Characteristic Functions

(1) Definition and Properties:

<u>Def</u>: ch.f of r.v. X is $\varphi_X(t) = E(e^{itX})$.

① Elementary properties:

i) $\varphi(0)=1$. $|\varphi(t)| \le 1$.

ii) $\varphi(t)$ is uniformly conti on $\mathbb{R}'$.

<u>pf</u>: $|\varphi(t+h)-\varphi(t)| \le E|e^{ihX}-1|$

iii) $\varphi_X(-t) = \varphi_{-X}(t) = \overline{\varphi_X(t)}$. So. φ_X is real $\Leftrightarrow$

X is symmetric about origin.

iv) $X.Y$ are indept $\Rightarrow \varphi_{X+Y}(t) = \varphi_X(t)\varphi_Y(t)$.

<u>Rmk</u>: Converse is false: $X=Y \sim$ Cauchy (1).

<u>Thm</u>. $X.Y$ are indept $\Leftrightarrow \varphi_{(X,Y)}(s,t) = \varphi_X(s)\varphi_Y(t)$.

<u>Pf</u>: $(\Leftarrow)$ p.m $\mu_{(X,Y)}$ has same ch.f as $\mu_X \times \mu_Y$.

By Inversion and Uniqueness:

$\mu_{(X,Y)} = \mu_X \times \mu_Y$. $\therefore X.Y$ are indept.

<u>Cor</u>. $E(g(X)f(Y)) = E(g(X))E(f(Y))$. for

$\forall g.f \in C_B \Leftrightarrow X.Y$ are indept.

<u>pf</u>: Chose e^{itx}. Alternative: approxi by I_B.

v) $\{\varphi_k\}_{k\in\mathbb{Z}^+}$ are ch.f. $\sum_1^\infty a_k = 1$. Then so is $\sum_1^\infty a_k\varphi_k$ (For $\{\varphi_k\}_1^n$, $\sum_1^n a_k = 1$. it's trivial)

<u>Pf</u>: Set Z. r.v. $P(Z=k) = a_k$.

Then $\varphi_{X_Z} = \sum a_k\varphi_k$. ch.f of X_Z.

<u>Rmk</u>: d.f's $\{F_k\}_1^n$. $\sum_1^n a_k = 1 \Rightarrow F = \sum_1^n a_k F_k$. d.f.

vi) $\varphi(t)$ is ch.f. So are $|\varphi|^2$. $\mathrm{Re}\,\varphi$.

<u>Pf</u>: 1°) Set X. Y. i.i.d. $\varphi_{X-Y} = |\varphi|^2$.

2°) $\mathrm{Re}\,\varphi$ corresponds d.f $F_X + F_{-X}/2$.

<u>Rmk</u>: $|\varphi|$ isn't necessary to be ch.f.

vii) If $|\varphi(t)| \equiv 1$. Then $\varphi(t) = e^{ibt}$. i.e. $X \sim \delta_b$.

<u>Pf</u>: Let X. Y. i.i.d. Then $Z = X-Y$ has ch.f $|\varphi|^2 = 1$.

$\therefore Z \sim \delta_0$ $\quad Var(Z) = 2Var(X) = 0$. $\Rightarrow X \sim \delta_b$.

viii) Exists ch.f that is nowhere differentiable.

<u>Pf</u>: Set r.v. X : $P(X = 5^k) = 1/2^{k+1}$. $k \geq 0$.

$\therefore \varphi(t) = \sum_1^\infty e^{it5^k}/2^{k+1}$. $\sum |e^{it5^k}/2^{k+1}| < \infty$.

$\therefore \varphi'(t) = \sum (5^k/2^{k+1})\cdot ie^{it5^k}$. if $\varphi'(t)$ exists.

But $\sum |5^k/2^{k+1} \cdot ie^{it5^k}| = \infty$. contradict!

7 Local and Global Properties:

Thm. (Extending)

$$\mathrm{Re}(1-\varphi(t)) \geq \frac{1}{4}\mathrm{Re}(1-\varphi(2t)) \geq \cdots \geq \frac{1}{4^n}\mathrm{Re}(1-\varphi(2^n t))$$

Pf: $E(e^{itX}) = E(\sin tX) + i\,E(\cos tX)$.

