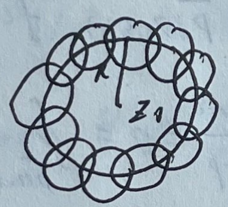
Holomorphic Extension

(1) Extension by Series:

Thm. $f(z)$ expands at $z=z_0$ is: $\sum_0^\infty a_n(z-z_0)^n$ with convergent radius R. Then f has at least one pole on $|z-z_0|=R$.

Pf: If not. $f \in \mathcal{O}(|z-z_0| \leq R)$.

Expand f on $|z-z_0| \leq R$.



$\therefore f$ can be extended to F on $|z-z_0| \leq R_1$ where $R_1 > R$. By Uniqueness. $f=F$.

$\therefore$ the convergent radius $> R$.

which is a contradiction.

Thm. $f(z) = \sum_0^\infty a_n(z-z_0)^n$ coverges on circle $|z-z_0| \leq R$.

If f has a pole ζ_0 on $|z-z_0|=R$. Then f diverges on $|z-z_0|=R$.

Pf: If exist one point η_0 on $|z-z_0|=R$.

$f(\eta_0) = \sum a_n(\eta_0-z_0)^n$ converges.

Then $a_n(\eta_0-z_0)^n \to 0 \quad \therefore |a_n R^n| \to 0$.

1°) Note that: $\lim\limits_{z\to\zeta_0} (z-\zeta_0)f(z) \neq 0$

2°) Suppose $\zeta_0 = z_0 + Re^{i\theta_0}$

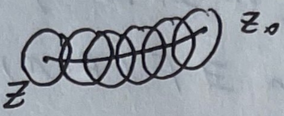
prove: $\lim\limits_{r\to R^-} (re^{i\theta_0} - Re^{i\theta_0}) f(re^{i\theta_0}+z_0) = 0$

since $|re^{i\theta_0} - Re^{i\theta_0}| |\sum a_n r^n e^{i\theta_0}| \leq$

$$|r-R|\sum_{1}^{N}|a_n|r^n + |r-R|\sup_{r\ge N}|a_n|R^n\sum_{N}^{\infty}\left(\frac{r}{R}\right)^n$$

$$= \varepsilon C + \varepsilon|r-R|\frac{1}{1-\frac{r}{R}} = C\varepsilon.$$

Method: From z to z_0. f holomorphic at z.



Then use the expansion of series. we can extend f by disc. from z to z_0

An example:

$f(z) = \sum_{k=1}^{\infty} z^{k!}$ converges on $|z|<1$.

But on $|z|=1$. every point is pole of f.

Pf: since poles of f is closed set.

Prove: it's a dense set on $|z|=1$.

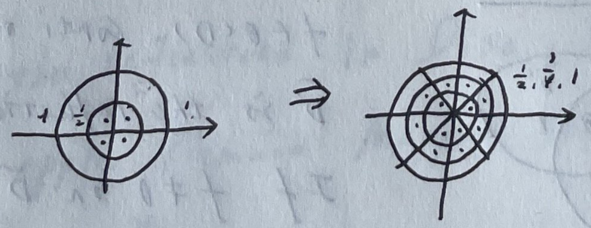
For $\{e^{2\pi q i} \mid q\in\mathbb{Q}\} \underset{dense}{\subseteq} |z|=1$

$\{k!\}$ will cover the period of $e^{2\pi q i}$

Thm. (Alternative)

$D \underset{open}{\subseteq} \mathbb{C}$. $\exists f\in\mathcal{O}(D)$. $f\not\equiv c$. s.t. f can't be extend across any boundary point of ∂D.

Pf:



"." is a z_k

Construct $\{z_k\}_1^{\infty}$. isolated in int$D$.

But accumulate at every point $\in\partial D$.

By Weierstrass Thm. $\exists f\in\mathcal{O}(D)$. Vanishes only on $\{z_k\}$.

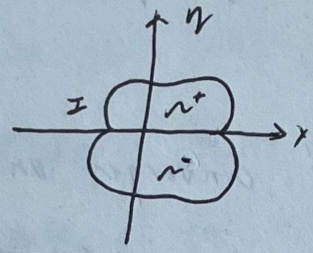
Check f can't be extended outside $\bar{D}$!

(2) Reflection Principle:

① Schwartz Reflection:

Thm. $f \in \mathcal{O}(\Omega^+)$, conti on I, and $f(x) \in \mathbb{R}$ when $x \in I$. Then $\exists F \in \mathcal{O}(\Omega)$, s.t. $F|_{\Omega^+} = f$.

Pf:



$(\Omega = \Omega^+ \cup \Omega^- \cup I)$

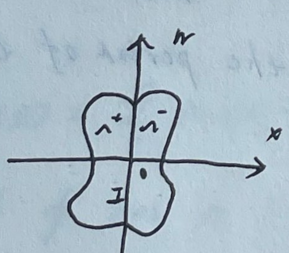
Let $F(z) = \begin{cases} f(z), & z \in \overline{\Omega^+} \\ \overline{f(\bar z)}, & z \in \overline{\Omega^-} \end{cases}$

Note that $\frac{\partial}{\partial \bar z} \overline{f(\bar z)} = \overline{\frac{\partial}{\partial z} f(\bar z)} = 0$.

$\therefore F(z) \in \mathcal{O}(\Omega^-)$.

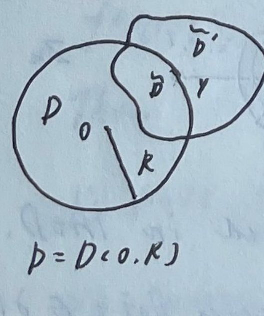
By Morera, check $F \in \mathcal{O}(\Omega)$

Cor.



In this case: Let $F(z) = \begin{cases} f(z), & z \in \overline{\Omega^+} \\ -\overline{f(-\bar z)}, & z \in \Omega^- \end{cases}$

② Case in Disc:



$D = D(0, R)$

$f \in \mathcal{O}(D)$, conti on γD.
$\tilde D'$ is the reflection of $\tilde D$.
If $f \not\equiv 0$ on $\tilde D$. Then $\exists F \in \mathcal{O}(\overline{\tilde D} \cup \tilde D')$. $F|_{\tilde D} = f$.

Pf: Let $F(z) = \begin{cases} f(z), & z \in \overline{\tilde D} \\ \dfrac{R^2}{\overline{f\left(\frac{R^2}{\bar z}\right)}}, & z \in \tilde D' \end{cases}$. check!

Remark: If $\tilde{D} \cap \{0\} \neq \emptyset$. Then $\tilde{D}'$ will tend to ∞.

If f has zero on $\tilde{D}$. Then f can only be extended meromophically since F has a pole at zero of f.

(3) Application:

By Uniquess of holomorphic function. The extension will coincide with the original one. Moreover, the method of reflection will endow f will special form

prop. $f \in \mathcal{O}(\mathbb{C})$. $f: \mathbb{R} \to \mathbb{R}$. $i\mathbb{R} = \{ix \mid x \in \mathbb{R}\}$.

$f: i\mathbb{R} \to i\mathbb{R}$. Then $f(z) = -f(-z)$

Pf: By the two reflection in (1).

$\Rightarrow f(z) = \overline{f(\bar{z})}$. $f(z) = \overline{-f(-\bar{z})}$

$\therefore f(z) = -f(-z)$.

② For the reflection in Disc. Sometimes we can apply Riemann Mapping Thm on D.

Let $D \xrightarrow{\varphi} \mathbb{H}$. Reflect $f \circ \varphi^{-1}$. repeatedly!