Topological Manifolds

Generally speaking, a manifold is a space that "locally looks like $\mathbb{R}^n$". Actually, we often picture it as a subset of larger vector space.

(1) <u>Definitions</u>:

① <u>Def</u>: A coordinate chart $(U, \tilde{U}, f)$ of topo space X is : $U \underset{open}{\subseteq} X$. $\tilde{U} \underset{open}{\subseteq} \mathbb{R}^n$. and $U \xrightarrow[f]{\simeq} \tilde{U}$. homeomorphism.

<u>Def</u>: A topo space X is manifold if

i) It's C_2 ii) It's Hausdorff

iii) $\forall x \in X$. exists a coordinate chart at X.

<u>Remark</u>: i) For $x, y \in X$. $U_x \cap U_y \neq \emptyset$. Assume:

$U_x \xrightarrow{\simeq} \tilde{U}_x \subseteq \mathbb{R}^{n(x)}$. $U_y \xrightarrow{\simeq} \tilde{U}_y \subseteq \mathbb{R}^{n(y)}$.

Then $n(x) = n(y)$. it's called invariant of domain. From:

For $\forall A, B \subseteq S^n$. If $A \simeq B$. and $A \underset{open}{\subseteq} S^n$. Then $B \underset{open}{\subseteq} S^n$.

<u>Cor</u>. $V \underset{open}{\subseteq} \mathbb{R}^{n_1}$. $V' \underset{open}{\subseteq} \mathbb{R}^{n_2}$. $V \simeq V'$. Then $n_1 = n_2$

<u>Pf</u>: $V \underset{open}{\subseteq} \mathbb{R}^{n_2}$ as well by above.

$\therefore n_2 = n_1$. Otherwise, $n_1 > n_2$, $\exists \prod_1^{n_1} I_k \underset{open}{\subseteq} V$.

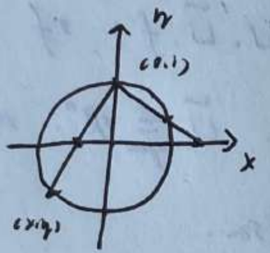
But $\prod_1^{n_1} I_k \not\subseteq \mathbb{R}^{n_2}$. contradict !

ii) If $\forall x \in X$, $r(x) = n$. We call it n-dimension manifold. (Next, our discussion restrict on it)

e.g. i) $S^1 = \{x^2+y^2=1\} \subset \mathbb{R}^2$.

Chart: $\begin{cases} U_1 = S^1/(0,-1), & f_1 = \dfrac{x}{1+y} \\ U_2 = S^1/(0,1), & f_2 = \dfrac{x}{1-y} \end{cases}$

It's called stereographic projection



$\therefore S^1$ is one-dimension manifold.

ii) $S^n = \{(x_0, x_1 \cdots x_n) \in \mathbb{R}^{n+1} \mid \sum_{0}^{n} x_k^2 = 1\}$.

Chart $\begin{cases} U_1 = S^n/(\vec{0},1), & f_1 = \left(\dfrac{x_0}{1-x_n}, \cdots \dfrac{x_{n-1}}{1-x_n}\right) \\ U_2 = S^n/(\vec{0},-1), & f_2 = \left(\dfrac{x_0}{1+x_n}, \cdots \dfrac{x_{n-1}}{1+x_n}\right) \end{cases}$

(Consider projection on each $x_n O x_k$, $0 \le k \le n-1$)

② Manifold with boundary:

Def: A topo space X is n-dimension topo manifold with boundary if $\forall x \in X$, $\exists x \in U \underset{\text{open}}{\subseteq} X$, st. $U \xrightarrow[f]{\sim} \tilde{U} \subseteq \mathbb{R}_{\le 0} \times \mathbb{R}^{n-1}$. f is homeomorphism.

Remark: i) It's not a manifold. It locally looks like a half-space $\{x_1 \le 0\} \subseteq \mathbb{R}^n$. So, $\tilde{U}$ may not be open in $\{x_1 \le 0\}$.

ii) For the point $x \in X$. s.t. $f(x) \in \mathbb{R}_{<0} \times \mathbb{R}^{m}$.

We call it interior point.