Note: $1-\cos 2tX = 2\sin^2 tX = 2(2\sin\frac{tX}{2}\cos\frac{tX}{2})^2$

$$\leq 8\sin^2\frac{tX}{2} = 4(1-\cos tX).$$

Cor. i) $1-|\varphi(t)|^2 \geq \frac{1}{4}(1-|\varphi(2t)|^2) \geq \cdots \geq \frac{1}{4^n}(1-|\varphi(2^n t)|^2)$

ii) $1-|\varphi(t)| \geq \frac{1}{8}(1-|\varphi(2t)|) \geq \frac{1}{8^n}(1-|\varphi(2^n t)|)$

Pf: i) Apply Thm on $|\varphi|^2$.

ii) $\because 1-|\varphi(2t)| \leq (1-|\varphi(2t)|)(1+|\varphi(2t)|)$

$$\leq 4(1-|\varphi(t)|^2) \leq 8(1-|\varphi(t)|)$$

Cor. If $|\varphi(t)| \leq a < 1$ for $|t| \geq b > 0$. Then $\forall\, |t| < b$.

$|\varphi(t)| \leq 1-ct^2 \leq e^{-ct^2}$. $c = (1-a^2)/8b^2$.

Pf. $\exists\, m \in \mathbb{Z}^+$ st. $b/2^m \leq |t| < 2b/2^m$. for $|t|<b$.

$1-|\varphi(t)|^2 \geq \frac{1}{4^m}(1-|\varphi(2^m t)|^2) \geq (\frac{1}{2^m})^2(1-a^2)$.

$$\geq (\frac{t}{2b})^2(1-a^2) = 2ct^2.$$

$\therefore |\varphi(t)| \leq (1-2ct^2)^{\frac{1}{2}} \leq 1-ct^2$.

RMK: It means the behavior far away from 0 can control behavior near 0.

Cor. Under Cramer's condition: $\overline{\lim\limits_{|t|\to\infty}} |\varphi(t)| < 1$. Then for $\forall \delta > 0$. $\exists \lambda \in (0,1)$. s.t. $|\varphi| \le \lambda$. $\forall |t| \ge \delta$.

Pf: $\exists\, a < 1$. $b > 0$. $|\varphi(t)| \le a < 1$. $\forall |t| \ge b$

$\Rightarrow |\varphi(t)| \le 1 - ct^2 \le 1 - c\delta^2$. for $|t| \in [\delta, b]$

Thm. X is nondegenerate r.v. Then $\exists \delta > 0$. $\exists \varepsilon > 0$. s.t.

$|\varphi(t)| \le 1 - \varepsilon t^2$. $\forall |t| \le \delta$.

Pf: Let X, Y i.i.d. $Z = X - Y$. Then:

$$1 - |\varphi| \ge \frac{1 - |\varphi|^2}{2} = \frac{1}{2}(1 - \varphi_Z) = \frac{1}{2} E(1 - \cos tZ).$$

$$1 - \cos t \ge \frac{t^2}{2!} - \frac{t^4}{4!} \quad \text{for } |t| < 1.$$

$$\therefore E(1 - \cos tZ) \ge E\left(\frac{t^2}{2}\left(Z^2 - \frac{t^2 Z^4}{12}\right) I_{\{|tZ| \le 1\}}\right)$$

$$\ge \frac{t^2}{2}\left(1 - \frac{1}{12}\right) E\left(Z^2 I_{\{|Z| \le \frac{1}{t}\}}\right).$$

Thm. $\forall t, h \in \mathbb{R}^1$. $|\varphi(t+h) - \varphi(t)|^2 \le 2(1 - \mathrm{Re}(\varphi(h)))$

Pf: $$|\varphi(t+h) - \varphi(t)|^2 \le E(|e^{ihX} - 1|^2)$$

$$= 2E(1 - \cos hX).$$

Cor. ch.f's $\varphi_n(t) \xrightarrow{\forall t} g(t)$. g is conti at 0.

Then g is uniformly conti on $\mathbb{R}^1$.

Pf: $$|g(t+h) - g(t)|^2 = \lim_n |\varphi_n(t+h) - \varphi_n(t)|^2$$

$$\le \lim_n 2(1 - \mathrm{Re}(\varphi_n(h)))$$

$$= 2(1 - \mathrm{Re}\, g(h))$$

Cor. If $\exists\, \delta > 0$. s.t. ch.f's $|\varphi_n| \to 1$. $\forall |t| < \delta$.

Then $|\varphi_n(t)| \to 1$. $\forall t \in \mathbb{R}'$.

Pf. $\big||\varphi_n(2t)| - |\varphi_n(t)|\big|^2 \le 2(1 - \operatorname{Re}\varphi_n(t))$, $\forall |t| \le \delta$.

$\therefore \lim\limits_n |\varphi_n(2t)| = 1$. $\forall |t| \le \delta$. By induction.

