Law of Large Number

(1) Simple Limit Thm:

① **Thm.** If X_k's are correlated. $\sup_k E(X_k^2) \leq M < \infty$.

Then i) $\bar{X} - \bar{\mu} \to 0$ in L^2. $\quad \bar{X} = \sum_1^n X_i / n$

ii) $\bar{X} - \bar{\mu} \to_p 0$ $\quad \bar{\mu} = \sum_1^n E(X_i)/n$.

iii) $\bar{X} - \bar{\mu} \to 0$. a.s.

Pf: i), ii) are trivial to check.

iii) $\therefore P(|\bar{X} - \bar{\mu}| > \varepsilon) \leq \frac{Var(S_n)}{n^2 \varepsilon^2} \leq \frac{M}{n\varepsilon^2}$.

$\therefore$ For subseq : $\frac{S_{n^2} - \mu n^2}{n^2} \to 0$. a.s.

Fill the gap by estimate : $D_n = \sup_{n^2 \leq k < (n+1)^2} |S_k - S_{n^2}|$

$$E(D_n^2) \leq \sum_{n^2}^{(n+1)^2-1} E((S_k - S_{n^2}))$$

$$= \sum_{k=n^2}^{(n+1)^2-1} \sum_{n^2+1}^{k} E(X_i^2) \leq 4n^2 M.$$

$\Rightarrow P(|D_n| \geq \varepsilon n^2) \leq \frac{4M}{n^2 \varepsilon^2}$. $\quad \therefore \frac{D_n}{n^2} \to 0$. a.s.

$\forall k$. if $n^2 \leq k < (n+1)^2$ Then :

$$\left|\frac{S_k}{k}\right| \leq \frac{|S_{n^2}|}{k} + \frac{|S_k - S_{n^2}|}{k} \leq \frac{|S_{n^2}| + D_n}{n^2} \to 0 \text{ . a.s.}$$

Rmk: i) It's optional : $S_n / n^\alpha \to 0$. a.s. holds.

for $\alpha > \frac{3}{4}$

ii) $Var(S_n) = o(n^2) \Rightarrow S_n/n - \mu \to_p 0$.

By Chebyshev. directly.

③ Application:

Def: For $w = 0.\overline{x_1 x_2 \cdots x_n \cdots}$. Denote $V_k^{(n)}(w)$ is the number of digits $= k$ in the first n digits. $0 \le k \le 9$. Define: $\varphi_k(w) = \lim_{n \to \infty} \frac{V_k^{(n)}(w)}{n}$

w is simply normal if $\varphi_k(w) = \frac{1}{10}$, $\forall 0 \le k \le 9$.

Thm. Almost every number in $[0,1]$ is simply normal.

Pf: Denote $w = 0.\overline{\xi_1 \xi_2 \cdots \xi_n \cdots}$. ξ_k is r.v. indept.

$p(\xi_k = i) = \frac{1}{10}$, by symmetry. $\forall 0 \le i \le 9$.

$X_n^k = I_{\{\xi_n = k\}}$ $\therefore E(X_n^k) = E((X_n^k)^2) = \frac{1}{10}$

$\therefore \sum_k^n X_n^k / n \to \frac{1}{10}$. a.s. $\therefore p(\varphi_k = \frac{1}{10}) = 1$.

$\Rightarrow p(\bigcap_0^9 \varphi_k(w) = \frac{1}{10}) = 1$.

(2) WLLN:

① Truncation:

Def: r.v.'s $\{X_n\}$, $\{Y_n\}$ on $(\Omega, \mathcal{A}, P)$ are equivalent.

$\Leftrightarrow \sum p(X_n \ne Y_n) < \infty$.

Thm. If $\{X_n\}$, $\{Y_n\}$ are equivalent. Then, we have:

i) $\sum (X_n - Y_n)$ converges. a.s. ii) $\frac{1}{a_n} \sum_1^n (X_k - Y_k) \to 0$. $a_n \uparrow \infty$.

Pf: From $P(X_n \neq Y_n, \text{i.o.}) = 0 \Rightarrow P(X_n = Y_n, \text{ult}) = 1$.

Cor. The property of convergence w.r.t $\sum X_n$, $\frac{1}{a_n}\sum X_k$.

is same as $\sum Y_n$, $\frac{1}{a_n}\sum_1^n Y_k$. in a.s. sense.

② Common Forms:

i) Thm. $\{X_n\}$ pairwise indept. i.d. $m = E(X_1) < \infty$. Then

$$\bar{X} = S_n/n \xrightarrow{p} m.$$

Pf: 1°) $E(|X_1|) < \infty \Rightarrow \sum P(|X_1| \geq n) < \infty$.

Set $Y_n = X_n I_{\{|X_n| \leq n\}}$. equi. with X_n's

2°) $$P(|\bar{Y} - m| \geq \varepsilon) \leq \frac{E(|\bar{Y}-m|^2)}{\varepsilon^2} = \frac{Var(\bar{Y}) + (E(\bar{Y})-m)^2}{\varepsilon^2}$$

prove: $Var(\bar{Y}) \to 0$. $E(\bar{Y}) \to m$.

