Central Limit Thm.

(1) Matrix of r.v.'s:

Def: Follows is called double array : $\{k_n\} \subseteq \mathbb{Z}^+$.

$$\begin{matrix} X_{11} & X_{12} & \cdots & X_{1,k_1} \\ X_{21} & X_{22} & \cdots & X_{2,k_2} \\ & \vdots & & \\ X_{n1} & X_{n2} & \cdots & X_{n,k_n} \\ & \vdots & & \end{matrix}$$

s.t. each r.v. in row is indept.

Denote: $E(X_{ni}) = \sigma_{ni}$. $S_n = \sum_1^{k_n} X_{n,i}$. $s_n^2 = \sigma^2(S_n)$.

$E(|X_{nj}|^3) = \gamma_{nj}$. $\Gamma_n = \sum_1^{k_n} \gamma_{nj}$.

prop. (Negibility)

i) $\forall i$. $p(|X_{ni}| \geq \varepsilon) \to 0$ $(n \to \infty)$ for $\varepsilon > 0$.

ii) $\max_{1\leq k\leq k_n} p(|X_{nk}| \geq \varepsilon) \to 0$ $(n\to\infty)$ for $\varepsilon > 0$.

iii) $p(\max_{1\leq k\leq k_n} |X_{nk}| \geq \varepsilon) \to 0$ $(n\to\infty)$ for $\varepsilon>0$.

iv) $\sum_1^{k_n} p(|X_{ni}| \geq \varepsilon) \to 0$ $(n\to\infty)$. for $\varepsilon > 0$.

Then iv) $\Rightarrow$ iii) $\Rightarrow$ ii) $\Rightarrow$ i).

Rmk: If $\{X_{nk}\}_1^{k_n}$ indept. Then iii) $\Rightarrow$ iv)

Since $p(\max_{1\leq k\leq k_n} |X_{nk}| \geq \varepsilon) = p(\bigcup_1^{k_n} \{|X_{nk}| \geq \varepsilon\})$

$= \sum_1^{k_n} p(|X_{nk}| \geq \varepsilon) \to 0$.

Def: If $(X_{n,k})$ satisfies ii). Then call it holospondic.

Thm. (X_{nk}) is holospondic $\Leftrightarrow$ $\forall t \in \mathbb{R}'$. $\max_{1\le k\le k_n} |\varphi_{nj}(t)-1| \to 0$.

where (φ_{nj}) are the correspond ch.f's.

Pf: ($\Rightarrow$) $|\varphi_{nj}(t)-1| \le \int |e^{itx}-1| dF_{nj}(x) = \int_{|x|\ge\varepsilon} + \int_{|x|<\varepsilon}$

$\le 2\int_{|x|\ge\varepsilon} dF_{nj}(x) + \int_{|x|\le\varepsilon} |tx| dF_{nj}(x)$

($\Leftarrow$) By Lemma: $P(|X_{ni}|\ge\varepsilon) \le \frac{\varepsilon}{2}\int_{-\frac{2}{\varepsilon}}^{\frac{2}{\varepsilon}} (1-\varphi_{ni}) dt$.

Then apply DCT.

(2) Liapounov's CLT:

① Lemma:

For $\{\theta_{nj}\}_{n\ge1}^{1\le j\le k_n} \subseteq \mathbb{C}$. Satisfies: for $\theta<\infty$.

i) $\max_j |\theta_{nj}| \to 0$ $(n\to\infty)$ ii) $\sum_j |\theta_{ni}| \le M < \infty$. $\forall n$.

iii) $\sum_j^{k_n} \theta_{nj} \to \theta$ $(n\to\infty)$. Then $\prod_i^{k_n}(1+\theta_{ni}) \to e^{\theta}$.

Pf. $e^{\ln \prod_i^{k_n}(1+\theta_{ni})} = e^{\sum \ln(1+\theta_{ni})}$. By Taylor expansion.

$\sum \theta_{ni}^2 \le \max_i |\theta_i| \sum|\theta_{ni}| \le M \max_i |\theta_{ni}| \to 0$.

② Thm.

