Vague Convergence

(1) Definitions:

① **Def:** i) For μ on $(\mathbb{R}^1, \mathcal{B}_{\mathbb{R}^1})$. $\mu(\mathbb{R}^1) \leq 1$. Then:

it's called subprobability measure (s.p.m).

ii) μ_n. s.p.m's $\xrightarrow{v} \mu$. if $\exists$ dense set $D \subseteq \mathbb{R}^1$.

s.t. $\forall a.b \in D$. $\mu_n(a.b] \longrightarrow \mu(a.b]$. $(\mu_n \xrightarrow{v} \mu)$

Rmk: i) Introduce s.p.m : generally. $\{\mu_n\}$ p.m's.

$\mu_n \to \mu$. μ is not necessary to be p.m. $(\mu \leq 1)$

e.g. Choose d.f. : $F_n = a I_{\{x \geq n\}} + b I_{\{x \leq -n\}} + G(x)$.

$a.b > 0$. $F_n \to G(x)$. is not d.f.

ii) For p.m's : It corresponds a d.f.

$\mu_n \xrightarrow{v} \mu \iff F_n \xrightarrow{v} F \iff X_n \xrightarrow{d} X$. if

$X_n \sim F_n$. $F_n \sim \mu_n$.

② **Criteria:**

Thm. For s.p.m's $\{\mu_n\}$. μ. The followings are equi.

i) $\mu_n \xrightarrow{v} \mu$ ii) $\forall$ conti. interval $(a.b]$. of μ.

$\mu_n(a.b] \longrightarrow \mu(a.b]$.

iii) $\forall \varepsilon > 0$. $a.b \in \mathbb{R}^1$. $\exists n_0 = n_0(a.b.\varepsilon)$. $\forall n > n_0$. we have:

$\mu(a+\varepsilon. b-\varepsilon) - \varepsilon \leq \mu_n(a.b) \leq \mu(a-\varepsilon. b+\varepsilon) + \varepsilon$.

Rmk: Define Levy metric: $\rho(\mu,\lambda) = \|\mu-\lambda\|_\rho$

$\inf\{\delta>0 \mid \mu(-\infty, x-\delta]-\delta \le \lambda(-\infty, x] \le \mu(-\infty, x+\delta]+\delta, \forall x\}$.

Then $\mu_n \xrightarrow{v} \mu \iff \|\mu_n-\lambda\|_\rho \to 0 \ (n\to\infty)$

Thm. For p.m's $\{\mu_n\}$, μ: Replace iii) by:

iii*) $\forall \varepsilon>0$, $\forall \delta>0$, $\exists n_0 = n_0(\delta,\varepsilon)$, s.t. $\forall n > n_0$,

$\mu(a+\delta, b-\delta)-\varepsilon \le \mu_n(a,b) \le \mu(a-\delta, b+\delta)+\varepsilon$.

for $\forall a,b \in \mathbb{R}^1$.

It still holds. for Thm above.

Rmk: i) The problem is we can find $(a,b) \subseteq \mathbb{R}^1$. (for ε)

s.t. $|\mu_n(a,b)^c|$, $|\mu(a,b)^c| < \varepsilon$. But for s.p.m's.

It won't hold probably.

ii) Define metric $\tilde{\rho}(\mu,\lambda) = \|\mu-\lambda\|_{\tilde{\rho}} =$

$\inf\{\delta>0 \mid \mu(a+\delta, b-\delta)-\varepsilon \le \lambda(a,b) \le \mu(a-\delta, b+\delta)+\varepsilon, \forall \varepsilon>0, a,b\}$.

μ_n p.m's $\xrightarrow{v} \mu \iff \|\mu_n-\mu\|_{\tilde{\rho}} \to 0$.

iii) Require conti point is because d.f's are

right-conti. $\therefore \lim_n F_n(x) = F(x-) \ne F(x)$.

except for continuity. e.g. $\delta_{n^{-1}} \Rightarrow \delta_0$.

(2) Equi. Definition for Vague Convergence:

Thm. (Portmanteau).