(2) Inversion Formula:

① Lemma. i) $0 \le \operatorname{sgn}(\alpha) \int_0^{\eta} \frac{\sin \alpha x}{x} dx \le \int_0^{\pi} \frac{\sin x}{x} dx$

ii) $\int_0^{\infty} \frac{1 - \cos \alpha x}{x^2} dx = \frac{\pi}{2} |\alpha|$.

iii) $\int_0^{\infty} \frac{\sin \alpha x}{x} = \frac{\pi}{2} \operatorname{sgn}(\alpha)$.

Pf. i) Partition $[0, \eta)$.

ii) $1 - \cos \alpha x = \int_0^x \sin \alpha u\, du$

iii) $1/x = \int_0^{\infty} e^{-xu} du$.

② Formula:

Thm. Suppose $\varphi(t) = \int e^{-itx} d\mu(x)$. μ is p.m. Then:

$$\lim_{T \to \infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} \varphi(t)\, dt = \mu(a, b) + \frac{1}{2} \mu\{a, b\}.$$

for $\forall a < b$.

Pf: Apply Fubini, DCT.

Thm. $\mu(\{a\}) = \lim\limits_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e^{-ita} \varphi(t)\, dt$

Cor. If $\varphi \in L^1(\mathbb{R}^1)$. Then μ is conti.

Pf: By Riemann - Lebesgue Lemma:

$$\lim_T \frac{1}{2T} \int_{-T}^{T} e^{-ita} \varphi(t) = 0.$$

Thm. $\displaystyle \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\varphi(t)|^2 dt = \sum_{a\in\mathbb{R}^1} \mu(\{a\})^2$

Pf:
$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \varphi(t)\varphi(-t) = \lim_T \frac{1}{2T} \int_{-T}^{T} \int_{\mathbb{R}^1} e^{-itx} \varphi(t) \mu(dx) dt$$
$$= \int_{\mathbb{R}^1} \lim_T \frac{1}{2T} \int_{-T}^{T} e^{-itx} \varphi \, dt \, \mu(dx)$$
$$= \sum_{x\in\mathbb{R}^1} \mu(\{x\})^2.$$

Cor. μ is conti $\Leftrightarrow \displaystyle \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\varphi|^2 = 0.$

③ Correspondence:

Thm. There's one-to-one correspondence between ch.f's and d.f's.

Pf: If F_1, F_2 d.f's correspond φ. $\forall a, b \in C(F_1) \cap C(F_2)$.

$$F_1(b) - F_1(a) = \lim_T \frac{1}{2\pi} \int_{-T}^{T} \square = F_2(b) - F_2(a).$$

Let $a \to -\infty$ along $C(F_1) \cap C(F_2)$.

$\therefore F_1(b) = F_2(b)$. $\forall b \in C(F_1) \cap C(F_2) \underset{\text{dense}}{\subseteq} \mathbb{R}^1$.

Then by right-conti. of d.f's.

④ Integrability:

i) When $\varphi \in L^1(\mathbb{R}^1)$:

Then μ has density $f(x) \in C_B$.

$$\begin{cases} f = \hat{\varphi} \\ \varphi = f^{\vee} \end{cases}$$ connected by Fourier Transform.

Rmk: There're examples: $\int_{\mathbb{R}^1} |\varphi| = \infty$. but p.d.f. exists.

ii) when $\varphi \notin L^1(\mathbb{R}^1)$:

Thm. $P(X \in \{b + h\mathbb{Z}\}) = 1$. Then $\forall x = b + hk$.

$$P(\{x\}) = \frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} e^{-itx} \varphi(t)\, dt.$$

(3) Levy Continuity Thm:

Lemma. $\forall a > 0$. we have: $P(|X| \geq \frac{2}{a}) \leq \frac{1}{a} \int_{-a}^{a} (1 - \varphi(t))\, dt$.

Pf:
$$\begin{aligned}\int_{-a}^{a} (1 - \varphi(t)) &= 2a - \int_{-a}^{a} E(e^{itX})\, dt \\ &= 2a - E(\int_{-a}^{a} e^{itX}\, dt) \\ &= 2a\, E(1 - \frac{\sin aX}{aX}) \\ &\geq 2a\, E((1 - \frac{\sin aX}{aX}) I_{\{|aX| > 2\}})\end{aligned}$$

Rmk: It means tail prob can be controled by the behaivor of ch.f near origin.

Lemma. F_n d.f's. correspond ch.f's $\{\varphi_n\}$. If $\varphi_n \to g$

g conti at 0. Then $\{F_n\}$ is tight.

Pf: Apply Lemma above. DCT.