3°) $$|E(\bar{Y}) - m| = \left|\sum_1^n E(X_k I_{\{|X_k| > k\}})/n\right|$$

$$\leq \sum_1^n E(|X_k| I_{\{|X_k| > k\}})/n \to 0 \ (\text{Stolz}).$$

4°) $$Var\left(\sum_1^n Y_k\right) \leq \sum_1^n E(X_1^2 I_{\{|X_1| \leq k\}}) = \sum_1^{a_n} + \sum_{a_n+1}^n E(X_1^2 I_{\{|X_1| \leq k\}}).$$

$$\leq \sum_1^{a_n} E(X_1^2 I_{\{|X_1| \leq k\}}) + \sum_{a_n+1}^n E(X_1^2 I_{\{|X_1| \leq a_n\}}) + n\sum_{a_n+1}^n E(|X_1| I_{\{|X_1| \geq a_n\}})$$

$$\leq a_n \sum_1^n E(|X_1|) + n^2 E(|X_1| I_{\{|X_1| \geq a_n\}}).$$

Choose $a_n = o(n)$. (e.g. $[\sqrt{n}]$).

RMK: i) Truncate is for tail Expectation:

$$E(|X_1| I_{\{|X_1| \geq a_n\}}) \to 0 \ (a_n \to \infty).$$

ii) Directly: $\sum_{k=1}^{n} E(X_i^2 I\{|X_i|\le k\}) = \sum_{k=1}^{n}\sum_{i=1}^{k} E(\square I\{i-1\le |X_i|\le i\})$

$\le \sum\sum i^2 p(i-1\le |X_i|\le i) \le \sum_{k=1}^{n}\sum_{i=1}^{k}\sum_{j=1}^{i} 2p(i-1\le |X_i|\le i)$

$= 2\sum_{k=1}^{n}\sum_{j=1}^{k} 2p(j-1\le |X_i|\le k) \le 2\sum_{j=1}^{n}(n-j)p(j-1\le |X_i|)$

$\le 2nE(|X_i|)$. $\therefore Var(\bar{Y}) \to 0$.

(Abel Transf will not work: Last term losts!)

ii) <u>For symmetric r.v's:</u> (Only need indept.)

<u>Thm.</u> (Kolmogrov)

$\{X_n\}$ indept. $\{b_n\} \uparrow +\infty$. If: ($S_n = \sum_1^n X_k$)

$\sum_1^n p(|X_k| \ge b_n) \to 0$ $(n\to+\infty)$

$\frac{1}{b_n^2}\sum_1^n E(X_k^2 I\{|X_k|\le b_n\}) \to 0$ $(n\to\infty)$. Then:

$\frac{1}{b_n}(S_n - a_n) \to_p 0$. $a_n = \sum_1^n E(|X_k| I\{|X_k|\le b_n\})$

<u>Pf:</u> 1°) $Y_k^n = X_k I\{|X_k|\le b_n\}$. $T_n =: \sum_{k=1}^n Y_k^n$.

2°) By condition: $\begin{cases} \sum_1^n p(Y_k^n \ne X_k) \to 0 \\ \sum_1^n E((\frac{Y_k^n}{b_n})^2) \to 0 \end{cases}$

$\Rightarrow p(T_n \ne S_n) \le p(\bigcup_{k=1}^n \{Y_k^n \ne X_k\}) \to 0$.

$\sigma^2(T_n/b_n) \le \sum E((\frac{Y_k^n}{b_n})^2) \to 0$

$\therefore \frac{T_n - E(T_n)}{b_n} \to_p 0$. $S_n - T_n/b_n \to_p 0$.

<u>Remark</u>: i) Application: $p(X_n = n) = p(X_n = -n) = \frac{c}{n^2 \log n}$

$E(|X_i|) = \infty$. But $S_n/n \to_p 0$. holds.

ii) If $\exists \lambda > 0$. $p(X_n \le 0)$. $p(X_n \ge 0) \ge \lambda$. Then converse holds

iii) **Thm.** $\{X_n\}$ pairwise indept. i.d. satisfies:

$$E(X_1 I_{\{|X_1| \le n\}}) \to 0. \quad nP(|X_1| \ge n) \to 0. \text{ Then:}$$

$$S_n/n \to_p 0. \quad (\underline{pf}: b_n = n \text{ on Kolmogorov ii)})$$

Cor. (For i.i.d. r.v.'s)

$\{X_n\}$ i.i.d. Then the followings are equi.:

(a) $\bar{X} \to_p c$. $c \in$ const.

(b) $nP(|X_1| \ge n) \to 0$. $E(X_1 I_{\{|X_1| \le n\}}) \to c$

(c) $\varphi'_{X_1}(0) = ic$. (φ_{X_1} is ch.f of X_1. diff at 0).

Remark: When c be ∞. we can write it in:

$$\exists a_n. \ \bar{X} - a_n \to_p 0 \iff nP(|X_1| > n) \to 0.$$

(Actually c may not exist)

(3) SLLN:

① Maximum Inequilities:

Note that for dealing with a.s. convergence: $\forall \varepsilon > 0$.

Test: $P(\sup_{n \ge m} |X_n - X| \ge \varepsilon) = \lim_{r \to \infty} P(\max_{m \le n \le r} |X_n - X| \ge \varepsilon) = 0$.

