Linear Fractional Transform

· **Def.** $\varphi(z) = \dfrac{az+b}{cz+d}$. $a,b,c,d \in \mathbb{C}$. $ad-bc \neq 0$.

Then easy to check: $\overline{\mathbb{C}}_\infty \underset{\varphi}{\cong} \overline{\mathbb{C}}_\infty$

(1) Properties:

① Elementary Group:

φ is generated by the following four kinds elementary transformations:

i) $e^{i\theta}z$ ii) $z+a$

iii) $\frac{1}{z}$ iv) az.

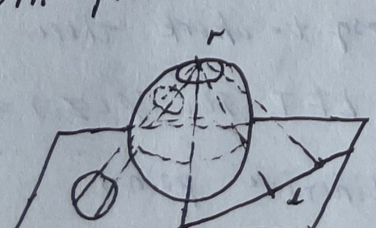
② Fixed Points:

For $\varphi(z_0) = z_0$. z_0 is called fixed point.

φ has at most 2 fixed points except $\varphi = id = z$.

③ Generalized cycle:

We see a line pass ∞ as a circle since it projects on Riemann Surface S^2 will produce a circle cross "N" on S^2.



Thm. C is a circle on $\overline{\mathbb{C}}_\infty$. Then φ will map C to another circle $\tilde{C}$ on $\overline{\mathbb{C}}_\infty$

Pf: Check 4 kinds of LFT in order.

(2) Cross Ratio :

Def: $[z_1, z_2, z_3, z_4] = \dfrac{z_1 - z_2}{z_1 - z_4} \Big/ \dfrac{z_3 - z_2}{z_3 - z_4}$

is cross ratio of distinctive $\{z_i\}_1^4$

Thm. $[z_1, z_2, z_3, z_4] = [\varphi(z_1), \varphi(z_2), \varphi(z_3), \varphi(z_4)]$

For φ is LFT.

Pf: $f(z) = [z, \varphi(z_2), \varphi(z_3), \varphi(z_4)]$

For $h(z) = T \circ \varphi^{-1}$. $T(z) = [z, z_2, z_3, z_4]$.

Then $h \circ f^{-1}$ fix 3 points $\therefore h = f$.

$\therefore T(z) = [\varphi(z), \varphi(z_2), \varphi(z_3), \varphi(z_4)]$

Thm. $\{z_i\}_1^4$ distinctive, on a same circle

$\Leftrightarrow [z_1, z_2, z_3, z_4] \in \mathbb{R}$.

Pf: We can directly check $\arg(\text{Cross Ratio}) = \pi$ or 0.

By Thm above: $[z_1, z_2, z_3, z_4] = [\varphi(z_1), 0, 1, \infty]$

$\therefore [z_1, z_2, z_3, z_4] \in \mathbb{R} \Leftrightarrow \varphi(z_1) \in \mathbb{R} \Leftrightarrow z_1$ on $C(\{z_i\}_2^4)$

Remark: It easy to check there exists unique φ is LFT. st. $\varphi(z_i) = w_i$, $1 \le i \le 3$. distinctive points.

(3) Aut $(\overline{\mathbb{C}_\infty})$:

$\mathrm{Aut}(\overline{\mathbb{C}_\infty}) = \{\varphi \mid \varphi \text{ is a LFT}\}$.

Pf: 1°) For $\varphi(x) = \dfrac{az+b}{cz+d}$. $\varphi \in \mathcal{O}(\mathbb{C})$. when $z \neq -\frac{d}{c}$

we only need to consider $\infty \to \infty$

or finite point $\to \infty$, or $\infty \to$ finite point.

Near ∞. choose coordinate: $(U(\infty), \frac{1}{z})$.

2°) $\forall \varphi \in \mathrm{Aut}(\overline{\mathbb{C}_\infty})$.

Suppose $\varphi: 0, 1, \infty \to \alpha, \beta, \gamma$.

Set $T(z) = [z, \alpha, \beta, \gamma]$. $\therefore T \circ \varphi \in \mathrm{Aut}(\overline{\mathbb{C}_\infty})$

$T \circ \varphi: 0, 1, \infty \to 0, 1, \infty$

Since $T \circ \varphi|_{\mathbb{C}} \in \mathrm{Aut}(\mathbb{C})$. $\therefore T \circ \varphi|_{\mathbb{C}} = az+b$.

(4) Symmetry:

Thm. C is a circle on $\overline{\mathbb{C}_\infty}$. $z_2, z_3, z_4 \in C$. distinctive

Then z_1, z^* are symmetric w.r.t C $\Leftrightarrow$

$[z_1, z_2, z_3, z_4] = \overline{[z^*, z_2, z_3, z_4]}$

Pf: 1°) C is a line.

Note that translation and rotation

won't change the relative position of

z_1 and z^*. Suppose C is x-axis.

2°) For C is a cycle. $|z-a|=r$.

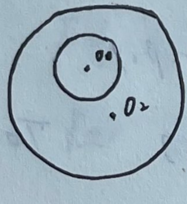
Choose $\varphi(z)=z-a$. retain the cross ratio!