Thm. X_n r.v's $\sim F_n$. d.f's. with ch.f's $\{\varphi_n\}$. $\forall n \in \overline{\mathbb{Z}}^+$

i) $X_n \to_d X_\infty \Rightarrow \varphi_n \to \varphi_\infty$

ii) $\varphi_n \to \varphi$. If φ conti at 0. Then $\exists$ r.v. X.

s.t. $X_n \to_d X$.

Pf: i) By DCT.

ii) By Lemma. $\{F_n\}$ is tight.

$\forall$ convergent subseq $\{F_{n_k}\} \subseteq \{F_n\}$. $F_{n_k} \to \tilde{F}$

Where $\tilde{F}$ is d.f. By i). $\therefore \varphi = \varphi_{\tilde{F}}$.

$\therefore F_n \xrightarrow{v} F$. Since it has unique limit.

Cor. r.v.'s $\{X_n\}$. X. $X_n \to_d X \Leftrightarrow \varphi_{X_n} \to \varphi_X$. $\forall t$.

(4) Moments and ch.f's:

① Lemma. $\forall n \geq 0$. $\forall t \in \mathbb{R}^1$.

$$\left|e^{it} - 1 - it - \cdots - \frac{(it)^n}{n!}\right| \leq \min\left\{\frac{|t|^{n+1}}{(n+1)!}, \frac{2|t|^n}{n!}\right\}.$$

Pf: $e^{it} = 1 + (e^{it} - 1) = 1 + i\int_0^t e^{is}\,ds$

$= 1 + i\int_0^t e^{is}\,d(s-t) = 1 + it + i^2\int_0^t (t-s)e^{is}\,ds$

$= \cdots = 1 + it + \cdots \frac{(it)^n}{n!} + \frac{i^{n+1}}{n!}\int_0^t (t-s)^n e^{is}\,ds.$

estimate the residue $R_n(t)$.

Cor. For $n \geq 0$ $\left| e^{it} - 1 - it - \frac{(it)^2}{2!} - \dots - \frac{(it)^n}{n!} \right| \leq \frac{2|t|^{n+\delta}}{n!}$

$\forall \delta \in [0,1]$. It holds.

③ Derivates:

Thm. $E(|X|^n) < \infty \Rightarrow \varphi^{(n)}(t)$ exists. uniformly conti on $\mathbb{R}^1$.

Besides. $\varphi^{(k)}(t) = i^k E(X^k e^{itX}) = i^k \int_{\mathbb{R}} x^k e^{itx} \, dF_X$. $\forall k \leq n$.

Pf: By induction. DCT.

Thm. If n is even. $\varphi^{(n)}(0)$ exists. finite. Then $E|X|^n < \infty$.

Pf: By induction:

1°) $n = 2$:

$$- \varphi^{(2)}(0) = - \lim_{h \to 0} \frac{\varphi(h) + \varphi(-h) - 2\varphi(0)}{h^2}$$

$$= 2 \lim_{h \to 0} E\left(\frac{1 - \cos hX}{h^2} \right)$$

$$\geq 2 E\left(\lim_{h \to 0} \frac{1 - \cos hX}{h^2} \right) = E(X^2)$$

By Fatou's Lemma. in the last "$\geq$".

2°) $n = 2(j+1)$. Apply $\varphi^{(2j)}(t)$.

Rmk: i) $g^{(2)}(0)$ exists $\Rightarrow \lim_{h} \frac{g(h) + g(-h) - 2g(0)}{h^2}$ exists

and $= g^{(2)}(0)$. by L'Hospital. Thm.

But converse is wrong. e.g. $g(t) = a_0 + a_1 t + \frac{1}{2} a_2 t^2 + D(t) t^3$. $D(t)$ is Dirichlet Func.

$g \notin C(\mathbb{R}^1)$.

ii) For ch.f $\varphi(t)$:

$\varphi^{(2)}(0)$ exists $\Leftrightarrow \lim_{h} \frac{\varphi(h)+\varphi(-h)-2\varphi(0)}{h^2}$ exists.

and they're equal. since the latter implies

$E(X^2)$ exists, so $\varphi^{(2)}(t)$ exists.

iii) For n is odd. It doesn't hold:

e.g. $P(X=\pm n) = \frac{c}{2n^2 \log n}$. $\varphi'(0)$ exists. $E(|X|)=\infty$.

③ Taylor Expansions:

Thm. i) If $E(|X|^{n+\delta})<\infty$. for some $n \geq 0$. $\delta \in [0,1]$.

