Random Variables

Def: i) A r.v. X is measurable: $(\Omega, \mathcal{A}) \longrightarrow (\mathbb{R}^1, \mathcal{B}_{\mathbb{R}^1})$.

i.e. $\forall B \in \mathcal{B}_{\mathbb{R}^1}$, $X^{-1}(B) \in \mathcal{A}$.

ii) random vector $\vec{X} = (X_1(\omega), \cdots X_n(\omega))$, s.t.

X_k is r.v. $\forall 1 \le k \le n$.

Remark: Sometimes the def requires more:

$P(|X| = \infty) = 0$.

(1) σ-algebra generated by r.v.:

Def: For $\{X_\lambda : \lambda \in \Lambda\}$, family of r.v.'s on $(\Omega, \mathcal{A})$

$\sigma(X_\lambda, \lambda \in \Lambda) = \sigma(\bigcup_{\lambda \in \Lambda} X_\lambda^{-1}(\mathcal{B}_{\mathbb{R}^1}))$ is the σ-algebra generated by $\{X_\lambda : \lambda \in \Lambda\}$.

① Discrete r.v.'s:

Prop. For $(A_k)_1^n \subset \mathcal{A}$, disjoint elements, $\sum A_i = \Omega$. Then:

$\sigma(\{A_k\}_1^n) = \sigma(\{A_k\}_0^n) = \{\bigcup_{i \in I} A_i \mid I \subset \{0, 1, 2 \cdots n\}\}$.

where $A_0 = \emptyset$. $\therefore \# \sigma(\{A_k\}_1^n) = 2^n$.

Remark: Sometimes $A_k := \{\omega \mid X(\omega) = x_k\}$.

For general case:

If $\{A_k\}_1^n$ are not exclusive for all pairs.

We use a technique: Disjointization.

Note that: $\bigcap_1^n \tilde{A}_k$. $\tilde{A}_k = A_k$ or A_k^c.

It forms a disjoint family. Besides. $\sum \bigcap_1^n \tilde{A}_k = \Omega$.

e.g. $n=2$. $\Omega = A\cap B + A\cap B^c + A^c\cap B + A^c \cap B^c$.

$\therefore \# \{\bigcap_1^n \tilde{A}_k\} = 2^n$. $\# \sigma(\{A_k\}_1^n) = 2^{2^n}$.

③ Arti. r.v:

Thm. $\{X_k\}_1^n$ are r.v's on $(\Omega, \mathcal{A})$. Then Y on Ω is $\sigma(\{X_k\}_1^n)$-measurable. $\Leftrightarrow$ $Y = f(X_1, \cdots X_n)$.

$f: \mathbb{R}^n \longrightarrow \mathbb{R}^1$. Borel-measurable.

(2) Distribution:

Thm. r.v. X on $(\Omega, \mathcal{A}, P)$ induces another prob. space $(\mathbb{R}^1, \mathcal{B}_{\mathbb{R}^1}, P_X)$. s.t. $P_X(B) = P(X\in B)$. for $\forall B \in \mathcal{B}_{\mathbb{R}^1}$.

Pf: check P_X is measure on $\mathcal{B}_{\mathbb{R}^1}$

Def: d.f of X is $F_X(x) = P_X(-\infty, x]$.

Thm. X is discrete r.v $\Leftrightarrow$ F_X is discrete.

Pf. Note: $P_X\{a\} = F_X(a) - F_X(a-)$.

Def: For random vector $\vec{X} = (X_1, \cdots X_n)$.

i) Distribution: $P_X(B) = P(\vec{X} \in B)$. $\forall B \in \mathcal{B}_{\mathbb{R}^n}$.

ii) d.f of $\vec{X}$: $F_X(\vec{x}) = P(X_1 \le x_1, \cdots X_n \le x_n)$.

Thm. $\vec{X}$ is discrete $\iff$ X_k is discrete. $\forall 1 \le k \le n$.

Pf: **Lemma.** For $1 \le N \le \infty$. If $P(A_k) = 1$. $\forall 1 \le k \le N$.

Then $P(\bigcap_1^N A_k) = 1$.

Pf: $P((\bigcap_1^N A_k)^c) = P(\bigcup_1^N A_k^c) \le \sum_1^N P(A_k^c) = 0$.

For $C \subset \mathbb{R}^n$. $C_i = \{x_i \mid \vec{x} \in C\}$. $P(\vec{X} \in C) = 1$

$\iff P(\bigcap_1^n \{X_i \in C_i\}) \iff P(X_i \in C_i) = 1$. $\forall 1 \le i \le n$.

(3) **Quantile:**

Def: For d.f. F. Its quantile is: $F^{-1}(u) = \inf\{t \mid F(t) \ge u\}$.

Remark: F^{-1} jumps when F is flat. F^{-1} is flat when F jumps. Actually, $F^{-1}(u)$ is mirror image of $F(t)$ along $u = t$.