We may estimate: $P(\max_{m \le n \le r} |X_n - X| \ge \varepsilon)$

Thm. (Hajek - Renyi)

$\{X_n\}$ indept. r.v.'s. $E(X_n) = 0$. $E(X_n^2) = \sigma_n^2 < \infty$.

$S_n = \sum_1^n X_k$. $\{c_k\} \subseteq \mathbb{R}^+$ nonincreasing. Then $\forall \varepsilon > 0$:

$$p(\max_{m\le k\le n} c_k|S_k|\ge\varepsilon)\le\frac{1}{\varepsilon^2}(c_m^2\sum_1^m\sigma_k^2+\sum_{m+1}^n c_k^2\sigma_k^2)$$

Pf: 1°) Apply some separation:

$$E_m=\{c_m|S_m|\ge\varepsilon\}.\quad E_j=\{\max_{m\le k<j}c_k|S_k|<\varepsilon,\ c_j|S_j|\ge\varepsilon\}$$

$$\therefore A=\{\max_{m\le k\le n}c_k|S_k|\ge\varepsilon\}=\sum_m^n E_j$$

Observe RHS: set $Y=c_m^2S_m^2+\sum_{m+1}^n c_k^2(S_k^2-S_{k-1}^2)$

2°) It suffices to show: $\sum_m^n p(E_j)\le E(Y)$

3°) $Y\ge 0$. By Abel Transformation.

$$\therefore E(Y)\ge E(YI_A)=\sum_m^n E(YI_{E_k})$$

4°) $E(YI_{E_j})\ge\sum_{k=j}^{n-1}(c_k^2-c_{k+1}^2)E(S_k^2I_{E_j})+c_n^2E(S_n^2I_{E_j})$.

$$\begin{aligned}E(S_k^2I_{E_j})&=E((S_k-S_j+S_j)^2I_{E_j})\\&=E([(S_k-S_j)^2+S_j^2]I_{E_j})\\&\ge E(S_j^2I_{E_j})\ge\varepsilon^2p(E_j)/c_j^2\end{aligned}$$

5°) Sum over: $E(YI_{E_j})\ge\varepsilon^2p(E_j)$.

Thm. (Kolmogorov Maximum)

$\{X_k\}$ indept. r.v's. $E(X_k)=0$. $E(X_k^2)=\sigma_k^2<\infty$. $\forall\varepsilon>0$.

i) (Upper Bound) $p(\max_{1\le k\le n}|S_k|\ge\varepsilon)\le Var(S_n)/\varepsilon^2$

ii) (Lower Bound) $p(\max_{1\le k\le n}|S_k|\ge\varepsilon)\ge 1-\frac{(c+\varepsilon)^2}{Var(S_n)}$

provide $|X_n|\le c<\infty$. $\forall n$.

Pf: i) is from $\{c_k\}=\{1\}$.

ii) Similarly. use the notations above ($m=1$).

1°) $Var(S_n I_A) = \sum_1^n Var(S_n I_{E_j})$

$$= \sum_1^n E((S_n-S_j)^2 I_{E_j}) + E(S_j^2 I_{E_j})$$

$$\le \sum_{j=1}^n \sum_{k=j+1}^n E(X_k^2 I_{E_j}) + E((|S_{j-1}|+|X_j|)^2 I_{E_j})$$

$$\le c((c+\varepsilon)^2 + Var(S_n))\, p(A).$$

2°) $Var(S_n I_A) = E(S_n^2) - E(S_n^2 I_{A^c})$

$$\ge E(S_n^2) - \varepsilon^2 p(A^c)$$

3°) Solve $p(A)$ from 1°). 2°)

<u>RMK</u>: Chebyshev is its special case of ii).

<u>Cor.</u> (One-side)

With the same assumptions: $p(\max_{1\le k\le n} S_k \ge \varepsilon) \le \dfrac{\sigma^2(S_n)}{\varepsilon^2+\sigma^2(S_n)}$

<u>Pf</u>: $p(A) = \sum_1^n p(E_i) \le \sum_1^n \int_{E_i} (\frac{S_i+\lambda}{\varepsilon+\lambda})^2 dM_{x_i}$

$$E\left(\left(\frac{S_n+\lambda}{\varepsilon+\lambda}\right)^2 I_{E_i}\right) = E\left[\left(\frac{S_n-S_i}{\varepsilon+\lambda}\right)^2 + \left(\frac{S_i+\lambda}{\varepsilon+\lambda}\right)^2\right] I_{E_i}$$

$$\ge E\left(\left(\frac{S_i+\lambda}{\varepsilon+\lambda}\right)^2 I_{E_i}\right)$$

$$\therefore p(A) \le \sum_1^n E\left(\left(\frac{S_n+\lambda}{\varepsilon+\lambda}\right)^2 I_{E_i}\right) \le E\left(\left(\frac{S_n+\lambda}{\varepsilon+\lambda}\right)^2\right)$$

<u>Cor.</u> (general case)

$\{X_n\}$ indept $\subset L^1$. $|X_n - E(X_n)| \le A$. Then $\forall \varepsilon > 0$

$$p(\max_{1\le k\le n} |S_k| \ge \varepsilon) \ge 1 - \frac{(2A+4\varepsilon)^2}{Var(S_n)}. \quad (A \in \mathbb{R}^+)$$

② Convergence of Series:

i) Variance Criterion:

<u>Thm.</u> $\{X_n\}$ indept. r.v.'s. $E(X_n)=0$. $\sigma_n^2 = E(X_n^2) < \infty$.