Then $\varphi(t)$ ch.f of X. have expansion:

$$\varphi(t) = \sum_{k=0}^{n} EX^k \frac{(it)^k}{k!} + \theta \frac{2E|X|^{n+\delta}|t|^{n+\delta}}{n!}. \quad |\theta| \leq 1.$$

$$\varphi(t) = \sum_{k=0}^{n} E(X^k) \frac{(it)^k}{k!} + o(t^n). \ (t \to 0).$$

ii) If $\varphi(t)$ can be written in:

$$\varphi(t) = \sum_{0}^{n} a_k \frac{(it)^k}{k!} + o(t^n) \ (n \to 0)$$

Then $E(|X|^{2[\frac{n}{2}]})$ exists. $a_k = E(X^k)$. for $k \leq 2[\frac{n}{2}]$.

Pf: i) The first is from Lemma.

Test second one:

$$\varphi(t) = \sum_{0}^{n-1} \frac{\varphi^{(k)}(0)}{k!} t^k + \frac{\varphi^{(n)}(\theta t)}{n!} t^n$$

$$= \sum_{0}^{n} \frac{\varphi^{(k)}(0)}{k!} t^k + R_n(t).$$

$\Rightarrow$ check $R_n(t)/t^n \to 0 \ (t \to 0)$.

ii) By induction:

$n=2$. Check $-\lim_{h} \frac{\varphi(h)+\varphi(-h)-2\varphi(0)}{h}$ exists. $= \lambda_2$.

$\Rightarrow \varphi^{(2)}(t)$ exists. $\Rightarrow$ Repeat the process.

Thm. (Carleman's condition on Moment problem)

If $\{F_n\}$ d.f's with moments $\{m_n^r\}$. $m_n^r \to m^r (n\to\infty)$.

Besides. $\lim_{k\to\infty} \frac{m^k}{k!} t^k = 0$. $\forall t \in \mathbb{R}^1$. Then $\exists$ d.f. F. st.

$F_n \xrightarrow{v} F$.

Pf: Expansion: $\varphi_n(t) = E(e^{itX_n}) = \sum_0^k \frac{m_n^k}{k!}(it)^k + \theta \frac{m_n^{k+1} t^{k+1}}{(k+1)!}$

Set $\varphi(t) = \sum_0^k \frac{m^j}{j!}(it)^j + \theta \frac{m^{k+1} t^{k+1}}{(k+1)!}$. check $\varphi_n \to \varphi$.

(5) Representation Theory:

① Positive definite:

Def: $f: \mathbb{R}^1 \to \mathbb{C}$. If $\forall (t_i)_1^n \subseteq \mathbb{R}^1$. $(z_i)_1^n \subseteq \mathbb{C}$.

$\sum_i \sum_j f(t_i - t_j) z_i \bar{z}_j \geq 0$. Then call f is positive definite.

prop. i) $f(-t) = \overline{f(t)}$. $|f(t)| \leq f(0)$

ii) If f is conti at 0. Then it's uniformly conti. in $\mathbb{R}^1$.

iii) If f conti at 0. $\forall \xi \in C(\mathbb{R}^1, \mathbb{C})$. $\forall T>0$.

Then $\int_0^T \int_0^T f(s-t)\xi(s)\bar{\xi}(t)\,ds\,dt \geq 0$.

Pf. Test $n=1$. $t_1=0$. $z_1=1$. $n=2$, $t_1=0$, $t_2=t$. $z_1=z_2=1$.

$n=3$. $t_1=0$, $t_2=t$, $t_3=t+h$. arbitrary $\vec{z}$.

For the last: By Riemann Sum. Take $n\to\infty$

Thm. (Bochner)

f is conti. positive definite $\Leftrightarrow$ $\hat{f}(x)\geq 0$. $\forall x\in\mathbb{R}^1$.

Pf: ($\Leftarrow$). $F(w)=\int_{\mathbb{R}} f(x)e^{-iwx}dx$. $f(x)=\frac{1}{2\pi}\int F(w)e^{iwx}dw$.

follows from Fourier Transform $\because f\in C(\mathbb{R}^1)$.

1°) e^{iwx} is positive definite. $\forall w\in\mathbb{R}^1$.

Since $\sum\sum e^{iw(t_i-t_j)}\bar{\alpha}_i\alpha_j=|\sum e^{iwx_i}\bar{\alpha}_i|^2\geq 0$.

2°) For $(a_k)_1^n\subseteq\mathbb{R}^+$. By Linearity:

$\sum_1^n a_k e^{iw_kx}$ is positive definite.