The followings are equi.

i) X_n. r.v's $\to_d X$. i.e. $\mu_n \xrightarrow{v} \mu$. p.m's.

ii) $\underline{\lim}\, P(X_n \in G) \geq P(X \in G)$. $\forall G$. open.

$\overline{\lim}\, P(X_n \in K) \leq P(X \in K)$. $\forall K$. closed.

$\lim P(X_n \in A) = P(X \in A)$. for $P(X \in \partial A) = 0$.

iii) $E(g(X_n)) \to E(g(X))$. $\forall g \in C_B$.

iv) $E(g(X_n)) \to E(g(X))$. $\forall g \in C_K$.

v) $\varphi_{X_n} \to \varphi_X$. for ch.f's of $\{X_n\}$. X.

Pf: i) $\Rightarrow$ ii) By Skorokhod's Representation

$\exists Y_n \sim X_n \to Y \sim X$. a.s. Check: $\underline{\lim}\, I_G(Y_n) \geq I_G(Y)$. a.s.

Apply Fatou's Lemma:

$P(X \in G) = E(I_G(Y)) \leq E(\underline{\lim}\, I_G(Y_n)) \leq \underline{\lim}\, P(X_n \in G)$.

For closed set. set $K^c = G$.

And Note that $\partial A = \bar{A} - A^\circ$. $\therefore P(\partial A) = P(\bar{A}) - P(A^\circ) = 0$

Apply above on $\bar{A}$. A°.

ii) $\Rightarrow$ i). Choose $x \in C(\mu_X)$. $\mu_X(x) = 0$.

Let $A = (-\infty, x]$. it holds for $C(\mu_X)$!

i) $\Rightarrow$ iii) By Skorokhod. DCT.

iii) $\Rightarrow$ i). Set $g_{x,\varepsilon} = \begin{cases} 1 & . \ y \leq x \\ 0 & . \ y \geq x+\varepsilon \\ \text{linear} & . \ \text{others.} \end{cases}$ $g_{x,\varepsilon} \geq I_{\{y \leq x\}}$.

$\therefore \overline{\lim} P(X_n \le x) \le \overline{\lim} E(g_{x,\varepsilon}(X_n)) = E(g_{x,\varepsilon}(X))$

$\le E(I_{\{X \le x+\varepsilon\}}) = P(X \le x+\varepsilon)$

Let $\varepsilon \to 0$. $\therefore \overline{\lim} P(X_n \le x) \le P(X \le x)$.

Converse is analogous. By $g_{x-\varepsilon,\varepsilon}$.

i) $\Rightarrow$ iv) Approxi by $I_{(a,b]}$. $a, b \in C(\mu)$.

iv) $\Rightarrow$ i). For $b, a \in C(\mu)$. $f = I_{(a,b]} \in C_K$.

$P(X_n \le b) \ge P(a \le X_n \le b) \quad (n \to \infty)$.

$\to P(a \le X \le b) \to P(X \le b)$

(Let $a \to -\infty$, through $C(\mu)$)

Similarly. $P(X_n \ge b') \ge P(X \ge b') - \varepsilon$. for $b' > b \in C$

$\therefore P(X \le b) - \varepsilon \le P(X_n \le b) \le P(X_n \le b') \le P(X \le b') + \varepsilon$.

Let $n \to \infty$. $\varepsilon \downarrow 0$, $b' \downarrow b$ through $C(\mu)$.

Rmk: i) For iv). We can extend to C_0. Since $\overline{C_K} = C_0$

ii) For s.p.m $\{\mu_n\}$, μ. $\mu_n \xrightarrow{v} \mu \Leftrightarrow$ iv).

But may not iii).

iii) ii) is equi. with followings:

$\forall f$ is l.s.c : $\underline{\lim} E(f(X_n)) \ge E(f(X))$

$\forall f$ is u.s.c : $\overline{\lim} E(f(X_n)) \le E(f(X))$

Pf: Note that for f l.s.c. $f \ge 0$.

$\exists f_n \in C_K$. $f_n \uparrow f$. And $-f$ is u.s.c.

Then prove general f. It holds

(3) Helly's Selection:

Lemma. G is bounded nondecreasing on $D \underset{\text{dense}}{\subseteq} \mathbb{R}^1$.