If $\sum \sigma_n^2 < \infty$ Then. $\sum X_n$ converge. a.s.

<u>Pf</u>: $p(\max_{m\le k\le M} |S_k - S_m| \geq \varepsilon) \leq \dfrac{\sum_{n=1}^{M} \sigma_k^2}{\varepsilon^2} \to \dfrac{\sum_{n=1}^{\infty} \sigma_k^2}{\varepsilon^2}$

$\therefore \lim_m p(\max_{m\le k} |S_k - S_m| \geq \varepsilon) \leq \lim_m \dfrac{\sum_n \sigma_k^2}{\varepsilon^2} = 0$

$\therefore p(\max_{m,n\geq M} |S_m - S_n| \geq \varepsilon) \leq p(\max_{m\geq M} |S_m - S_M| \geq \frac{\varepsilon}{2}) +$

$p(\max_{n\geq M} |S_n - S_M| \geq \frac{\varepsilon}{2}) \to 0$.

<u>Cor.</u> (Kolmogorov SLLN)

$\{X_n\}$ indept. r.v's. $m_k = E(X_k)$. $\sigma_k^2 = Var(X_k)$.

If $\sum \sigma_k^2/k^2 < \infty$. Then $\bar{X} \to \bar{m}$. a.s.

<u>Pf</u>: By Kronecker Lemma.

ii) Three Series Thm.

<u>Thm.</u> $\{X_n\}$ indept. r.v's. Then. $\sum X_n$. converges. a.s.

$\Leftrightarrow \exists A > 0$. $Y_n = X_n I_{\{|X_n| \leq A\}}$. The followings

converge : (a) $\sum p(|X_n| > A) < \infty$.

(b) $\sum E(Y_n) < \infty$

(c) $\sum Var(Y_n) < \infty$.

Pf. ($\Leftarrow$) By Criteria: $\sum (Y_n - E(Y_n)) < \infty$. a.s.

($\Rightarrow$) (a) $\because X_n \to 0$. a.s. $\therefore p(|X_n| > A.\ i.o) = 0$. $\forall A > 0$.

$\therefore \sum p(|X_n| > A) < \infty$. a.s.

(b) X_n's equi with Y_n's $\therefore \sum Y_n < \infty$. a.s.

By Kolmogrov Maximal:

$$p(\max_{n \le k \le m} |\sum_n^k Y_j| \ge \varepsilon) \ge 1 - \frac{(2A+4\varepsilon)^2}{Var(\sum_n^k Y_j)}$$

If $\sum Var(Y_n) = \infty$. Let $m \to \infty$. Then:

$$p(\sup_{k \ge n} |\sum_n^k Y_j| \ge \varepsilon) \ge 1.$$ contradict!

(c) By criteria: $\sum Y_n - E(Y_n) < \infty$ a.s.

$\therefore \sum E(Y_n) < \infty$. a.s.

Cor. Replace $\sum Var(Y_n) < \infty$ with $\sum E(Y_n^2) < \infty$.

It still holds for ($\Leftarrow$).

Pf: $\sum E(Y_n^2) \ge \sum E^2(Y_n)$ $\therefore \sum Var(Y_n) < \infty$.

Cor. $\{X_n\}$ indept. $\sum E(|X_n|^{p_n}) < \infty$. $0 < p_n \le 2$. $\forall n$.

Besides. $E(X_n) = 0$ if $p_n > 1$. $\Rightarrow \sum X_n < \infty$. a.s.

Pf: Separate $\sum X_n$ into $\{p_n > 1\}$. $\{p_n \le 1\}$.

1') Set $Y_n = X_n I_{\{|X_n| \le 1\}}$.

$p(Y_n \ne X_n) = p(|X_n| > 1) \le E(|X_n|^{p_n})$.

2') $|\sum E(Y_n)| \le |\sum_{\{p_n > 1\}}| + |\sum_{\{p_n \le 1\}}|$

Note that: $E(Y_n) = E(X_n I_{\{|X_n| > 1\}})$.

if $p_n > 1$.

3°) $\sum Var(Y_n) < \infty$ is direct.

iii) <u>Two Series:</u>

<u>Thm.</u> $\{X_n\}$ indept. $\sum |X_n| < \infty$. a.s $\iff \exists c > 0$.

s.t. $\sum p(|X_n| \geq c) < \infty$. $\sum E(|X_n| I_{\{|X_n| \leq c\}}) < \infty$.

<u>Pf:</u> ($\Leftarrow$). $E(X_n^2 I_{\{|X_n| \leq c\}}) \leq c\, E(|X_n| I_{\{|X_n| \leq c\}})$

<u>Cor.</u> $\{X_n\}$ indept. r.v.'s. $\sum E(|X_n|^r) < \infty$. $0 < r \leq 1$.

$\Rightarrow \sum |X_n| < \infty$. a.s.

<u>Pf:</u> 1°) $p(|X_n| \geq 1) \leq E(|X_n|^r)$.