3°) Set $a_k=F(w_k)$. $(w_k)=\{\frac{k}{n}\}_{-n^2}^{n^2}$. Let $n\to\infty$

Since limit of p.d Func. is p.d. $\Rightarrow f$ is p.d.

($\Rightarrow$). Choose $\alpha_k=e^{-iwx_k}$. $x_k=\frac{k}{n}T$. $\forall 1\leq k\leq n$.

$\therefore \alpha^T(f(x_i-x_j))_{n\times n}\alpha\geq 0$. Let $n\to\infty$.

$\therefore \int_0^T\int_0^T f(s-t)e^{-i(s-t)w}ds\,dt\geq 0$.

i.e. $\int_{-T}^T(1-\frac{|t|}{T})f(t)e^{-itx}dt\geq 0$. Let $T\to\infty$.

③ Criteria.

i) Thm. φ is ch.f $\Leftrightarrow$ φ is p.d. $\varphi(0)=1$. conti at 0.

Rmk: Since $\varphi(x) = \int e^{itx} \mu(dt)$, $\varphi(0) = \mu(\mathbb{R}^1)$.

ii) Thm. If $f(t) \geq 0$. $f(t) = f(-t)$. $f(0)=1$. f is conti. $\downarrow (t \geq 0)$

convex on $\mathbb{R}$. Then f is ch.f.

③ Applications:

i) Spectral Dist:

For $\{X_t\}_{t\in\mathbb{R}^+}$. r.v's. satisfies $\begin{cases} E(X_t^2)=1 \\ \lim_{t\downarrow 0} E((X_t-X_0)^2)=0 \\ \exists\, r(x) \text{ on } \mathbb{R}^1.\ E(X_sX_t)=r(s-t). \end{cases}$

Then $r(x)$ is ch.f. (w.r.t spectral dist)

Pf. $\sum\sum r(t_i-t_j) z_i \bar{z}_j = E(|\sum X_{t_i} z_i|^2) \geq 0$.

$|r(t)-r(0)|^2 = E^2(X_0(X_0-X_t)) \leq E(X_0^2)\, E((X_0-X_t)^2) \to 0$

$\therefore \exists\, \nu(\cdot)$. p.m. st. $\nu(x) = \int e^{itx} \nu(dt)$.

ii) Stable Dist:

$\forall \alpha \in [0,2]$. $\varphi_\alpha(t) = e^{-|t|^\alpha}$ is ch.f.

Pf: Consider density: $p_\alpha(x) = \begin{cases} \alpha/2|x|^{\alpha+1}, & |x| \geq 1 \\ 0, & |x|<1 \end{cases}$

Correspond ch.f:

$\varphi = 1 - c_\alpha|t|^\alpha + O(t^2)$. $\varphi^n(t/n^{\frac{1}{\alpha}}) \to e^{-c_\alpha|t|^\alpha}$.

$e^{-c_\alpha|t|^\alpha}$ is ch.f since it's conti at 0.

Let $Y = c_\alpha^{-\frac{1}{\alpha}} X$. $\therefore e^{-|t|^\alpha}$ is ch.f.

Rmk: i) $\forall \gamma_\alpha \in \mathbb{C}$. $e^{-\gamma_\alpha |t|^\alpha}$ is ch.f. $0 \le \alpha \le 2$.

ii) For $\alpha > 2$. It doesn't hold:

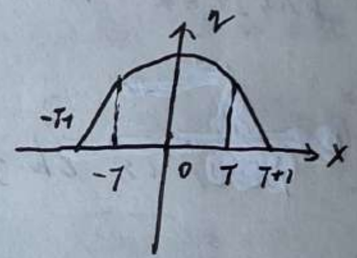
since $\varphi(t) = 1 + o(t^2)$ isn't ch.f.

Pf: check $-\varphi''(0) = 0 \Rightarrow \varphi(0) = 0$.

$\therefore X \equiv c$ if φ corresponds X. contradict!

iii) Construct two different ch.f which equals on $[-T, T]$:

For η is ch.f. sym. convex.



Let $f(t) = \begin{cases} \eta(t). & t \in [-T, T] \\ 0. & t \in [-T-1, T+1]^c \\ \text{linear.} & \text{others} \end{cases}$

$f(t)$ satisfies criteria ii). So it's ch.f.

iv) Poisson Transformation:

Thm. If f is ch.f. Then so $e^{\lambda(f-1)}$ is. $\forall \lambda > 0$.

Pf. $1 - \frac{\lambda}{n} + \frac{\lambda}{n} f(t)$ is convex combination of ch.f

$\therefore$ It's ch.f.