Define: $F(x) = \lim\limits_{y \in D, y \to x^-} G(y)$. $H(x) = \lim\limits_{y \in D, y \to x^+} G(y)$.

Then : i) $F(x)$ is left-conti. $C(F) \supseteq C(G)$

$H(x)$ is right-conti. $C(H) \supseteq C(G)$.

ii) $\forall x \in C(G)$. $F(x) = G(x) = H(x)$.

Pf: It's easy to check along $C(G)$.

Thm. $\{\mu_n\}$ s.p.m's. $\exists \{\mu_{n_k}\} \subseteq \{\mu_n\}$. st. $\mu_{n_k} \xrightarrow{v} \mu$.

where μ is also s.p.m.

Pf: Suppose $\{F_n\}$ are correspond d.f's of $\{\mu_n\}$.

Choose $D = \mathbb{Q}$. By diagonalisation method:

$\exists G(r) = \lim\limits_k F_{kk}(r)$. $\forall r \in D$. nondecreasing.

Extend G for right-conti: $F(x) = \lim\limits_{\substack{y \to x^+ \\ y \in D}} G(y)$

Check $\forall x \in C(F)$. $\forall \varepsilon > 0$. $\exists n_k$, k is large.

$|F_{n_k}(x) - F(x)| < \varepsilon$. (Note: $F(x) \geq G(x)$).

Cor. If any vaguely convergent subseq $\{\mu_{n_k}\}$

$\subseteq \{\mu_n\}$ s.p.m's $\to \mu$. Then $\mu_n \xrightarrow{v} \mu$.

Pf: By contradiction.

(4) Tightness:

Def: $\{\mu_n\}$ s.p.m's is tight. if $\forall \varepsilon > 0$. $\exists M \in \mathbb{Z}^+$.

s.t. $\underline{\lim}\, \mu_n[-M, M] \geq 1-\varepsilon$.

Thm. If $\exists \varphi \geq 0$. $\varphi \to \infty$ as $|x| \to \infty$. for $\{\mu_n\}$.

$C = \sup_n \int \varphi(x)\, d\mu_n(x) = \sup_n E(\varphi(X_n)) < \infty$.

Then $\{\mu_n\}$ is tight s.p.m's.

Pf. $C \geq \sup_n \int_{[-M,M]^c} \varphi(x)\, d\mu_n \geq \inf_{|x| \geq M} \varphi \cdot \mu_n[-M,M]^c$.

$\therefore \mu_n[-M,M]^c \leq C / \inf_{|x| \geq M} \varphi \to 0$.

Thm. $\{\mu_n\}$ p.m's $\xrightarrow{v} \mu$. Then μ is p.m $\Leftrightarrow$ $\{\mu_n\}$ is tight.

Pf: ($\Rightarrow$) By contradiction: $\exists \varepsilon_0 > 0$. $\{n_k\} \subseteq \mathbb{Z}^+$.

$\mu_{n_k}[-k, k]^c \geq \varepsilon_0$. $\forall k \in \mathbb{Z}^+$. holds.

Choose $s, r \in C(\mu)$. $\exists k_j$. $s < -k_j < k_j < r$.

Then $\mu[s, r]^c \geq \varepsilon_0$. $\forall s, r \in C(\mu)$.

($\Leftarrow$). Similarly. check $\mu[-M, M] \to 0$ $(M \to \infty)$

(4) Polya Thm.

Thm. If F_n d.f $\to F$. d.f. conti. Then: we have:

$\lim_n \sup_t |F_n(t) - F(t)| = 0$. i.e. $F_n \xrightarrow{u} F$. if $F \in C(\mathbb{R}^1)$.

Pf: Fix M. N large enough. s.t. $1-F(N)$. $F(-M) < \varepsilon$.

$\sup_t |F_n - F| \leq \sup_{[-\infty,-M]} F_n + F + \sup_{[-M,N]} |F_n - F| +$

$\sup_{[N,\infty]} 1-F_n + 1-F$

$= F_n(-M) + F(-M) + 1 - F_n(N) + 1 - F(N) + \sup_{[-M,N]} |F_n - F|$

since $F_n \to F$. $\therefore F_n(-M) + 1 - F_n(N) < 2\varepsilon$. for n large.

prove: $\sup_{[-M,N]} |F_n - F| \to 0$.

