Independence.

(1) Definitions: (All in $(\Omega, \mathcal{A}, P)$).

① **Def**: i) Event: $\{A_i\}_{i\in I} \subset \mathcal{A}$ are indept. if

$$P(\bigcap_{i\in J} A_i) = \prod_{i\in J} P(A_i).\ \forall J \subseteq I.\ |J| < \infty.$$

ii) Classes: $\{\mathcal{A}_i\}_{i\in I}$. indept. if:

$$P(\bigcap_{i\in J} A_i) = \prod_{i\in J} P(A_i).\ \forall J \subseteq I.\ \#J < \infty.\ \forall A_i \in \mathcal{A}_i.$$

iii) r.v.'s: $\{X_i\}_{i\in I}$. are indept. if:

$$P(\bigcap_{i\in J} \{X_i \in B_i\}) = \prod_{i\in J} P(X_i \in B_i).\ \forall J \subseteq I.\ \#J < \infty.$$

$\forall B_i \in \mathcal{B}_{\mathbb{R}^1}$.

Remark: i) We have $P(\bigcap_1^\infty A_n) = \prod_1^\infty P(A_n)$. for indept events $\{A_n\}_{n\in\mathbb{Z}^+}$. By limit.

ii) Written in measure language:

$\mu(d\vec{x}) = \prod_1^n \mu(dx_i)$. If $\{X_i\}_1^n$ r.v.'s are indept. $\vec{x} = (x_1, x_2 \cdots x_n)$.

② **Check Indepdent**:

i) **Events**:

Thm. $\{A_i\}_1^n$ are indept $\Leftrightarrow$ $\{I_{A_n}\}$ r.v.'s. indept.

i.e. $P(\bigcap_{i\in I} \tilde{A}_i) = \prod_{i\in I} P(\tilde{A}_i)$. $\tilde{A}_i = A_i$ or A_i^c.

Pf: By reordering and induction.

ii) r.v.'s:

Lemma. $\mathcal{G}$ and $\mathcal{D}$ are indept classes. $\mathcal{D}$ is π-class.

Then $\mathcal{G}$, $\sigma(\mathcal{D})$ are indept.

Pf: Define: $\mathcal{D}_B^* = \{A \mid A \in \sigma(\mathcal{D}),\ p(A \cap B) = p(A)p(B)\}$.

for any $B \in \mathcal{G}$.

Check $\mathcal{D}_B^*$ is λ-class. contain $\mathcal{D}$.

Cor. $\{\mathcal{A}_i\}_1^n$ indept π-classes. Then $\{\sigma(\mathcal{A}_i)\}_1^n$ indept.

Thm. r.v's $\{X_k\}_1^n$ are indept $\Leftrightarrow F_{\vec{X}}(\vec{t}) = \prod_1^n F_{X_i}(t_i)$

for $\forall\ t_i \in \mathbb{R}^1$, $1 \le i \le n$.

Pf: ($\Rightarrow$) Take $B_i = (-\infty, t_i]$

($\Leftarrow$) By Lemma: $\mathcal{A}_i = \{(-\infty, t_i] \mid t_i \in \mathbb{R}^1\}$,

are π-classes. $1 \le i \le n$. So $\sigma(\mathcal{A}_i) = \mathcal{B}_{\mathbb{R}^1}$.

Thm. (Discrete r.v's)

r.v's $\{X_k\}_1^n$ take values on countable set C.

They're indept $\Leftrightarrow p(\bigcap_1^n \{X_i = a_i\}) = \prod_1^n p(X_i = a_i)$, $\forall a_i \in C$.

Pf: ($\Rightarrow$) it's trivial. $B_i = \{a_i\}$.

($\Leftarrow$) Check $F_{\vec{X}}(\vec{t}) = \prod_1^n F_{X_i}(t_i)$

since $p(\bigcap \{X_i \le t_i\}) = \sum_{a_i \le t_i} p(\bigcap \{X_i = a_i\})$

Thm. (Absolutely conti r.v's)

$\vec{X} = (X_1, \cdots X_n)$ absolutely conti. r.v. Then :

$\{X_k\}_1^n$ indept $\Leftrightarrow$ $f_{\vec{X}}(\vec{t}) = \prod_1^n f_{X_i}(t_i)$. $\forall t_i \in \mathbb{R}^1$.