2°) $E(|X_n| I_{\{|X_n| \leq 1\}}) \leq E(|X_n^r|)$.

<u>Cor.</u> (Remove indept).

$\{X_n\} \subset L^1$. $X_n \geq 0$. $\forall n$. $\sum E(X_n) < \infty$. Then

$S_n = \sum_1^n X_k$ converges. a.s.

<u>Pf:</u> $p(|S_m - S_n| \geq \varepsilon) \leq \frac{1}{\varepsilon} E|S_m - S_n| \to 0$.

$\therefore \{S_n\}$ is Cauchy in pr.

$\exists \{S_{n_k}\} \subseteq \{S_n\}$. $S_{n_k} \to S_\infty$. a.s.

Since $\exists k$. $S_{n_k} \leq S_n \leq S_{n_{k+1}}$. By MCT.

$\therefore S_n \to S_\infty$. a.s. $S_n < \infty$. a.s.

<u>Remark:</u> It can apply Levy's Thm. directly.

iv) Levy's Thm:

Thm: ΣX_n converges a.s $\Leftrightarrow$ ΣX_n converges in pr. for indept. r.v's $\{X_n\}$.

Pf: Denote $S_{m,n} = \sum_m^n X_k$. For ($\Leftarrow$):

We have: $\forall \varepsilon > 0.\ \forall \delta > 0.\ \exists m_0.\ \forall m,n > m_0.$

$p(|S_{m,n}| > \varepsilon) < \delta$. ($\overline{\lim\limits_{m,n}}\ p(|S_{m,n}| > \varepsilon) = 0$)

Lemma. $p(\max\limits_{m<k\le n} |S_{m,k}| \ge 2\varepsilon) < \frac{\delta}{1-\delta}$. $\delta = \delta(m,n)$

Pf: 1') Partition: $A = \{\max\limits_{m<k\le n} |S_{m,k}| \ge 2\varepsilon\}$.

$E_j = \{\max\limits_{m<k\le j} |S_{m,k}| < 2\varepsilon.\ |S_{m,j+1}| \ge 2\varepsilon\}$

$\therefore A = \sum_m^{n-1} E_j \quad p(A) = \Sigma p(E_j)$.

2') $\Sigma p(E_j) = \Sigma p(E_j . |S_{j+1,n}| > \varepsilon) + p(E_j . |S_{j+1,n}| \le \varepsilon)$

$\le \Sigma p(E_j . |S_{m,n}| > \varepsilon) + p(E_j)\, p(|S_{j+1,n}| > \varepsilon)$

$\le p(A . |S_{m,n}| > \varepsilon) + \delta\, p(A)$.

$\le \delta + \delta\, p(A)$.

$\Rightarrow \lim\limits_{m,n} p(\max\limits_{m<k\le n} |S_k| \ge 2\varepsilon) = 0$. $\therefore S_\infty < \infty$. a.s

③ Common Forms:

i) Kronecker Lemma.

Thm. For $a_n \uparrow \infty$. $\sum \frac{\eta_n}{a_n} < \infty \Rightarrow \frac{1}{a_n} \sum_1^n \eta_k \to 0$

Pf: $x_n = \frac{\eta_n}{a_n}$. Then $\frac{1}{a_n} \sum_1^n \eta_k = \frac{1}{a_n} \sum_1^n a_k x_k$.

$$\frac{1}{a_n} \sum_1^n a_k x_k = \frac{1}{a_n} \sum_1^n a_k (S_k - S_{k-1})$$

$$= S_n - \frac{\sum_1^n (a_k - a_{k-1}) S_{k-1}}{\sum_1^n (a_k - a_{k-1})} \to 0$$

ii) For indept. r.v.'s:

Thm. $\{X_n\}$ indept. r.v.'s. $\{g_n(x)\}$ even. positive. nondecreasing if $x>0$

Func's. Besides. for $\forall n$. at least one holds follows:

(a) $x/g_n(x) \uparrow$ if $x>0$.

(b) $x/g_n(x) \downarrow$ $x^2/g_n(x) \uparrow$. $E(X_n) = 0$. $x>0$.

(c) $x^2/g_n(x) \uparrow$. X_n is symmetric about 0. $\forall n$.

Then for $\{a_n\} \subseteq \mathbb{R}^+$. $\sum \frac{E(g_n(X_n))}{g_n(a_n)} < \infty \Rightarrow \sum \frac{X_n}{a_n} < \infty$ a.s

(So if $a_n \uparrow \infty$. then $\frac{1}{a_n} \sum_1^n X_k \to 0$. a.s)

Pf: Set $Y_n = X_n I_{\{|X_n| \le a_n\}}$. $\therefore \frac{Y_n}{a_n} = \frac{X_n}{a_n} I_{\{|\frac{X_n}{a_n}| \le 1\}}$.

prove: $\sum p(|X_n| > a_n) < \infty$.

$$\sum E(\frac{Y_n}{a_n}) < \infty$$

$$\sum E(\frac{Y_n^2}{a_n^2}) < \infty.$$

1°) $p(|X_n| > a_n) \le \frac{E(g_n(X_n))}{g_n(a_n)}$.