<u>Cor.</u> z is symmetric with z^*. w.r.t circle C

φ is LFT. Then $\varphi(z)$ is symmetric

with $\varphi(z^*)$. w.r.t $\varphi(C)$.

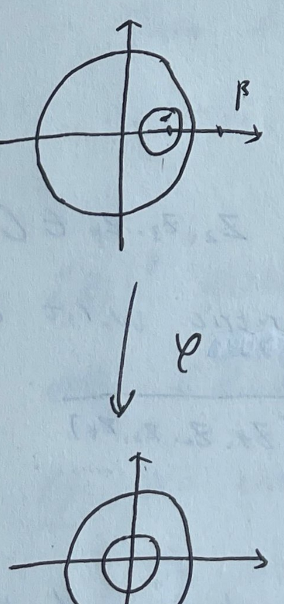
(5) <u>Application</u>:

① 

$\cong$ A. A is some annulus.

<u>Pf</u>: WLOG. suppose $C_1: |z-a|\le r_1$, $C_2: |z|\le r_2$.

$r_2>2r_1$. $a<r_2-r_1$. (By rotation, translation)



Find α. β on x-axis, st.

$\bar{\alpha}\beta = r_2^2$. $(\bar{\alpha}-a)(\beta-a)=r_1^2$

Then α. β. sym. w.r.t C_1. C_2

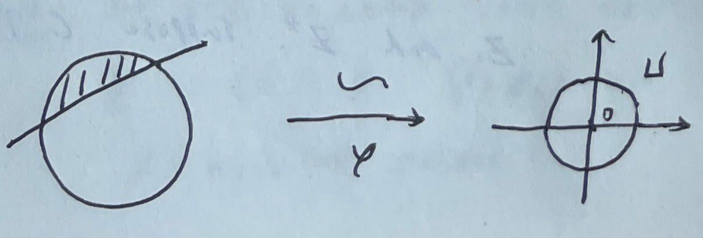
Note that $\varphi(z)=\dfrac{z-\alpha}{z-\beta}$ map

$\alpha\to 0$. $\beta\to\infty$. sym. w.r.t C_1. C_2

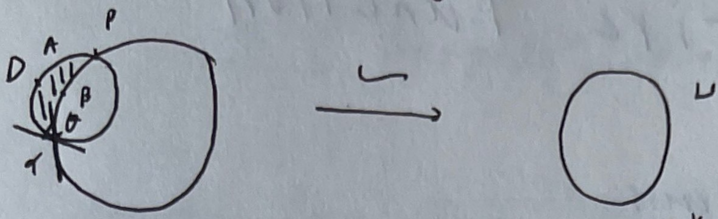
$\therefore$ $\varphi(C_1)$. $\varphi(C_2)$ are circles

with center at origin.

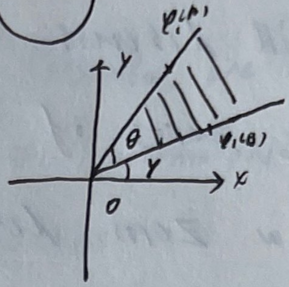
<u>Remark</u>: For C_1 is a line (Degenerate)



(2) Find a biholomorphism φ:



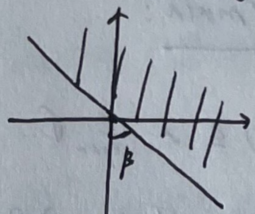
i) $\varphi_1(z) = \dfrac{z-\alpha}{z-\beta} : D \longrightarrow$



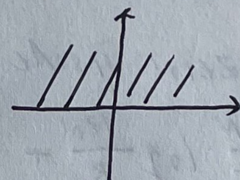
$(0 < \theta < \frac{\pi}{2})$

ii) Consider a holomorphic branch: $\varphi_2(z) = z^{\frac{\pi}{\theta}} = e^{\frac{\pi}{\theta}\log z}$

$\varphi_2 : \varphi_1(D) \longrightarrow$



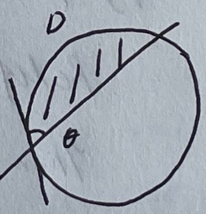
iii) $\varphi_3 : \varphi_2 \circ \varphi_1(D) \longrightarrow$



$\varphi_3 = e^{i(\pi-\beta)}$

iv) By Cayley Transform: $\varphi_4 = \dfrac{-z+i}{z+i} \longrightarrow U$.

Remark: Degenerate Case:

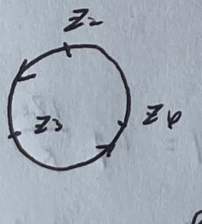


$\cong U$

(6) Orientation:

. For C is a cycle. $z_1, z_2, z_3 \in C$. distinct.

Then (z_2, z_3, z_4) defines an orientation of C.



Left side: $\{z \mid \operatorname{Im}[z, z_2, z_3, z_4] > 0\}$

Right side: $\{z \mid \operatorname{Im}[z, z_2, z_3, z_4] < 0\}$

Oriental Principle:

φ is LFT. w.r.t (z_2, z_3, z_4) of C. $(\varphi(z_2), \varphi(z_3), \varphi(z_4))$ of $\varphi(C)$. Then the left side of C correspond left side of $\varphi(C)$