Pf: ($\Rightarrow$) Differentiate $F_{\vec{X}}(\vec{t})$ ($\Leftarrow$) Integrate $f_{\vec{X}}$.

(2) Functions of indept r.v.'s:

① Transformation properties:

Thm. $\{X_k\}_1^n$ indept r.v's. $\{g_k\}_1^n$ Borel-measurable.

Then $\{g_k(X_k)\}_1^n$ are indept r.v.'s.

Cor. $1 \le n_0 < n_1 < n_2 \cdots < n_k = n$. $\{X_k\}_1^n$ indept

$\Rightarrow \{g_i(X_{n_i+1}, \cdots X_{n_{i+1}})\}_{i=0}^{k}$ indept.

Pf: For simplicity. $Z_1 = (X_1 \cdots X_m)$. $Z_2 = (X_{m+1}, \cdots X_n)$

$B_1 = \{A \in B_{\mathbb{R}^m} \mid p(Z_1 \in A, Z_2 \in B) = p(\square)p(\square). \forall B \in B_{\mathbb{R}^{n-m}}\}$

$\{\prod_1^m A_k \mid A_k \in B_{\mathbb{R}^1}\} \subseteq B_1$. check B_1 is λ-class.

② Convolution:

Thm. X, Y indept. absolutely conti r.v's. Then

$X + Y \sim f_{X+Y} = f_X * f_Y$.

Rmk: For d.f: $F_{X+Y} = F_X * F_Y = \int_{\mathbb{R}^1} F_X(x-y)\, d F_Y(y)$.

Thm. X, Y indept. nonnegative. take integer values.

Then $\forall n \geq 0$. $P(X+Y=n) = \sum_{k=0}^{n} P(X=k)P(Y=n-k)$.

③ Correlation:

Thm. If $X, Y \in L^1$. indept $\Rightarrow$ $Cor(X, Y) = 0$.

Pf: prove it by four step as usual.

Cor. $\{X_k\}_1^n \subset L^1$. indept $\Rightarrow$ $E(\prod_1^n X_k) = \prod_1^n E(X_k)$.

Thm. u, v are both nondecreasing or increasing on $I = (a, b)$.

$a, b \in \overline{\mathbb{R}}$. $P(X \in I) = 1$. Then $E(u(X)v(X)) \geq E(u(X))E(v(X))$.

if these means exist.

Pf: Denote $Y \sim X$. indept with X.

Then take expection: $(u(X) - u(Y))(v(X) - v(Y)) \geq 0$.

④ Maximum and Minimum:

Thm. $\{X_i\}_1^n$ indept r.v's. $X_i \sim F_i$. d.f. Then

$\max_{1 \leq i \leq n} X_i \sim \prod_1^n F_i$. Similar for $\min_{1 \leq i \leq n} X_i$

Pf: $P(\max X_i \leq x) = P(\bigcap_1^n \{X_i \leq x\}) = \prod_1^n F_{X_i}(x)$

Take complement to attain minimum.

(3) Lemmas and Laws:

① Borel-Cantelli Lemma:

Lemma. $\{A_n\}_{n\geq 1}$ indept events. $p(A_n)<1$, $\forall n$.

If $p(\bigcup_1^\infty A_n)=1$. Then $p(A_n. i.o)=1$.

Pf: $1-p(\bigcap_1^\infty A_n^c)=1$. $\therefore p(\bigcap_1^\infty A_n^c)=0$.

Since $p(A_n^c)>0$. $\therefore p(\bigcap_{n\geq k}^\infty A_n^c)=0$

$p(A_n^c. w.l.t)=\lim_k p(\bigcap_{n\geq k} A_n^c)=0$.

Thm. i) $\sum p(A_n)<\infty \Rightarrow p(A_n. i.o)=0$.

ii) $\sum p(A_n)=\infty$. $\{A_n\}$ indept $\Rightarrow p(A_n. i.o)=1$.

