Preliminaries.

(1) Holomorphic Functions:

① i) For: $f(z)$ on $\mathbb{C}$. $f(z) = u(z) + iv(z)$.

with norm $|f| = (u^2+v^2)^{\frac{1}{2}}$

ii) Actually. $(\mathbb{R}^2, J) \cong \mathbb{C}$. where J is

the complex structure: $\mathbb{R}^2 \xrightarrow{J} \mathbb{R}^2$. one-to-one.

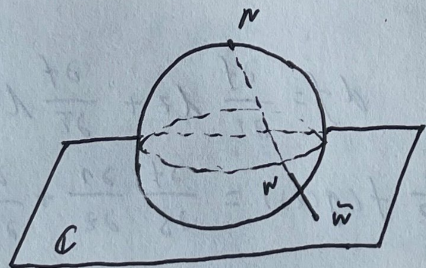
$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Define $ix = Jx$, $x \in \mathbb{R}^2$.

Note that $J^2 = -id$

Then $u(z) \triangleq u(x,y)$, when $z = x+yi$.

iii) Geometrically, $\mathbb{C} \cong S^2/\{N\}$. where S^2

is Riemann Sphere.



Note that $\mathbb{C}$ is a LCH space. Then we can use one-point-compactification $\mathbb{C} \cup \{\infty\} = \overline{\mathbb{C}}_\infty$

Then $S^2 \underset{f}{\cong} \overline{\mathbb{C}}_\infty \cong \mathbb{C}(\mathbb{P}^1)$

where f is sterographic projection $f: w \mapsto \tilde{w}$.

② Continuity and

differentiability:

i) $f(z) = u + iv$ is conti $\Leftrightarrow$ u, v conti.

e.g. $\arg(z)$ conti. on $\mathbb{C}/\{z\le 0\}$.

ii) If $f(z)$ is differentiable at $z_0 = x_0 + iy_0$

Then it's neccessary that: for $h = h_1 + ih_2$

$$\lim_{\substack{h_1 = 0 \\ h_2 \to 0}} \frac{f(z_0+h) - f(z_0)}{h} = \lim_{\substack{h_2 = 0 \\ h_1 \to 0}} \frac{f(z_0+h) - f(z_0)}{h}$$

i.e. $\dfrac{\partial f}{\partial x} = \dfrac{1}{i}\dfrac{\partial f}{\partial y}$. we obtain: $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

called Cauchy - Riemann Equation

Def: $\dfrac{\partial}{\partial z} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} + \dfrac{1}{i}\dfrac{\partial}{\partial y}\right)$. $\dfrac{\partial}{\partial \bar{z}} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} - \dfrac{1}{i}\dfrac{\partial}{\partial y}\right)$

Remark: $4\dfrac{\partial^2}{\partial z \partial \bar{z}} = \Delta$ (Laplace Operator)

Def: $f'(z_0) = \dfrac{\partial f}{\partial z}\Big|_{z=z_0}$

Remark: We have $df = \dfrac{\partial f}{\partial z} dz + \dfrac{\partial f}{\partial \bar{z}} d\bar{z}$.

Then $\dfrac{\partial}{\partial z} f(g(z)) = \dfrac{\partial f}{\partial \zeta}\dfrac{\partial g}{\partial z} + \dfrac{\partial f}{\partial \bar{\zeta}}\dfrac{\partial \bar{g}}{\partial z}$

when f is differentiable. f is derivable.

Thm. f is derivable at z_0 $\Leftrightarrow$ u, v are differentiable at z_0. satisfies Cauchy - Riemann Equation.

Remark: Derivability is concerning local of one dimension. $f(z)$ may be derivable on a line: e.g. $x^2 - iy$

Pf: For: u, v are differentiable. Then:

$$f(z) = \frac{\partial f}{\partial z}(z - z_0) + \frac{\partial f}{\partial \bar{z}}(\overline{z - z_0}) + f(z_0) + o(|z - z_0|)$$

Thm. f is holomorphic in $D \underset{\text{open}}{\subseteq} \mathbb{C}$. $\Leftrightarrow$

$u, v \in C^1[D]$. Satisfies C-R equation.