$\left(1 - \frac{\lambda}{n} + \frac{\lambda}{n} f(t)\right)^n = \left(1 + \frac{\lambda}{n}(f-1)\right)^n \to e^{\lambda(f-1)}$ conti.

$\therefore e^{\lambda(f-1)}$ is ch.f.

Rmk: If X_k. i.i.d. with ch.f φ. $N \sim \text{Poisson}(\lambda)$.

Then $\sum_1^N X_k$ has ch.f $e^{\lambda(\varphi - 1)}$.

(6) Lattices:

Def: If $p(\{X \in \{a+k\lambda\}_{k\in\mathbb{Z}}\}) = 1$. for some $a, \lambda \in \mathbb{R}^1$.

Then we say X has Lattice dist.

Rmk: λ may not be unique. Since $\forall \lambda' | \lambda$. It holds as well. But If λ is the maximum one. Then it's the span. (unique)

① **Thm.** For $\lambda \neq 0$. The following are equi.

i) $\varphi(\lambda) = 1$. ii) $\varphi(t+n\lambda) = \varphi(t)$ $\forall t \in \mathbb{R}^1$. $\forall n \in \mathbb{Z}^+$.

iii) $p(X \in \{kh\}_{k\in\mathbb{Z}}) = 1$. $h = 2\pi/\lambda$.

Pf: iii) $\Rightarrow$ ii) : $\varphi(t) = \sum_{k\in\mathbb{Z}} p_k e^{ikht}$

ii) $\Rightarrow$ i) : $n=1$. $t=0$.

i) $\Rightarrow$ iii) : $E(i\sin(\lambda X) + \cos(\lambda X)) = 1$

$\therefore E(1-\cos(\lambda X)) = 0. = \int (1-\cos\lambda x)\, d\mu$.

$\therefore \forall x \in \mathbb{R}^1$. $\mu(\{x\}) \neq 0 \Rightarrow x = 2k\pi/\lambda$

Cor. Replace X by $X-b$. Follows equi.

i) $\varphi(\lambda) = e^{ib\lambda}$

ii) $\varphi(t+n\lambda) = e^{in\lambda b}\varphi(t)$.

iii) $p(X \in \{b+kh\}_{k\in\mathbb{Z}}) = 1$. $h = 2\pi/\lambda$

Thm. (Classification):

There're only three cases:

i) $|\varphi(t)| \equiv 1$. $\forall t$. $\varphi = e^{ibt}$. (degenerate)

ii) $|\varphi(\lambda)| = 1$. $|\varphi(t)| < 1$. $\forall 0 < t < \lambda$. (lattice with span $\frac{2\pi}{\lambda}$)

iii) $|\varphi(t)| < 1$. $\forall t \neq 0$. (non-lattice dist.)

<u>Rmk</u>: Since if X is nondegenerated. then.

$\exists \delta, \varepsilon > 0$. s.t. $|\varphi(t)| \leq 1 - \varepsilon t^2$. $\forall |t| \leq \delta$.

$\therefore$ "$\exists (\lambda_n) \to 0$. $|\varphi(\lambda_n)| = 1$." doesn't exists.

② <u>Strong nonlattice d.f.</u>

<u>Def</u>: X is strongly nonlattice if Cramer's condition holds: $\overline{\lim\limits_{|t| \to \infty}} |\varphi(t)| < 1$.

<u>Rmk</u>: $\exists X$. r.v. nonlattice. but not strongly:

<u>e.g.</u> $p(X=0) = p_0$. $p(X=1) = p_1$. $p(X=a) = p_2$

$a \in \mathbb{R}/\mathbb{Q}$ (e.g. $a = \sqrt{2}$). $\sum p_i = 1$.

$\Rightarrow X$ is nonlattice. definitely.

But X isn't strongly nonlattice.

<u>Thm.</u> If d.f F has nonzero absolutely conti. component. Then it's strongly nonlattice.

<u>Pf</u>: $F = aF_d + bF_{ac} + cF_s$. $a+b+c=1$. $b \neq 0$.

$\therefore \varphi = a\varphi_1 + b\varphi_2 + c\varphi_3$. Suppose $f = dF_{ac}/dt$.

$\varphi_2 = \int e^{itx} dF_{ac} = \int e^{itx} f(t) dt \to 0$ $(x \to \infty)$.

$\therefore \overline{\lim} |\varphi| < 1$. By Riemann-Lebesgue Lemma.

Rmk: For d.f. F is conti. but not a.c.

It's not necessary that F is strongly.

Lattice. e.g. $S = \{ X_{kj} = j/3^k.\ k.j \in \mathbb{Z}^+ \} \cap [0,1]$.

$p(X_{kj}) = 1/2^{kj} \cdot c$. Then its d.f is singular. but

not a.c. choose $t_m = 2\pi \cdot 3^m$. $|\varphi| \to 1\ (t_m \to \infty)$.

(7) Essen's Smoothing Lemma:

· Note by CLT: $\sup_x | P(T_n \le x) - \phi(x) | \to 0.\ (n \to \infty)$

The question is how fast the limit $\to 0$.

① **Smoothing**:

For any r.v. X. can be perturbed by a indept. conti

r.v. Y. i.e. X + Y is also conti. r.v. :

$F_{X+Y}(t) = \int_{\mathbb{R}^1} F_Y(t-y) \, dF_X(y)$. The Smoothness depend on Y.

Choose V_T is d.f of density : $v_T(x) = \frac{1-\cos(Tx)}{\pi T x^2}$.

Denote $\Delta^T = \Delta * V_T = \int_{\mathbb{R}^1} \Delta(t-x) v_T(x) dx$.

Rmk: This is due to the difficulty that research

on discrete case : $| F(x) - G(x) |$.

② **Estimations**:

Lemma: $\frac{x}{1+x^2} e^{-\frac{x^2}{2}} \le \int_x^{+\infty} e^{-\frac{y^2}{2}} dy \le \frac{1}{x} e^{-\frac{x^2}{2}}$.

Pf. $\int_x^{+\infty} d\left(\frac{-t}{1+t^2} e^{-\frac{t^2}{2}}\right) \le \Box \le \int_x^{+\infty} \frac{y}{x} e^{-\frac{y^2}{2}} dy$.

Lemma. $X \sim F$. $Y \sim G$. d.f. $E(|X|)$. $E(|Y|) < \infty$.

$$\Rightarrow \int_{\mathbb{R}^1} |F-G| < \infty.$$

Pf: $\int_{\mathbb{R}^1} = \int_0^\infty + \int_{-\infty}^0 \le \int_0^{+\infty} (1-F(x)) + \int_{-\infty}^0 F(x) +$

$$\int_0^\infty (1-G(x)) + \int_{-\infty}^0 G(x)$$

$$= E|X| + E(|Y|) < \infty.$$

Lemma. $X \sim F$. d.f. $E(X) = 0$. ch.f is φ_F. $F-G$ vanish at ∞.

G satisfies G' exists. $\sup_x |G'(x)| \le \lambda$. has C'. Fourier

Transform: φ_G. $\varphi_G(0) = 1$. $\varphi_G'(0) = 0$. Then $\forall T > 0$.

$$\sup_x |F-G| \le \frac{1}{\pi} \int_{-T}^{T} \left| \frac{\varphi_F(t) - \varphi_G(t)}{t} \right| dt + \frac{24\lambda}{\pi T}.$$

Rmk: It means LHS can be controlled by ch.f's.

Commonly choose $T = \sqrt{n}$.

(8) Laplace Transform:

Def: For $X \sim F$. define $F^L = E(e^{-\lambda X})$. where

$X \ge 0$. ($\therefore F^L(0) = 1$. $F^L(\infty) = 0$)

① Uniqueness:

Thm. $F_1^L = F_2^L \Leftrightarrow F_1 = F_2$

Pf: By Weierstrass Thm:

Approxi by $\{e^{-\lambda x}\}_{\lambda \ge 0}$. in C_B.

② Convergence:

Thm. $\{F_n\}$ s.d.f support on $\mathbb{R}^+$. with $\{\hat{F}_n\}$ Then:

$$F_n \to F_\infty.\ F_\infty \text{ is d.f} \iff \begin{cases} \lim_n \hat{F}_n(\lambda) \text{ exists. } \forall \lambda \\ \lim_{\lambda \downarrow 0} \lim_n \hat{F}_n(\lambda) = 1. \end{cases}$$

Pf: ($\Rightarrow$) is trivial. ($\Leftarrow$). Denote $G(\lambda) = \lim_n \hat{F}_n(\lambda)$.

Check any convergence subseq of $\{F_n\}$.

③ Characterization:

Def: f is completely monotonic if $(-1)^n f^{(n)}(x) \geq 0 \quad \forall n \geq 0$.

Thm. For f supports on $\mathbb{R}^+$. F is d.f.

Then $f = \int_{\mathbb{R}^+} e^{-\lambda x} dF(x) \iff f$ is completely monotic and $\lim_{\lambda \downarrow 0^+} f(\lambda) = 1$

Pf. Apply expansion of $f(\lambda)$.