If $\sum_i^{k_n} \sigma_{ni}^2 = 1$. $a_{ni}=0$. $\forall 1\le i\le k_n$. $\gamma_{ni} \le M<\infty$. $\forall i,n$.

Besides $\Gamma_n \to 0$ $(n\to\infty)$. Then $S_n = \sum_i^{k_n} X_{ni} \xrightarrow{d} N(0,1)$.

Pf. $\varphi_{ni}(t) = 1 - \frac{1}{2}\sigma_{ni}^2 t^2 + \Lambda_{ni}\gamma_{ni}|t|^3$. $|\Lambda_{ni}| \le \frac{1}{6}$

$\theta_{ni} = -\frac{1}{2}\sigma_{ni}^2 t^2 + \Lambda_{ni}\gamma_{ni}|t|^3$. directly check.

Cor. If $\sum^{k_n} \sigma_{ni}^2 = 1$. $|X_{ni}| \le M_{ni}$ a.s. $\max_i |M_{ni}| \to 0$.

Then $S_n - E(S_n) \to_d N(0,1)$. $S_n = \sum_1^{k_n} X_{ni}$.

Pf: Replace by $X_{ni} - E(X_{ni})$.

$E(|X_{ni} - E(|X_{ni}|)|^3) \le 2M_{ni}\sigma_{ni}^2$. $\therefore I_n \to 0$.

Rmk: For $\{X_n\}$ indept. $\sigma_n^2 < \infty$. $\gamma_n < \infty$. Let X_{ni}

$= X_i / (\sum_1^n \sigma_k^2)^{\frac{1}{2}}$. If $I_n / (\sum_1^n \sigma_k^2)^{\frac{3}{2}} \to 0 \ (n \to \infty)$

Then: $\sum_1^n X_k - E(X_k) / (\sum_1^n \sigma_k^2)^{\frac{1}{2}} \to_d N(0,1) \ (n \to \infty)$

(3) Lindeburg Feller's CLT:

① Thm.

For $\{X_{nk}\}$. $E(X_{nk}) = 0$. $\sum \sigma_{nk}^2 = 1$. Then

the followings are equi.

i) $\forall \varepsilon > 0$. $\lim_n \sum_1^{k_n} E(|X_{ni}|^2 I_{\{|X_{ni}| \ge \varepsilon\}}) = 0$.

ii) $\max_k \sigma_{nk}^2 \to 0 \ (n \to \infty)$. $S_n \to_d N(0,1)$.

Pf: i) $\Rightarrow$ ii).

$\sigma_{nk}^2 = E(X_{nk}^2 (I_{\{|X_{nk}| \ge \varepsilon\}} + I_{\{|X_{nk}| < \varepsilon\}}))$

$\le \varepsilon^2 + \sum E(X_{nk}^2 I_{\{|X_{nk}| \ge \varepsilon\}})$

Next. check $\prod \varphi_{ni} \to e^{-\frac{t^2}{2}}$.

1°) $\sum | \ln \varphi_{nk} - (\varphi_{nk} - 1)| \le$

$\sum |\varphi_{nk} - 1|^2 \le \max_k |\varphi_{nk} - 1| \sum_1^{k_n} |\varphi_{nk} - 1|$

check: $\max_k |\varphi_{nk}-1| \to 0$. $\sum |\varphi_{nk}-1| \le \frac{t^2}{2}$.

$\therefore \sum |\ln\varphi_{nk}+1-\varphi_{nk}| \to 0$. $(n\to\infty)$.

2°) $|\sum(\varphi_{nk}-1)+\frac{t^2}{2}| \to 0 \quad (n\to\infty)$

$$LHS = |\sum E(e^{itX_{nk}}-1-itX_{nk}-\frac{1}{2}(itX_{nk})^2)|$$

$$\le \sum E(\min\{t^2X_{nk}^2, \frac{1}{6}|tX_{nk}|^3\})$$

$$\le \sum E(t^2X_{nk}^2 I_{\{|X_{nk}|\ge\varepsilon\}}) + \frac{|t|^3\varepsilon}{6}\sum E(X_{nk}^2 I_{\{|X_{nk}|\le\varepsilon\}})$$

$$\to \frac{|t|^3}{6}\varepsilon \to 0.$$

ii) $\Rightarrow$ i):

Note that $\max\sigma_{nk}^2 \to 0 \Rightarrow \sum|\ln\varphi_{nk}+1-\varphi_{nk}| \to 0$ in 1°).