For the point $x \in X$. s.t. $f(x) \in \{X_1 = 0\}$. We call it boundary point. Denote the set by ∂X.

e.g. $X = \overline{B_n(0,1)}$, $\partial X = S^{n-1}$.

(2) Atlas:

· How can we switch between the two coordinate systems? Note that
$$\begin{cases} f_1 : U_1 \cap U_2 \xrightarrow{\sim} f_1(U_1 \cap U_2) \subseteq \widetilde{U}_1 \\ f_2 : U_1 \cap U_2 \xrightarrow{\sim} f_2(U_1 \cap U_2) \subseteq \widetilde{U}_2 \end{cases}$$

Def: The transition function between (U_1, f_1) and (U_2, f_2) is $\phi_{12} = f_1 \circ f_2^{-1} : f_2(U_1 \cap U_2) \xrightarrow{\sim} f_1(U_1 \cap U_2)$

Remark: $\phi_{12}^{-1} = \phi_{21}$. ϕ depends on the order.

Def: For X is a topo manifold. An atlas for X is collection of coordinate charts : $f_i : U_i \xrightarrow{\sim} \widetilde{U}_i$.

s.t. $\bigcup\limits_{i \in I} U_i = X$.

An atlas is C^k if $\forall (U_\alpha, f_\alpha)$, (U_β, f_β).

the transition func $\phi_{\alpha\beta}$ is C^k.

Remark: i) For a smooth atlas. Then $\forall \phi$ transition function is diffeomorphism

(i.e. ϕ, ϕ^{-1} are smooth)

ii) Smooth bijection may not be diffeomorphism.

e.g. $f(x) = x^3 : \mathbb{R}^1 \longrightarrow \mathbb{R}^1$

e.g. i) $T^1 = \mathbb{R}^1/\mathbb{Z}^1$. one-dimension torus.

Actually: $T^1 = [0,1]/_{0\sim 1}$. Consider:

$q: \mathbb{R}^1 \longrightarrow \mathbb{R}^1/\mathbb{Z}^1 = T^1$. quotient map

1°) $\forall\, U \underset{open}{\subseteq} T^1$. $q^{-1}(U) = \bigcup_{k\in\mathbb{Z}} U+k$ open

2°) $\forall\, W \underset{open}{\subseteq} \mathbb{R}^1$. $q^{-1}(q(W)) = \bigcup W+k$ open

$\Rightarrow q$ is conti. open mapping.

Construct smooth atlas from q:

$$\begin{cases} \tilde{U}_1 = (0,1).\ U_1 = (0,1).\ f_1 = q^{-1} \\ \tilde{U}_2 = (-\frac{1}{2}, \frac{1}{2}).\ U_2 = T^1/\{\frac{1}{2}\}.\ f_2 = q^{-1} \end{cases} \Rightarrow$$ Check ϕ_{12} and ϕ_{21} are smooth.

Remark: $T^1 \xrightarrow[f]{\sim} S^1$. $f(x) = (\cos 2\pi x, \sin 2\pi x)$

ii) Generally. For $T^n = \mathbb{R}^n/\mathbb{Z}^n$. n-dim torus.

Consider $\tilde{U}_i = \prod_1^n A_i$. $A_i = \begin{cases} (0,1) \\ (-\frac{1}{2}, \frac{1}{2}) \end{cases}$ or

with $f_i = q^{-1}$. check it's smooth atlas.

Remark: $T^n \cong \prod_1^n T^1 \cong S^1\times S^1\times\cdots\times S^1 \ncong S^n$.

(3) Smooth Structure:

① Compatible:

Def: For X is a Topo manifold. A is a smooth atlas. (U, f) is another coordinate chart for X.

(U, f) is compatible with A iff $A \cup \{(U, f)\}$ is still a smooth atlas.

<u>Remark</u>: i) It's not important to know which cochart really in A. The important one is which one is compatible with A.

ii) Check whether (U, f) is compatible with A: only need to consider locally in U.

<u>Lemma.</u>

For (U, f) in a smooth atlas A.

i) $\forall V \underset{open}{\subseteq} U$. $(V, f|_V)$ is compatible.

ii) $\tilde{V} \underset{open}{\subseteq} \mathbb{R}^n$. $g: \tilde{U} \longrightarrow \tilde{V}$. diffeomorphism.