Pf. i) $p(\bigcup_k^\infty A_n)\leq \sum_k^\infty p(A_n)\to 0\ (k\to\infty)$

ii) $p(A_n^c. w.l.t)=\lim_{k\to\infty} p(\bigcap_{n\geq k}^\infty A_n^c)$

$=\lim_{k\to\infty}\prod_k^\infty (1-p(A_n))$

$\leq \lim_k e^{-\sum_k^\infty p(A_n)}=0$.

Since $e^{-x}\geq 1-x$. $\forall x\in\mathbb{R}^1$.

Remark: ii) may not hold if $\{A_n\}$ not indept.

e.g. $A_n=A$. $p(A)\in(0,1)$.

Cor. $\sum p(A_n)=\infty$. $\{A_n\}$ pairwise indept

$\Rightarrow p(A_n. i.o)=1$.

Pf: Denote $I_n = I_{A_n}$. prove: $\sum E(I_n) = \infty \Rightarrow p(\sum I_n = \infty) = 1$.

$S_k = \sum_1^k I_n$. $\sigma^2(S_k) = \sum_1^k p(A_n) - p^2(A_n) \le E(S_k)$.

$\therefore \sigma(S_k) \sim o(E^{\frac{1}{2}}(S_k))$. $(k \to \infty)$.

$\therefore$ If k large enough: (fix A).

$$p(S_k \ge \tfrac{1}{2} E(S_k)) > p(|S_k - E(S_k)| < A\sigma(S_k))$$
$$\ge 1 - \frac{\sigma^2(S_k)}{A^2 \sigma^2(S_k)} = 1 - \frac{1}{A^2} \to 1.$$

Let $k \to \infty$. $A \to \infty$. Done.

Cor. $\{A_n\}$ pairwise indept. Then

$$p(A_n. i.o) = \begin{cases} 0 & \Leftrightarrow \sum p(A_n) < \infty \\ 1 & \Leftrightarrow \sum p(A_n) = \infty. \end{cases}$$

Cor. $\{A_n\}$ pairwise indept. $A_n \to A$. Then $p(A) = 0$ or 1.

Pf: $p(A) = p(\lim A_n) = p(\overline{\lim} A_n) = p(A_n. i.o)$.

Cor. $\{X_n\}$ pairwise indept. $X_n \to 0$. a.s $\Leftrightarrow \forall \varepsilon > 0$. $\sum p(|X_n| > \varepsilon) < \infty$.

Cor. If $p(\{X_n < a, i.o\} \cap \{X_n > b. i.o\}) = 0$. $\forall b > a$. $b. a \in \mathbb{R}'$.

Then $\lim X_n$ exists a.e. (may be infinite).

Pf: $p(\bigcup_{\substack{a.b \in \mathbb{Q} \\ a<b}} (\{X_n < a, i.o\} \cap \{X_n > b. i.o\}))$

$= p(\underline{\lim} X_n < \overline{\lim} X_n) = \sum p(\{\square\} \cap \{\square\}) = 0$

$\therefore p(\underline{\lim} X_n = \overline{\lim} X_n) = 1$.

Cor. $\{X, X_n\}$ pairwise indept. i.d. Then.

$$E(|X|^r) < \infty \iff X_n = o(n^{\frac{1}{r}}).\ a.e.\ (r>0).$$

③ Kolmogorov 0-1 Laws:

Def: Tail σ-algebra of $\{X_n\}_{n\geq 1}$. r.v's on $(\Omega, \mathcal{A}, P)$ is $\bigcap_{n\geq 1} \sigma(X_j, j\geq n) =: \tilde{\sigma}$.

Events in $\tilde{\sigma}$ are called tail events.

Measurable Func's w.r.t $\tilde{\sigma}$ is tail Func's

Remark: Intuitively, A is tail event iff change finite number of values won't affect the occurrence. It depends entirely on tail series.

e.g.

i) $\{A_n\ i.o\}$ is tail event. since $\{A_n\ i.o\} = \bigcap \bigcup \{A_n\}. \in \bigcap \sigma(X_i, i\geq n)$. $X_i = I_{A_i}$.

ii) $S_n = X_n + X_{n-1} + \cdots + X_1$

(a) $\{\lim S_n \text{ exists}\}$ is tail event.

since $\{\square\} = \bigcap \{\sum_{k\geq n} X_k \text{ converges}\}$.