$\therefore$ From $S_n \to_d N(0,1)$. We have: $\sum(\varphi_{nk}-1)+\frac{t^2}{2} \to 0$.

$$\therefore 0 \leftarrow Re(\sum(\varphi_{nk}-1)+\frac{t^2}{2}) = \sum E(\cos(tX_{nk})-1+\frac{1}{2}t^2X_{nk}^2)$$

$$\ge \sum E(\square\, I_{\{|X_{nk}|\ge\varepsilon\}}). \text{ since } \cos x-1+\frac{x^2}{2}\ge 0$$

$$\ge \sum E(\frac{1}{2}t^2X_{nk}^2-2\, I_{\{|X_{nk}|\ge\varepsilon\}})$$

$$\ge \sum E(X_{nk}^2(\frac{t^2}{2}-\frac{2}{\varepsilon^2})I_{\{|X_{nk}|\ge\varepsilon\}}). \text{ fix } t^2>\frac{4}{\varepsilon^2}.$$

<u>Cor.</u> For $\{X_{nk}\}$. $a_{nk}=0$. $\sum_1^{k_n}\sigma_{nk}^2=1$. If $\sum_1^{k_n}E(|X_{nk}|^{2+\delta})\to 0$ as $n\to\infty$. for $\delta\ge 0$. Then $S_n\to_d N(0,1)$.

<u>Pf</u>: $E(|X_{nk}|^{2+\delta}) \ge \varepsilon^\delta E(|X_{nk}|^2 I_{\{|X_{nk}|\ge\varepsilon\}})$.

<u>Rmk</u>: $\max\sigma_{nk}^2 \to 0 \Leftarrow \{X_{nk}\}$ is holospoudic

<u>Pf</u>: $(\Rightarrow)$ $P(|X_{nk}|\ge\varepsilon) \le \frac{\sigma_{nk}^2}{\varepsilon^2} \to 0$

$(\Leftarrow)$. $\varphi_{nk}-1 = -\frac{\sigma_{nk}^2}{2}t^2+o(t^2) \to 0$

$\therefore \max\sigma_{nk}^2 \to 0$. by $\max|\varphi_{nk}-1|\to 0$.

Thm. (general form)

For $\{X_{nk}\}$, indept. Then follows are equi.

i) $\exists (a_n)$ seq. st. $S_n - a_n \to_d N(0,1)$

and $\{X_{n,k}\}$ is holospondic

ii) $\sum_1^{k_n} E(X_{nk}^2 I_{\{|X_{nk}|\le\varepsilon\}} - E^2(X_{nk} I_{\{|X_{nk}|\le\varepsilon\}})) \to 1$

and $\sum_1^{k_n} E(I_{\{|X_{nk}|\ge\varepsilon\}}) \to 0$. $\forall \varepsilon > 0$.

② Apply in indept. r.v's:

Thm. For $\{X_n\}$ indept. nondegenerated. $E(X_n)=0$. $\sigma_n^2 < \infty$.

Set $S_n = \sum_1^n X_k$. $B_n^2 = \sum_1^n \sigma_k^2$. Then i), ii) equi.

i) $\forall \varepsilon > 0$. $B_n^{-2} \sum_1^n E(X_k^2 I_{\{|X_k|\ge \varepsilon B_n\}}) \to 0$

ii) $\max_k (\sigma_k^2 / B_n^2) \to 0$, $S_n / B_n \to_d N(0,1)$.

Pf: Take $X_{n,k} = X_k / B_n$.

Rmk: $\max_k (\sigma_k^2/B_n^2) \to 0$ is avoiding some X_k

has a dominated variance. Then: $S_n/B_n \not\to_d \varphi$.