Remark: Holomorphic (Analytic) is defined in an open neighbour. If f is derivable in $D \Rightarrow f \in \theta(D)$. (holomorphic)

i.e. Holomorphic is two-dimensional local

Cor. $f \in \theta(D) \Leftrightarrow u, v \in C^1(D)$, $\frac{\partial f}{\partial \bar{z}} = 0$. $\forall z \in D$

Pf: Since u, v satisfies C-R equation

$$\Leftrightarrow \frac{\partial f}{\partial \bar{z}} = 0. \ \forall z \in D.$$

Prop. If $\operatorname{Re} f$ or $\operatorname{Im} f$ or $|f| \equiv$ Const. Then $f \equiv$ const. when $f \in \theta(D)$. $\forall z \in D$.

$\rightarrow$ Alternative: open map Thm.

Pf: By C-R equation. The first two ✓

And $\frac{\partial f\bar{f}}{\partial z} = \frac{\partial |f|^2}{\partial z} = 0 \quad \therefore \overline{\frac{\partial f}{\partial z}} f = 0.$

Cor. $f: D \longrightarrow \mathbb{R}$. $f \in \theta(D)$. $D \subseteq \mathbb{C}$. Then $f \equiv$ const.

(2) Elementary Complex Func:

① Univariate Functions:

ex. $e^{z} = e^{x+iy} = e^{x}(\cos x + i\sin y)$

ex. $\cos z = \dfrac{e^{iz}+e^{-iz}}{2}$, $\sin z = \dfrac{e^{iz}-e^{-iz}}{2i}$

Remark: e^{z}, $\cos z$, $\sin z \in \mathcal{O}(\mathbb{C})$. They're unbounded. But the limit doesn't exist when $z \to \infty$.

They're well-def univalue Func.

prop. $e^{z_1}\cdot e^{z_2} = e^{z_1+z_2}$, $\cos^2 z + \sin^2 z = 1$

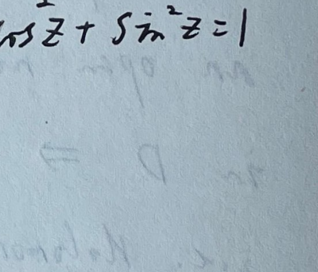
② Multivalue Functions:

i) Arg z:

ex. $\mathrm{Arg}(z) = \arg(z) + 2k\pi = \arg(\frac{y}{x}) + 2k\pi$.

where $k \in \mathbb{Z}$, $z = x+iy$, $\arg(z) \in (-\pi, \pi]$

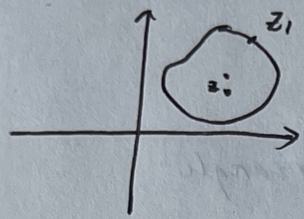
arg(z) is define commonly on $\mathbb{C}/\{z \le 0\}$.



$\lim\limits_{\substack{z\to 0\\ y\to 0^{+}}} \arg(z) = \pi$

$\lim\limits_{\substack{z\to 0\\ y\to 0^{-}}} \arg(z) = -\pi$

Def: If $f(z)$ move on a curve contouring the point z_0 one whole circle. Suppose it starts at z_1 with value $w_0 = f(z_1)$



But the second time it's back to z_1 with value w_1, $w_0 \neq w_1$. Then z_0 is called a pivot of $f(z)$.

The line connects two pivot is called the parting line.

e.g. 0 and ∞ are 2 pivots of $\arg(z)$.

② $\operatorname{Ln} z$:

※ $\operatorname{Ln} z = \ln|z| + i\arg(z) + 2k\pi i$. $k \in \mathbb{Z}$.

$z = 0, \infty$ is its pivot.

Prop. $\operatorname{Ln} z^\alpha = \alpha \operatorname{Ln} z \iff \alpha = \frac{1}{n}, n \in \mathbb{Z}$.

③ z^a ($a \in \mathbb{C}/\mathbb{Z}$):

※ $z^a = e^{a \operatorname{Ln} z} = e^{a(\ln|z| + i\arg(z) + 2k\pi i)}$. $k \in \mathbb{Z}$.

$z = 0, \infty$ is its pivot.