2°) (a) $E(\frac{Y_n}{a_n}) \le E(\frac{g_n(Y_n)}{g_n(a_n)})$

(b) $|\sum \frac{E(Y_n)}{a_n}| = |-\sum \frac{E(X_n I_{\{|X_n| \ge a_n\}})}{a_n}| \le \sum \frac{E(g_n(Y_n))}{g_n(a_n)}$

(c) $E(Y_n) = 0$. $\forall n$. It's trivial.

3°) prove: $\frac{X_n^2}{a_n^2} \le \frac{g_n(X_n)}{g_n(a_n)}$ $\therefore \sum E((\frac{Y_n}{a_n})^2) < \infty$.

<u>Cor.</u> $\{X_n\}$ indept. r.v.'s. $0 < a_n \uparrow \infty$. If:

$\sum E(|\frac{X_n}{a_n}|^r) < \infty$. for $0 < r \le 2$. Then:

$$\begin{cases} \frac{1}{a_n}\sum_1^n X_k \to 0. \text{ a.s.} & 0 < r \le 1. \\ \frac{1}{a_n}\sum_1^n X_k - E(X_k) \to 0. \text{ a.s} & 1 \le r \le 2. \end{cases}$$

<u>pf.</u> $g_n(x) = x^r$. By Cr-Inequility:

$E|\frac{X_n - E(X_n)}{a_n}|^r \le C_r(E|\frac{X_n}{a_n}|^r + |E(\frac{X_n}{a_n})|^r) < \infty$.

<u>Remark:</u> i) $r = 1$. We can drop $E(X_k)$:

since $\sum \frac{E(|X_n|)}{a_n} < \infty$. by Kronecker Lemma.

ii) $r > 2$, Cor may not hold.

<u>Thm.</u> (Neccessary and Sufficient conditions)

$\{X_n\}$ indept. r.v.'s. $0 < a_n \uparrow \infty$. $Y_{nk} = \frac{X_k}{a_n} I_{\{|X_k| < a_n\}}$.

If $\sum E(Y_{nn}^2) < \infty$. Then: $\frac{1}{a_n}\sum_1^n X_k \to 0$. a.s $\Leftrightarrow$

$\sum P(|X_n| \ge a_n) < \infty$. $\sum_{k=1}^n E(Y_{nk}) \to 0$

iii) For i.i.d. r.v.'s:

Thm. (Kolmogorov)

$\{X_n\}$ i.i.d. r.v's. $S_n = \sum_1^n X_k$. Then:

(a) $E(|X_n|) < \infty$. $\Rightarrow S_n/n \to E(X_1)$. a.s.

(b) $E(|X_n|) = \infty \Rightarrow \overline{\lim_n} S_n/n = \infty$. a.s.

Pf: (a) Set $Y_n = X_n I_{\{|X_n| \le n\}}$.

$\because E(|X_n|) \sim \sum p(|X_n| > n) < \infty$. $\therefore Y_n$. equi. X_n

1°) $\sum_1^n E(Y_k)/n \to E(X_1)$.

By Stolz: since $E(Y_n) \to E(X_1)$. by MCT.

2°) $\frac{1}{n}\sum_1^n Y_k - E(Y_k) \to 0$. a.s:

Check: $r=2$. $a_n = n$:

$$\sum \frac{E(Y_n^2)}{n^2} = \sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{1}{n^2} E(|X_1|^2 I_{\{k-1 \le |X_1| \le k\}})$$

$$\le \sum_{k=1}^{\infty}\sum_{n=k}^{\infty} \frac{1}{n^2} E(k|X_1| I_{\{k-1 \le |X_1| \le k\}})$$

$$= c\sum_k E(|X_1| I_{(k-1 \le |X_1| \le k)}) < \infty.$$

(b) $\forall A > 0$. $p(|X_n| > An.\ i.o) = 1$. Since $E(|X_1|/A) = \infty$.

$\therefore p(|S_n - S_{n-1}| > An.\ i.o) \le$

$p(\{|S_n| > \frac{An}{2}\} \cup \{|\frac{S_{n-1}}{n}| > \frac{A}{2}\}.\ i.o)$

$\therefore \exists N(A)$. null set. $\overline{\lim} \frac{S_n}{n} > \frac{A}{2}$

$\therefore \exists N = \bigcup_{m \in \mathbb{Z}^+} N(m)$. $\overline{\lim} \frac{S_n}{n} = \infty$. $\mu(N) = 0$.

Cor. Addition: If $E(X_1^+) = \infty$. $E(X_1^-) < \infty$.

Then $\lim_n S_n/n \to \infty$.

Pf: Set $X_n^m = X_n \wedge m$. $\therefore E(|X_n^m|) < \infty$.

$\therefore S_n^m/n \to E(X_n^m)$. a.s.

$\because \underline{\lim}\, S_n/n \geq \lim S_n^m/n = E(X_n^m) = E(X_n^{m+} - X_n^{m-})$. $\forall m$.

$E((X_n^m)^+) \uparrow \infty$. $(m \to \infty)$. $E((X_n^m)^-) < \infty$.

$\therefore \underline{\lim}\, S_n/n = \infty$. $\Rightarrow \lim S_n/n = \infty$.

Cor. $\{X_n\}$ i.i.d. r.v's. $S_n = \sum_1^n X_k$. Then:

$S_n/n \to c$. a.s $\Leftrightarrow E(X_1)$ exists. $E(X_1) = c$.