Cor. Under the same conditions above:

If $\frac{1}{B_n^{2+\delta}} \sum E(|X_n|^{2+\delta}) \to 0$. for $\delta \ge 0$.

Then $S_n / B_n \to_d N(0,1)$.

Rmk: $B_n^{-2} \sum_1^n E(X_k^2 I_{\{|X_k|\ge \varepsilon B_n\}}) \to 0$ is equi. with:

$B_n^{-2} \sum_1^n E(X_k^2 I_{\{|X_k|\ge \varepsilon B_k\}}) \to 0$.

Pf: ($\Leftarrow$) $I = \sum_{\{B_k < \delta B_n\}} + \sum_{\{\varepsilon B_k \geq \delta B_n\}}$

$$\leq \delta^2 + \sum_1^n E(X_k^2 I_{(|X_k| \geq \varepsilon\delta B_n)}) \to \delta^2 \to 0.$$

(3) For i.i.d. r.v.'s:

i) Levy Thm.

$\{X_k\}_1^n$ i.i.d. $E(X_k)=0$. $\sigma^2 < \infty$. Then: $S_n/\sigma\sqrt{n} \xrightarrow{d} N(0,1)$

Pf: $\varphi_{S_n/\sigma\sqrt{n}}(t) = \varphi^n(t/\sqrt{n}\sigma) = (1 - \frac{t^2}{2n} + o(n))^n \to e^{-\frac{t^2}{2}}$

ii) General Case:

Thm. $\{X_n\}$ i.i.d. Then $\exists B_n, A_n$ s.t. $\frac{\sum_1^n X_k - A_n}{B_n}$

$$\xrightarrow{d} N(0,1). \iff \lim_{x \to \infty} \frac{P(|X| \geq x)}{x^{-2} E(X^2 I_{(|X| \leq x)})} = 0.$$

Rmk: $E(|X|^2) < \infty \Rightarrow \lim_x \frac{P(|X| \geq x)}{x^{-2} E(X^2 I_{\{|X| \leq x\}})} = 0.$

But converse can only imply: $\forall \delta > 0$.

$E(|X|^{2-\delta}) < \infty$. but not $E(|X|^2) < \infty$.

(4) Barry - Esseen Bound:

(1) Uniform:

Thm. $\{X_n\}$ i.i.d. r.v.'s. Let $\delta \in (0,1]$.

$E(X_1) = 0$. $E(|X_1|^{2+\delta}) < \infty$. $E(X_1^2) = \sigma^2 > 0$

Then for $\forall n \in \mathbb{Z}^+$: $\sup_x |F_n - \Phi| \le \frac{A\,\ell_\delta}{n^{\frac{\delta}{2}}}$ where

$\ell_\delta = E(|X_1|^{2+\delta})/\sigma^{2+\delta}$.

Rmk: Even if r.v. has all moments of order.

The order of error is still $O(n^{-\frac{1}{2}})$.

e.g. Random walk in $\mathbb{Z}^1$.

② Non-uniform Case:

Thm. $\{X_n\}$ i.i.d. $E(|X_1|^{2+\delta}) < \infty$. $\delta \in (0,1]$. Then:

$$|F_n(x) - \Phi(x)| \le \frac{C_\delta E(|X_1|^{2+\delta})}{\sigma^{2+\delta}\, n^{\delta/2}} \frac{1}{1+|x|^{2+\delta}}, \quad \forall x,\ n \in \mathbb{Z}^+.$$

③ Edgeworth Expansion:

Thm. $\{X_k\}$ i.i.d. $E(X_1) = 0$. $E(X_1^2) = \sigma^2$. $\mu_3 = E(X_1^3) < \infty$.

If F is nonlattice. d.f. Then. we have:

$$\sup_x \left| F_n - \Phi + \frac{\mu_3}{6\sigma^3\sqrt{n}} H_2(x)\varphi(x) \right| = o(n^{-\frac{1}{2}}).$$

where $H_k(x)\varphi(x) = \frac{1}{2\pi}\int (it)^k e^{-\frac{t^2}{2}} e^{-itx}\,dt$

is the correction