(b) $\{\overline{\lim}\, S_n > X\}$ isn't tail event

since it depends on the initial value of X_n's.

Thm. B is tail event w.r.t $\{X_k\}$ indept r.v.'s.

Then $p(B) = 0$ or 1.

Pf: $\forall n \geq 1$: $\sigma(X_i, 1 \leq i \leq n)$ indept with $\sigma(X_i, i > n)$

so with $\bigcap_{n=0} \sigma(X_i, i > n) \subset \sigma(X_i, i > n)$

$\therefore A = \bigcup_{n \geq 1} \sigma(X_i, 1 \leq i \leq n)$ indept with $\bigcap \sigma(X_i, i > n)$

$\therefore \sigma(A)$ indept with $D =: \bigcap \sigma(X_i, i > n)$

$\because D \subset \sigma(A)$. $\therefore D$ indept with itself.

$p(B) = p(B \cap B) = p^2(B)$. $\therefore p(B) = 0, 1$. $\forall B \in D$.

Cor. Tail Func's of indept r.v's are degenerated.

Pf: Denote it by Y. $\{Y \leq c\}$ is tail Func.

$\therefore p(Y \leq c) = 0$ or 1. Let $c_0 = \inf\{c \mid p(Y \leq c) = 1\}$.

$\therefore p(Y = c_0) = 1$ $c_0 \in \overline{\mathbb{R}}$.

Cor. $\{X_n\}$ indept. r.v's. Then $\overline{\lim} X_n$, $\underline{\lim} X_n$ are

degenerated. a.s.

Pf: $Y_n = \sup_{k \geq n} X_k$ is $\sigma(X_j, j \geq n)$-measurable.

$\therefore \lim_n Y_n = \overline{\lim} X_k$ is $\sigma(X_j, j \geq n)$-measurable. $\forall n \in \mathbb{Z}^+$.

so it's $\bigcap \sigma(X_j, j \geq n)$-measurable. tail Func.

Cor. $\{X_n\}$ indept. r.v.'s. $p(\lim X_n$ exists$) = 0$ or 1.

Pf: $p(\overline{\lim} X_n = c_1 = c_2 = \underline{\lim} X_n) = 0$ or 1.

③ Hewitt - Savage 0-1 Law:

<u>Def</u>: i) $\tau : \mathbb{Z}^{\geq 0} \longrightarrow \mathbb{Z}^{\geq 0}$ is finite permutation

if $\tau(k) \neq k$ only finite terms

ii) For event A. generated by r.v.'s $(X_n)_{n \in \mathbb{Z}_{\geq 0}} \in S^{\mathbb{Z}_{\geq 0}}$

Denote : $X_n(w) = w_n$. define $\tau(w) = (w_{\tau(n)})$

We say A is permutable if $\tau^{-1}(A)$

$= \{ w \mid \tau w \in A \} = A$. for $\forall$ finite permutation τ

<u>Rmk</u>: i) Collection of such event in ii) is a

σ-algebra. denote by $\mathcal{E}(X) \triangleq \mathcal{E}$. called

exchangable σ-algebra. $X = (X_n)$.

ii) $\mathcal{T}(X) \subseteq \mathcal{E}(X)$ follows from finite permutation

doesn't work on $\mathcal{T}(X)$. $X = (X_n)$

<u>e.g.</u> Consider $S = \mathbb{R}^1$. $X_n(w)$ take value on S.

$S_n(w) = \sum_1^n X_k(w)$. Consider $\mathcal{E} = \mathcal{E}(X)$.

Then: $\{ w \mid S_n(w) \in B \text{ i.o} \} \in \mathcal{E}$. but not $\mathcal{T}$.

So $\mathcal{T} \subsetneqq \mathcal{E}$.

<u>Thm.</u> (Hewitt - Savage 0-1 Law)

If $X_1, X_2 \cdots$ i.i.d. $A \in \mathcal{E}(X)$. Then: $P(A) \in \{0,1\}$.