1°) For $(x_n) \uparrow x$ or $(x_n) \downarrow x$. $\Rightarrow F_n(x_n) \longrightarrow F(x)$.

<u>Pf</u>: $|F_n(x_n) - F(x)| \le |F_n(x_n) - F_n(x)| + |F_n(x) - F(x)|$

Fix n_0. s.t. $\mu(x_{n_0}, x] < \delta$. (suppose $x_n \uparrow x$)

Since $\mu_n(x_{n_0}, x] \longrightarrow \mu(x_{n_0}, x]$. $|F_n(x_n) - F_n(x)| \le \mu_n(x_{n_0}, x]$

$\therefore \lim_n |F_n(x_n) - F(x)| \le \delta$. $\forall \delta > 0$.

2°) By contradiction: if $\exists \varepsilon_0 > 0$. $(n_k) \subset \mathbb{Z}^+$.

$\sup_{[-M,N]} |F_{n_k} - F| \ge \varepsilon_0$. then $\exists (x_{n_k}) \subset [-M, N]$. s.t. $\exists$ some x.

$|F_{n_k}(x_{n_k}) - F(x_{n_k})| \ge \frac{\varepsilon_0}{2}$ $\exists (\tilde{x}_{n_k}) \subset (x_{n_k})$. $\tilde{x}_{n_k} \uparrow x$ or $\downarrow x$

$\Rightarrow$ Let $n \to \infty$ first. Then $\varepsilon \to 0$.

(5) Addition Topics:

① Stable convergence:

<u>Def</u>: $Y_n \to_d Y$. where Y_n are all on $(\Omega, \mathcal{F}, P)$.

we say it's stable convergence if:

i) $\forall E \in \mathcal{F}$. continuity of Y. $\lim_n P\{Y_n \le y\} \cap E$ $= Q_y(E)$ exists.

ii) $Q_y(E) \longrightarrow P(E)$. $y \to +\infty$.

Rmk: $P(\{\eta \geq Y_n\} \cap E) = P(Y_n \leq \eta) P(E \mid Y_n \leq \eta)$. $\Rightarrow$

It means the convergence depends on $\{Y_n\}$.

e.g. $X, \tilde{X}$, i.i.d. nondegenerated. $Z_n = \begin{cases} X, & n \text{ is odd} \\ \tilde{X}, & n \text{ is even} \end{cases}$

$\Rightarrow Z_n \xrightarrow{d} X$. But not stably.

check $E = \{X \leq \eta\}$. $P(Z_n \leq x, E)$ diverges.

Def: Denote $E(XY) = \langle X, Y \rangle$. $\{Z_n\}$ in L^1. if $\forall \eta$ measurable. $\langle Z_n, \eta \rangle \longrightarrow \langle Z, \eta \rangle$. We say it converges weakly in L^1.

Thm. L^1-convergence $\Rightarrow$ weakly converge in $L^1 \Rightarrow$ u.i.

② Moment Problem:

Thm. If $\exists$ unique d.f. F with moments $\{m^r\}_{r \geq 1}$. finite (F_n) seq of. d.f's with finite moments $\{m_n^r\}$. And $\lim_n m_n^r = m^r$. Then $F_n \xrightarrow{v} F$.

Pf: Check on $\forall \{\mu_{n_k}\} \subseteq \{\mu_n\}$. which is vaguely convergent to an identical p.m μ.

1°) $\lim_k \mu_{n_k}[-A, A] \geq 1 - \frac{m_{n_k}^2}{A^2} \to 1 - \frac{m^2}{A^2} \to 1$.

$\therefore \mu_{n_k} \to \mu$. μ is p.m.

2°) $m_{n_k}^r = \int x^r \, d\mu_{n_k} \longrightarrow \int x^r \, d\mu$.

By unique correspond. F is d.f of μ.

Rmk: Recall: $M_X = M_Y$. m.g.f $\Leftrightarrow m_X^r = m_Y^r$. which are all finite. $\forall r \in \mathbb{Z}^+$.