Pf: Note that $X_n/n \to 0$. a.s. $\therefore P(|X_n| > n$. i.o$) = 0$.

i.e. $E(|X_1|) < \infty$. Apply SLLN.

Thm. (Marcinkiewicz)

$\{X_n\}$ i.i.d r.v's. $0 < r < 2$. Then $\frac{1}{n^{\frac{1}{r}}} \sum_1^n (X_k - a) \to 0$. a.s.

$\Leftrightarrow E(|X_1|^r) < \infty$. where $a = \begin{cases} E(X_1). & 1 \le r < 2 \\ \text{arbitrary}. & 0 < r < 1 \end{cases}$

Pf: $(\Rightarrow)$ $X_n/n^{\frac{1}{r}} \to 0$. a.s. $\therefore P(|X_n| \geq n^{\frac{1}{r}}$. i.o.$) = 0$.

$(\Leftarrow)$ Set $Y_n = X_n I\{|X_n| \le n^{\frac{1}{r}}\}$. Y_n equi. with X_n.

(a) $r=1$. We have proved.

(b) $0<r<1$: prove: $\frac{1}{n^{\frac{1}{r}}} \sum_1^n Y_k \to 0$. Since $a n^{1-\frac{1}{r}} \to 0$.

Check $a_n = n^{\frac{1}{r}}$. $g_n(x) = |x|$.

$$\sum \frac{E(|Y_n|)}{n^{\frac{1}{r}}} = \sum \frac{E(|X_n| I\{|X_n| \le n^{\frac{1}{r}}\})}{n^{\frac{1}{r}}}$$

$$= \sum_{n=1}^{\infty} \sum_{k=1}^{n} E(|X_1| I\{k-1 \le |X_1|^r < k\}) / n^{\frac{1}{r}}$$

$$\le \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \frac{1}{n^{\frac{1}{r}}} \cdot k^{\frac{1}{r}-1} E(|X_1|^r I_{\Box})$$

$$= c \sum E(|X_n|^r I_{\{k-1 \le |X_n|^r \le k\}})$$

$$= c E(|X_1|^r) < \infty. \quad \therefore \sum_1^n Y_k / n^{\frac{1}{r}} \to 0 \ a.s.$$

(c) $1 < r < 2$: $c_n = E(X_1)$

Note: $\frac{1}{n^{\frac{1}{r}}} \sum_1^n (X_k - E(X_k)) = \frac{1}{n^{\frac{1}{r}}} \sum_1^n (X_k - Y_k) + \frac{1}{n^{\frac{1}{r}}} \sum_1^n (Y_k - E(Y_k)) +$

$\frac{1}{n^{\frac{1}{r}}} \sum_1^n (E(Y_k) - E(X_k))$.

1°) Check $a_n = n^{\frac{1}{r}}$. $g_n(x) = x^2$.

$$\sum \frac{E(Y_n^2)}{n^{\frac{2}{r}}} = \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{E(X_n^2 I_{\{k-1 \le |X_n|^r \le k\}})}{n^{\frac{2}{r}}}$$

$$\le \sum_{k=1}^{\infty} \sum_{n \ge k}^{\infty} \frac{1}{n^{\frac{2}{r}}} \cdot k^{\frac{2}{r}-1} E(|X_n|^r I_{\{k-1 \le |X_n|^r \le k\}})$$

$$= C E(|X_1|^r) < \infty.$$

2°) $$\left| \sum_{n=1} \frac{E(X_n) - E(Y_n)}{n^{\frac{1}{r}}} \right| \le \sum n^{-\frac{1}{r}} E(|X_1| I_{\{|X_1|^r \ge n\}})$$

$$= \sum_{n=1}^{\infty} n^{-\frac{1}{r}} \sum_{k=n}^{\infty} E(|X_1| I_{\{k \le |X_1|^r \le k+1\}})$$

$$\le \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} n^{-\frac{1}{r}} \cdot k^{\frac{1}{r}-1} E(|X_1|^r I_{\{\cdots\}})$$

$$\le C E(|X_1|^r) < \infty.$$

$$\therefore \sum_1^n E(X_k) - E(Y_k) / n^{\frac{1}{r}} \to 0.$$

Remark: For $r \ge 2$. It may not hold. But if $r = 2$.

$\{X_n\}$ i.i.d. $E(X_1) = 0$. $\sigma^2 = E(X_1^2) < \infty$. then:

$S_n / n^{\frac{1}{2}} (\log n)^{\frac{1}{2}+\varepsilon} \to 0$. a.s. $\forall \varepsilon > 0$

Pf: $\sum E(\frac{X_n^2}{a_n^2}) = \sigma^2 \sum \frac{1}{n(\log n)^{1+2\varepsilon}} + a_1^2 < \infty$.

where $a_n = n^{\frac{1}{2}} (\log n)^{\frac{1}{2}+\varepsilon}$.