① **Properties:**

i) $F^{-1}(u)$ is non-decreasing, left-conti.

ii) $F^{-1}(F(x)) \le x$. $F(F^{-1}(u)) \ge u$. $\forall x \in \mathbb{R}^1$. $u \in (0,1)$.

iii) $F^{-1}(u) \le t \iff u \le F(t)$.

iv) If F is conti. Then $F(F^{-1}(u)) = u$. $\forall u \in (0,1)$.

Pf: ii) $F^{-1}(F(x)) = \inf\{t \mid F(t) \geq F(x)\} \leq x$.

since $x \in \{t \mid F(t) \geq F(x)\}$.

Second. we claim: $\{t \mid F(t) \geq u\} = (a, \infty)$ or $[a, \infty)$.

since $F(r') \geq F(r) \geq u$. for $r' > r \in \{\cdots\}$.

Denote $\inf\{t \mid F(t) \geq u\} = a$. $\therefore a + \frac{1}{n} \in \{\cdots\}$.

By right-conti. $\lim_n F(a+\frac{1}{n}) = F(a) \geq u$.

i.e. $F(F^{-1}(u)) \geq u$.

Remark: By the proof: $\{t \mid F(t) \geq u\}$ has form: $[a, \infty)$.

③ Transformation:

Thm. F is d.f. For $U \sim$ Uniform $(0,1)$. Then we have:

$F^{-1}(U) \sim F$. (Note: F^{-1} is Borel-measurable.)

Cor. $X \sim F$. Then $E(X) = \int_0^1 F^{-1}(u)\,du$.

(since $F^{-1}(U) \sim F \sim X$)

Thm. r.v. X has conti. d.f F. Then $F(X) \sim U(0,1)$.

Pf: Claim: $F(X)$ is conti. r.v.

show $P(F(X) = t) = 0$. $\forall t \in (0,1)$.

$\{F = t\} = \{F \geq t\} \cap \{F \leq t\}. = [a, \infty) \cap (-\infty, b]$

$= [a, b]$. $F(a) = F(b) = t$.

Limit operation:

· Recall Fatou's Thm:

i) $X_n \geq Y$. a.s. $E(|Y|) < \infty$. Then $\underline{\lim}\, E(X_n) \geq E(\underline{\lim}\, X_n)$.

ii) $X_n \leq Y$. a.s. $E(|Y|) < \infty$. Then $\overline{\lim}\, E(X_n) \leq E(\overline{\lim}\, X_n)$.

Cor. $\mu(\underline{\lim}\, A_n) \geq \underline{\lim}\, \mu(A_n)$. $\mu(\overline{\lim}\, A_n) \leq \overline{\lim}\, \mu(A_n)$.

③ For general r.v.'s:

· Note: $X = X^+ - X^-$. $E(X) = E(X^+) - E(X^-)$.

prop. $E_\Omega(X) = E_A(X)$. for $P(A) = 1$.

pf: $|E_{A^c}(X)| \leq \max|X|\, P(A^c) \leq \infty \cdot 0 = 0$.

(2) Integration:

① Def: For nondecreasing, right-conti func on $\mathbb{R}^1$. f.

There exists unique measure μ: $\mu(a,b] = f(b) - f(a)$.

Define: $\int g\, df = \int g\, \mu(dx)$. L-S integral associated with f.

Remark: i) $\int_{(a,b]} f\, d\mu \neq \int_{(a,b)} f\, d\mu$. since μ may not be conti. at $x = b$.

ii) R-S integral require: f. g can't be disconti. at same point. But L-S integral needn't it.

② Some cases:

For: $\int_B f\,dG$. $B \in \mathcal{B}$: (L-S integral)

i) G is right-conti BV:

Note: $G = G_1 - G_2$. nondecreasing Func's difference.

$\therefore \int_{(a,b]} f\,dG \overset{\Delta}{=} \int_{(a,b]} f\,dG_1 - \int_{(a,b]} f\,dG_2$.

ii) G is discrete:

Suppose $\{X_k\}_1^\infty$ is its jumps. $\Delta G(X_n) = G(X_n) - G(X_n-)$.

Then: $\int_{(s,t]} f\,dG = \sum_{s<X_k\le t} f(X_k)\,\Delta G(X_k)$

iii) G is absolutely conti:

$\exists g$. $g = G'$ a.e. $m(s,t] = \int_{(s,t]} dG = \int_{(s,t]} g\,dx$.

Then: $\int_B f\,dG = \int_B fg\,dx$.

iv) G is mixture of ii). iii). right-conti:

Suppose $G(t) = G(a) + \int_a^t g(x)\,dx + \sum_{X_n\le t} \Delta G(X_n)$.

Then: $\int_{(s,t]} f\,dG = \int_{(s,t]} fg\,dx + \sum_{s<X_n\le t} f(X_n)\,\Delta G(X_n)$

③ Integration by part: