∂X of n-dimension manifold with

boundary X is $(n-1)$-dimension. For $(U, f) \in A$.

$\Rightarrow f: U \cap \partial X \xrightarrow{\sim} \tilde{U} \cap \{x_1 = 0\}$.

② Smooth Structure:

For X is n-dimension manifold. $Z \subseteq X$ is m-dimension

submanifold of X. (X is Smooth)

Lemma. $\forall (U_1, f_1)$. $(U_2, f_2) \in A_X$ (map Z to standard

subspace) induces chart (V_1, g_1). (V_2, g_2) of Z.

where $V_i = U_i \cap Z$. $g_i: V_i \xrightarrow{\sim} \mathbb{R}^m \cap \tilde{U}_i = \tilde{V}_i$

Then $\psi_{12} = g_1 \circ g_2^{-1}$ is smooth as well.

Pf: Let $U = U_1 \cap U_2$. $V = Z \cap U$.

By assumption: $f_1(V) = f_1(U) \cap \mathbb{R}^m$. $f_2(V) = f_2(U) \cap \mathbb{R}^m$.

Actually, the components of ψ_{12} equals = the first m components of $\phi_{12}|_{\mathbb{R}^m}$.

prop. Z is a m-dimension manifold carrying with a smooth structure induced from $\{A_x\}$ of X.

Pf: Apply lemma at every point $z \in Z$.

Besides, the smooth structure is indept with choice of A_x, by compatibility.

(3) Level set:

- Note that $S^1 = \{x^2+y^2=1\}$ is submanifold in $\mathbb{R}^2$.

generally, we can ask: (For $h: \mathbb{R}^n \to \mathbb{R}^k$)

when is $\{h(\vec{x}) = \tau\} \subseteq \mathbb{R}^n$ a submanifold?

① Def: i) $h: \mathbb{R}^n \to \mathbb{R}^k$. $h \in C^\infty$. For point $x \in \mathbb{R}^n$

is called regular point if $r(Dh(x)) = k$.

is called critical point if $r(Dh(x)) < k$.

ii) $h: \mathbb{R}^n \to \mathbb{R}^k$. $h \in C^\infty$. For value $\alpha \in \mathbb{R}^k$.

is called regular value if $h^{-1}(\alpha)$ is set of regular point. Otherwise it's called critical value.

iii) Standard Projection: $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^k$. $(k \le n)$.

$\pi(x_1, \cdots x_n) = (x_{n-k+1}, \cdots x_n)$. $\ker \pi = \mathbb{R}^{n-k}$.

Remark: Note that $D\pi = (0 | I_k) = \pi : \mathbb{R}^n \longrightarrow \mathbb{R}^k$

Thm. (general IFT)

$k \le n$. $U \underset{\text{open}}{\subseteq} \mathbb{R}^n$. $h : U \longrightarrow \mathbb{R}^k$. $h \in C^\infty(U)$. If z is a regular point of h. Then $\exists V$ neighbour of z. $\exists f : V \xrightarrow{\sim} \tilde{V} \underset{\text{open}}{\subseteq} \mathbb{R}^n$. diffeomorphism. st. $h \circ f^{-1} = \pi : \tilde{V} \longrightarrow \mathbb{R}^k$.

Remark: It said we can find f diffeomorphism. st. $f_{n-k+i} = h_i$. $\forall 1 \le i \le k$.

Pf: since $r(Dh|_z) = k$. reorder $(x_i)_1^n$:

st. $M = \dfrac{\partial(h_1 \cdots h_k)}{\partial(x_{n-k+1} \cdots x_n)}$ has rank k.

Let $f(x_1 \cdots x_n) = (x_1 \cdots x_{n-k}, h_1(\vec{x}) \cdots h_k(\vec{x}))$

since $Df|_z = \begin{pmatrix} I_{n-k} & 0 \\ 0 & M \end{pmatrix}$. Apply IFT.

prop:

$h : \mathbb{R}^n \longrightarrow \mathbb{R}^k$. smooth. If $k \le n$. $\alpha \in \mathbb{R}^k$ is regular value of h. Then the level set $Z_\alpha = h^{-1}(\alpha) \subset \mathbb{R}^n$ is $(n-k)$-dim submanifold of $\mathbb{R}^n$.

Remark: i.e. Z_α has codim k. Intuitively. Z_α has $n-k$ freedom.

Pf: $\forall z \in h^{-1}(c)$. By general IFT. $\exists f: V \rightleftarrows \tilde{V}$.

$\therefore f(Z_c \cap V) = f(Z_c) \cap f(V) = \lambda^{-1}(c) \cap \tilde{V}$

It forms a chart at z. So an atlas.

Remark: i) We can generalize it by considering $h: X \to \mathbb{R}^k$. $X \underset{\text{open}}{\subseteq} \mathbb{R}^n$. (Since the procedure before is local). Then Z_c is $(n-k)$-dim submanifold of X.

ii) For finding the chart of Z_c:
Caution about "reorder". If $r\left(\frac{\partial(h_1 \cdots h_k)}{\partial(x_{i_1} \cdots x_{i_k})}\right) = k$.
generally. Let $f_{i_j} = h_{i_j}$. $f_i = x_i$. $\forall i \notin (i_j)_1^k$
to obtain chart of $\mathbb{R}^n / X$.

② Sard's Thm:

For $f: X \underset{\text{open}}{\subseteq} \mathbb{R}^k \longrightarrow \mathbb{R}^m$. Smooth. $(m < k)$

Then the critical value of f is a Lebesgue null set.

Pf: Proceed by induction on k:
$k = 0$ it's trivial. Suppose it holds for $k < n$
For the case $k = n$:
Denote: $C = C_f$ (critical points of f)

$C_l = \{ a \in X \mid \frac{\partial^s f_j}{\partial x_{i_1} \cdots \partial x_{i_s}}(a) = 0.\ 1 \le j \le m.\ s \le l.\ (i_j)_1^s \subseteq \{1, 2, \cdots n\} \}$.

$\therefore \cdots C_l \subseteq C_{l-1} \cdots \subseteq C_1 \subseteq C$.

prove: i) $f(C/C_1)$ ii) $f(C_i/C_{i+1})$ iii) $f(C_l)$. $l > \frac{n}{m} - 1$

are all $\mathcal{L}$-null set.

Then: $f(C) = \bigcup_{0}^{p} (f(C_i/C_{i+1})) \cup f(C_{p+1})$. $p > \frac{n}{m}$

is a null set.

i) $m=1$. Then $C_1 = C$. trivial.

For $m>1$. $\forall a \in C/C_1$. We can find V nbd of a. $V \subset X$.

st. $f(V \cap (C/C_1))$ is $\mathcal{L}$-null.

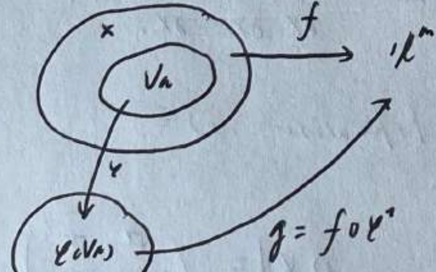
Then $f(C/C_1) = \bigcup_{a \in a^n} f((C/C_1) \cap V_a)$ is $\mathcal{L}$-null set.

WLOG. suppose $\frac{\partial f_1}{\partial x_1}(a) \neq 0$.

Define: $\varphi: X \longrightarrow \varphi(X)$ $\quad \therefore D\varphi(a) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & * \\ 0 & I_{n-1} \end{pmatrix}$

$(x_1 \cdots x_n) \longmapsto (f_1(x), x_2 \cdots x_n)$

Apply IFT: $\exists V_a \subseteq X$. st. $\varphi|_{V_a}$ is diffeomorphism.



$\because Dg|_x = Df|_{\varphi^{-1}(x)} \cdot D\varphi^{-1}|_x$

$\therefore f(V_a \cap (C/C_1)) \subseteq f(V_a \cap C)$

$= g(C_g)$

For C_g:

$\because g(t, x_2 \cdots x_n) = (t, g_2 \cdots g_m)$

$\therefore Dg|_{(t,x')} = \begin{pmatrix} 1 & 0 \\ * & D_{x'}\tilde{g} \end{pmatrix}$. $\tilde{g} = (g_2 \cdots g_m)$

$\therefore (t, x') \in C_g \iff x' \in C(\tilde{g}_t)$ (fix t)

$\therefore C_g = \bigcup \{t\} \times C(\tilde{g}_t)$

$\therefore g(C_g) = \bigcup \{t\} \times \tilde{g}_t(C_{\tilde{g}_t})$

By inductive assumption $m(\tilde{g}_t(C_{\tilde{g}_t})) = 0$.

Note that $C_g = \bigcup_{\substack{(j_k)_1^m \subseteq \\ \{1,2,\cdots,n\}}} \{a \in X \mid r(Dg|_a(\begin{smallmatrix} 1 & 2 & \cdots & m \\ j_1 & j_2 & \cdots & j_m \end{smallmatrix})) < m\}$.

$\because g \in C^\infty$. $\therefore C_g$ is closed.

For $V_a \underset{open}{\subseteq} \mathbb{R}^n$. Since $\mathbb{R}^n$ is σ-cpt. $V = \bigcup_n P_n$. cpt sets.

$\therefore g(C_g) = g(C_g \cap \varphi(V_a)) = g(\bigcup_n (C_g \cap \varphi(P_n)))$

$= \bigcup_n g(C_g \cap \varphi(P_n))$ is Borel-measurable

Apply Fubini Thm:

$m(g(C_g)) = \int_{\mathbb{R}} \int_{\mathbb{R}^{m-1}} \chi_{g(C_g)} = \int_{\mathbb{R}} (\int_{\mathbb{R}^{m-1}} \chi_{\{t\}} \chi_{\tilde{g}_t(C_{\tilde{g}_t})} dx')dt$

$= 0$.

ii) $\forall a \in C_i / C_{i+1}$. Suppose $\frac{\partial^{i+1} f_1(a)}{\partial x_1 \partial x_{i_1} \cdots \partial x_{i_i}} \neq 0$.

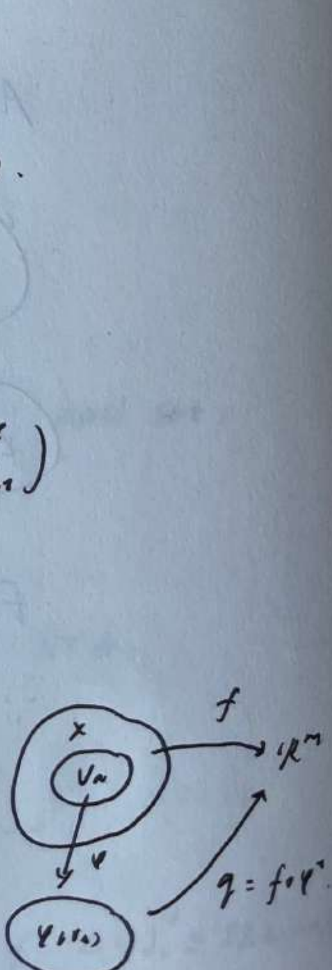
$u(x) = \frac{\partial^i f_1(x)}{\partial x_{i_1} \cdots \partial x_{i_i}}$ likewise i):

$\varphi: X \longrightarrow \varphi(X)$ $\quad D\varphi|_a = \begin{pmatrix} \square & * \\ 0 & I_{n-1} \end{pmatrix}$

$(x_1 \cdots x_n) \longmapsto (u, x_2 \cdots x_n)$

$\exists V \underset{open}{\subseteq} X$. $\varphi|_V$ is diffeomorphism.

$\because \varphi(V \cap C_i) \subseteq \{0\} \times \mathbb{R}^{n-1}$



Denote $\tilde{g}(y_2 \cdots y_n) = g(0, y_2 \cdots y_n)$

$\therefore\ f(V\cap C_i) \subseteq f(\varphi^{-1}(\{0\}\times C_{\tilde q})) = \eta(C_q)$

$\therefore\ V\cap C_i \subseteq \varphi^{-1}(\{0\}\times C_{\tilde q})$ $\quad\therefore\ \varphi(V\cap C_i) \subseteq \{0\}\times C_{\tilde q}$

$\Rightarrow f(V\cap C_i) = \eta\circ\varphi(V\cap C_i) \subseteq \eta(\{0\}\times C_{\tilde q}) = \tilde\eta(C_{\tilde q})$

By inductive assumption: $m(\tilde\eta(C_{\tilde q}))=0$ $\quad\therefore\ m(f(V\cap C_i))=0$

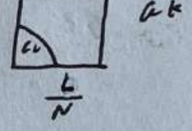
iii) Q is closed cube with width L. $Q\subseteq X$.

By Taylor expansion at a: $|f(x)-f(a)| \le C|x-a|^{l+1}$

for $\forall x\in Q$. $a\in C_l\cap Q$.

Subdivide Q into N^n closed identical cubes

Suppose $(Q_k)_1^r$ touch C_l



$f(Q_k)$ is contained in a close cube with width: $2C\left(\sqrt{\sum_1^n \left(\frac{L}{N}\right)^2}\right)^{l+1}$ (inside is diagonal length)

$\therefore$ Its volum is:

$$\left(2C\left(\sqrt{\sum_1^n\left(\frac{L}{N}\right)^2}\right)^{l+1}\right)^m = AN^{-(l+1)m}$$

Note that $(Q_k)_1^r$ cover C_l ($\dim C_l \le nl$. $\forall l\in\mathbb{Z}^+$)

$\therefore\ f(Q\cap C_l)$ can be covered by r such cubes.

$\therefore\ m^*(f(Q\cap C_l)) \le ArN^{-m(l+1)} \le A\cdot N^n\cdot N^{-m(l+1)}$

$\longrightarrow 0\ (N\to\infty)$. $\quad\therefore\ m(f(Q\cap C_l))=0$

i.e. $f(C_l)$ is L-null set for large $l\in\mathbb{Z}^+$.