For more general law of iterated logarithm:

$\{X_n\}$ i.i.d. r.v.'s. $E(X_1) = 0$. $\sigma^2 = Var(X_1)$. Then:

$$\overline{\lim_n}\ S_n / n^{\frac{1}{2}}(\log\log n)^{\frac{1}{2}} = \sqrt{2}\sigma \text{ a.s. (Require } \sigma^2 < \infty)$$

iv) <u>Feller's Extension:</u>

<u>Thm.</u> $\{X_n\}$ i.i.d. $E(|X_1|) = \infty$. $\{a_n\}_1^\infty \subseteq \mathbb{R}^+$. $\frac{a_n}{n} \uparrow$ $(a_0 = 0)$

Then
$$\begin{cases} \lim_n |S_n|/a_n = 0 \text{ a.s} & \text{if } \sum p(|X_n| \geq a_n) < \infty \\ \overline{\lim_n} |S_n|/a_n = \infty \text{ a.s} & \text{if } \sum p(|X_n| \geq a_n) = \infty. \end{cases}$$

<u>Pf:</u> Set $Y_n = X_n I_{\{|X_n| \leq a_n\}}$.

(a) $\sum p(|X_n| \geq a_n) < \infty \Rightarrow X_n$ equi. with Y_n.

prove: $\frac{\sum_1^n Y_k - E(Y_k)}{a_n}$, $\frac{\sum_1^n E(Y_k)}{a_n} \to 0$.

1°) $a_n/n \uparrow \infty$.

If $a_n/n \leq c$. Then since $E(|X_1|) \sim \sum p(|X| > n)$

$\sum p(|X| \geq cn) \leq \sum p(|X| \geq \frac{a_n}{n} \cdot n) < \infty$.

$\therefore E(|X|/c) < \infty$. contradict!

2°) $\sum_1^m E(Y_n)/a_m \to 0$

$$\sum_1^m E(X_1 I_{\{|X_1| \leq a_m\}})/a_m \leq \frac{m}{a_m}(a_N + E(|X_1| I_{\{a_N \leq |X_1| \leq a_m\}}))$$

$\frac{m}{a_m} \cdot a_N \leq \frac{a_N}{a_1}$. since $\frac{n}{a_n} \downarrow$.

$$\frac{m}{a_m} E(|X_1| I_{\{a_N \leq |X_1| \leq a_m\}}) = \frac{m}{a_m}\sum_N^{m-1} \square$$
$$\leq \frac{m}{a_m}\sum_N^{m-1} a_{k+1} p(a_k \leq |X_1| \leq a_{k+1})$$
$$\leq \sum_N^{m-1} (k+1) p(a_k \leq |X_1| \leq a_{k+1})$$

$$\le \sum_{N}^{\infty} p(|X_1| \ge a_n) \longrightarrow 0 \ (N \to \infty).$$

∴ For large N (fix). $\frac{m}{a_m} E(|X_1| I_{\{a_N \le |X_1| \le a_m\}}) \le \varepsilon$.

∴ Let $m \to \infty$. $\frac{m}{a_m} a_N \to 0$. $\lim \frac{\sum E(Y_n)}{a_m} \le \varepsilon$. $\forall \varepsilon > 0$

3°) $\sum E(\frac{Y_n^2}{a_n^2}) < \infty$.

$$\sum E(X_n^2 I_{\{|X_n| \le a_n\}} / a_n^2) = \sum_{n}^{\infty} \sum_{k=1}^{n} E(|X_1|^2 / a_n^2 \, I_{\{a_{k-1} \le |X_1| \le a_k\}})$$

$$\le \sum_{k \ge 1}^{\infty} \sum_{n=k}^{\infty} \frac{a_k^2}{a_n^2} p(a_{k-1} \le |X_1| \le a_k)$$

$$\le \sum_{k=1}^{\infty} \sum_{n \ge k}^{\infty} \frac{k^2}{n^2} p(a_{k-1} \le |X_1| \le a_k) < \infty.$$

∴ $\lim S_n/n = 0$. a.s.

(b). ∵ $a_n/n \uparrow$. ∴ $a_{kn} \ge k a_n$. $\forall k$ (fix) $\in \mathbb{Z}^+$.

$$\sum_{1}^{\infty} p(|X_1| \ge k a_n) \ge \sum_{1}^{\infty} p(|X_1| \ge a_{kn}) \ge \frac{1}{k} \sum_{m=k}^{\infty} p(|X_1| \ge a_m) = \infty.$$

Since $k\, p(|X_1| \ge a_{kn}) \ge \sum_{i=1}^{k} p(|X_1| \ge a_{kn+i})$.

∴ $p(\frac{|X_n|}{a_n} \ge k \text{ i.o.}) = 1 \Rightarrow p(\frac{|S_n|}{a_n} \ge k \text{ i.o.}) = 1$.

④ Application:

$f \in C[0,1]$. Bernstein polynomials: $p_n(x) = \sum_{k=1}^{n} f(\frac{k}{n}) \binom{n}{k} x^k (1-x)^{n-k}$

Then $p_n \xrightarrow{u} f$ in $[0,1]$

Pf: $p(X_n = 1) = x$, $p(X_n = 0) = 1 - x$. $S_n = \sum_{1}^{n} X_k$.

∴ $p_n(x) = E(f(\frac{S_n}{n})) \longrightarrow f(x)$. a.s. By SLLN.

Check uniform: $|p_n - f|$. separate: $I_{\{|\frac{S_n}{n} - x| \le \varepsilon\}}$