Then $(U, g \circ f)$ is compatible.

<u>Pf.</u> i) $\varphi \circ (f|_V)^{-1} = \varphi \circ f^{-1}|_{f(V \cap U_\varphi)}$. $(f|_V) \circ \varphi^{-1} = f \circ \varphi^{-1}|_{\varphi(V \cap U_\varphi)}$

ii) $(g \circ f) \circ \varphi^{-1} = g \circ (f \circ \varphi^{-1})$. $\varphi \circ (g \circ f)^{-1} = (\varphi \circ f^{-1}) \circ g^{-1}$

<u>Def.</u> For smooth atlas A, B for X. A and B are compatible $\Leftrightarrow$ $A \cup B$ is still smooth atlas.

<u>Lemma.</u> For A and B are compatible. If (U, f) is compatible with A. Then it's compatible with B as well. (Only need to check one in $[A]$)

Pf: For every $(U_j, f_j) \in B$. WLOG. $U_j \cap U \neq \emptyset$.

Consider $\phi_{j0} : f_j \circ f^{-1} : f(U \cap U_j) \rightleftharpoons f_j(U \cap U_j)$

prove: $\forall f(x) \in f(U \cap U_j)$. $\phi_{j0}|_{f(x)}$ is smooth.

Find $(U_i, f_i) \in A$. st. $W = U \cap U_i \cap U_j \neq \emptyset$.

$f(x) \in f(W)$. Then we obtain:

$$\phi_{j0}|_{f(W)} = \phi_{ji}|_{f_i(W)} \circ \phi_{i0}|_{f(W)}$$

It's easy to see: ϕ_{j0}. ϕ_{j0}^{-1} are smooth.

Cor. Compatibility is an equivalence relation

② Smooth manifolds:

Def: A smooth manifold is a topo manifold X together with an equivalent class $[A]$ of compatible smooth atlases (call it smooth structure) on X.

e.g. $\mathbb{R}^1$ with $[\{(id, \mathbb{R}^1)\}]$ is different with $\mathbb{R}^1$ with $[\{(g, \mathbb{R}^1)\}]$ $g = \begin{cases} x, & x \le 0 \\ 2x, & x > 0 \end{cases}$

Remark: i) Smooth atlas may not consist of smooth C^0 chart. Since transition function may not exist!

ii) There're C^∞-diffeomorphism:

$(\mathbb{R}^1, A_1) \xrightarrow{g^{-1}} (\mathbb{R}^1, A_2)$ (see in chart)

Exotic Smooth Structure:

For a pathconnected topo manifold X:

$S_X = \{ M \mid M$ is smooth manifold with the underlying topo $X \} / \simeq$

where "$\simeq$" is equivalance under C^∞- diffeomorphism.

Then: $|S_X| \leq 1$ if $\dim X = 1, 2, 3$

$|S_{\mathbb{R}^n}| = 1$ if $n \neq 4$. $|S_{S^n}| > 1$. $\forall n \in \mathbb{Z}^+$.

$|S_{\mathbb{R}^4}|$ is uncountably infinite.

(4) Pseudo-atlas:

· Actually, an smooth atlas doesn't just determine the smooth structure on X. It can also determine the underlying topology.

For X, a set merely. A pseudo-chart is:

$f: U \subseteq X \xrightarrow{\sim} \hat{U} \subseteq_{open} X$. bijection.

Then a pseudo-altas is $A = \{(U_i, f_i)\}_{i \in I}$. where

$\bigcup_{i \in I} U_i = X$. (U_i, f_i) is pseudo-chart. $\forall i \in I$.

Note that ϕ_{ij} doesn't need to be smooth/conti

prop. If for any two (U_1, f_1). (U_2, f_2) in

A. satisfies:

i) $f_1(U_1 \cap U_2) \underset{open}{\subseteq} \tilde{U}_1$. $f_2(U_1 \cap U_2) \underset{open}{\subseteq} \tilde{U}_2$

ii) ϕ_{21} is cont.

Then. there exists a unique topo on X

st. Each (U_i, f_i) is co-ordinate chart.

<u>Pf.</u> 1°) Uniqueness:

Firstly $\{U_i\}_{i\in I}$ must be open. by def

If $V \underset{open}{\subseteq} U$. Then $f_i(V \cap U_i) \underset{open}{\subseteq} \tilde{U}_i$. $\forall i \in I$.

$\therefore$ $V \cap U_i$ is open. $\forall i \in I$. (f_i is homeo)

Conversely. since $V = \bigcup_{i\in I} (V \cap U_i)$.

if V satisfies the rule. then V is open.

2°) Existence:

check : $V \underset{open}{\subseteq} X \iff f_i(V \cap U_i) \underset{open}{\subseteq} \tilde{U}_i$

determines a topo structure.

<u>e.g</u> i) $\mathbb{RP}^1$ is the set of lines through origin in $\mathbb{R}^2$. Any $(x, y) \neq \vec{0}$ lies in a line.

We denote : $\mathbb{RP}^1 = \{ x:y \mid (x, y) \neq \vec{0}.\ x:y \sim \lambda x : \lambda y.\ \forall \lambda \neq 0 \}$

$$\text{pesudo-chart}: \begin{cases} U_1 = \mathbb{RP}^1 / (0:1). & f_1(x:y) = y/x \to \mathbb{R}^1 \\ U_2 = \mathbb{RP}^1 / (1:0). & f_2(x:y) = x/y \to \mathbb{R}^1 \end{cases}$$

Note that ϕ_{12}, ϕ_{21} are smooth. determines a topo.

Remark: $T^1 = \mathbb{R}^1/\mathbb{Z}^1 \xrightarrow[f]{\sim} \mathbb{R}\mathbb{P}^1$. $f(x) = \cos 2\pi x : \sin 2\pi x$.

ii) Generally. $\mathbb{R}\mathbb{P}^n = \{ x_0 : x_1 : \cdots x_n \mid (x_0, x_1 \cdots x_n) \neq \vec{0} \in \mathbb{R}^{n+1}$

$\lambda x_0 : \lambda x_1 \cdots : \lambda x_n \sim x_0 : x_1 : \cdots x_n .\ \forall \lambda \neq 0 \in \mathbb{R} \}$.

We have: $U_i = \{ x_0 : x_1 : \cdots : x_n \mid x_i \neq 0 \}$ with

$f_i(x_0 : \cdots x_n) = \left(\frac{x_0}{x_i}, \frac{x_1}{x_i} \cdots \hat{i} \cdots \frac{x_n}{x_i} \right)$.

Check ϕ_{ij}'s are smooth.

iii) Grassmanian Manifolds:

Denote $Gr(k,n)$ is set of k-dimension subspaces of $\mathbb{R}^n$.

(e.g. $\mathbb{R}\mathbb{P}^n = Gr(1, n+1)$)

Claim: $Gr(k,n)$ has smooth structure of $k(n-k)$-dim topo.

1) $\forall S \in Gr(k,n)$. Fix basis of S. determine a rank k matrix. $M \in M^{k\times n}(\mathbb{R})$. Note that it equals to PM. $P \in GL_k(\mathbb{R})$. transitive matrix

$\therefore Gr(k,n) = \{ M \in M^{k\times n}(\mathbb{R}),\ r(M) = k \} / GL_k(\mathbb{R})$

2) For $M = (M' \mid M'') \in Gr(k,n)$:

Consider $U_J = \{ M \in Gr(k,n) \mid \exists \tilde{M}.\ M \sim \tilde{M}.$ st. $(\tilde{M}_{i_1}\ \tilde{M}_{i_2} \cdots \tilde{M}_{i_k}) = I_k \}$. $J = \{ i_j \}_1^k \subseteq \{1, 2, \cdots n\}$.

$f_J(M) = (\tilde{M}_{j_1} \cdots \tilde{M}_{j_{n-k}})$. $j_1 < j_2 < \cdots j_{n-k}$. $(j_n) \subseteq \{1, 2 \cdots n\}/J$.

e.g. $U_J = \{ M \in Gr(k,n) \mid M = (M' \mid M'') \sim (I_k \mid N) \}$. $J = \{i\}_1^k$. $f_J(M) = N \in M^{k\times(n-k)}(\mathbb{R})$. Where f_J is bijection. Check the pesudo-atlas is smooth. (determine by transform: $M \to \tilde{M}$.)