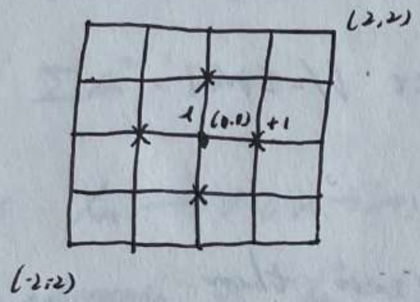
Ising Model.

(1) <u>Configuration</u>:

i) $\Lambda_n = \{-N, \cdots, 0, \cdots N\}^d$. d is dimension.

N is the length for the configuration

$N \geq 1$. $d \geq 1$. e.g. $N=2$. $d=2$.



ii) $\Omega_N = \{-1, 1\}^{\Lambda_n}$. the state space.

$\sigma \in \Omega_N$, i.e. $\sigma = \{\sigma_x : x \in \Lambda_n\}$.

where $\sigma_x = -1$ or 1 ($\underline{+1} : \{\sigma_x = 1 : \forall x \in \Lambda_n\}$)

iii) Hamiltonian: $H_N : \Omega_N \longrightarrow \mathbb{R}$
$\sigma \longmapsto H_N(\sigma)$

$$H_N(\sigma) = -\frac{1}{2} \sum_{x \in \Lambda_n} \sum_{\substack{y \sim x \\ y \in \Lambda_n}} \sigma_x \sigma_y . \quad y \sim x \Leftrightarrow \|x - y\| = 1.$$

<u>Remark</u>: "$\frac{1}{2}$" is because $y \sim x \Leftrightarrow x \sim y$.

then $\sigma_x \sigma_y$ will be counted twice.

"-1" is for convenience to find the ground states σ^*. st. $H_N(\sigma^*) = \min\{H_N(\sigma) | \sigma \in \Omega_N\}$. e.g.

$$H_N^h(\sigma) = H_N(\sigma) - \sum_{x \in \Lambda_n} h \sigma_x \quad \begin{cases} h=0. \text{ G.S } \sigma^* = \underline{+1} \text{ or } \underline{-1} \\ h>0. \text{ G.S } \sigma^* = \underline{+1} \\ h<0. \text{ G.S } \sigma^* = \underline{-1} \end{cases}$$

for $\eta \in \{-1,1\}^{\mathbb{Z}^d}$. Def: $\partial\Lambda_n = \{x \in \Lambda_n \mid \exists y \in \mathbb{Z}^d/\Lambda_n, \|x-y\|=1\}$

Def: $$H_N^{\eta}(\sigma) = H_N(\sigma) - \sum_{x\in\partial\Lambda_n} \sum_{\substack{y\in\mathbb{Z}^d/\Lambda_n \\ \|y-x\|=1}} \sigma_x \eta_y$$

for $\eta = \underline{+1}$, $H_N^{\eta}(\sigma) \triangleq H_N^{+}(\sigma)$, $\eta = \underline{-1}$, $H_N^{\eta}(\sigma) \triangleq H_N^{-}(\sigma)$

iv) Gibbs measure: $\beta > 0$, the inverse temperature.

$$\mu_N^{\beta,\eta,h}(\sigma) = \frac{e^{-\beta H_N^{\eta,h}(\sigma)}}{Z_N^{\eta,h}(\beta)}, \quad Z_N^{\eta,h}(\beta) = \sum_{\sigma\in\Lambda_N} e^{-\beta H_N^{\eta,h}(\sigma)}$$

Property: $$\mu_N^{\beta}(\sigma) \longrightarrow \begin{cases} 0, & \sigma \notin G.S \\ \frac{1}{|G.S|}, & \sigma \in G.S. \end{cases} \quad (\beta\to\infty)$$

$\mu_N^{0}(\cdot)$ is uniform dist.

v) Pressure: $$\psi_N^{\eta,h}(\beta) = \frac{1}{|\Lambda_N|} \log Z_N^{\eta,h}(\beta)$$

Property: $\psi_N(\beta)$ is convex

$$\forall \eta, \lim_{\Lambda_n \uparrow \mathbb{Z}^d} \psi_N^{\eta}(\beta) = \psi(\beta), \text{ convex.}$$

Pf: $$\frac{d}{d\beta}\psi_N = \frac{1}{|\Lambda_N|}\sum_{\sigma\in\Lambda_N} H_N(\sigma)\mu_N^{\beta}(\sigma) \triangleq \frac{\langle H_N\rangle_{\beta,N}}{|\Lambda_N|}$$

$$\frac{d^2}{d\beta^2}\psi_N = \frac{\langle H_N^2\rangle_{\beta,N} - \langle H_N\rangle_{\beta,N}^2}{|\Lambda_N|} \geq 0.$$

Def: Exists first order phase transition

if $\psi(\beta)$ isn't differentiable at some $\beta \in (0,+\infty)$

(2) <u>In dimension one</u>:

<u>Thm.</u> $\mu^{\eta}_{\beta,N}(\sigma_0=-1) \longrightarrow \frac{1}{2} \ (N \to +\infty)$ for

$\forall \beta > 0, \ \forall \eta \in \{-1,1\}^{\mathbb{Z}}$

<u>remark</u>: The boundary action will not influence the spin at origin.

<u>Pf</u>: <u>Lemma.</u> $Z_N^{\eta}(\beta) = \sum_{\sigma \in \Lambda_N} e^{-\beta H_N^{\eta}(\sigma)}$ can be

presented in: $Z_N^{\eta}(\beta) = (e^{\beta}+e^{-\beta})^{2N+2} \alpha_N$

where $\alpha_N \to \frac{1}{2} \ (N \to +\infty)$

<u>Pf</u>: $H_N^{\eta}(\sigma) = -\frac{1}{2}\sum\sum \sigma_x \sigma_y - \sigma_{-N}\eta_{-(N+1)} - \sigma_N \eta_{N+1}$

$$= -\sum_{z=-N}^{N-1} \sigma_z \sigma_{z+1} - \sigma_{-N}\eta_{-(N+1)} - \sigma_N \eta_{N+1}$$

$$Z_N^{\eta}(\beta) = \sum_{\sigma \in \Lambda_N} e^{\beta \eta_{-(N+1)}\sigma_{-N}} \prod_{-N}^{N-1} e^{\beta \sigma_z \sigma_{z+1}} e^{\beta \sigma_N \eta_{N+1}}$$

Construct a DTMC:

$$S = \{-1,1\}. \quad P = \begin{pmatrix} e^{\beta} & e^{-\beta} \\ e^{-\beta} & e^{\beta} \end{pmatrix} \frac{1}{e^{\beta}+e^{-\beta}}$$

It's aperiodic, irreducible with stationary state $\pi(1) = \pi(-1) = \frac{1}{2}$

Then by BLT. it's unique.

Then $p(\sigma_z, \sigma_{z+1}) = \dfrac{e^{\beta \sigma_z \sigma_{z+1}}}{e^{\beta}+e^{-\beta}}$

$$\therefore Z_N^{\eta}(\beta) = (e^{\beta}+e^{-\beta})^{2N+2} \sum_{\sigma \in \Lambda_N} p(\eta_{-(N+1)}, \sigma_{-N}) \cdot \prod_{-N}^{N-1} p(\sigma_z, \sigma_{z+1}) \cdot p(\sigma_N, \eta_{N+1})$$

$= (e^{\beta}+e^{-\beta})^{2N+2} P_{2N+2}(\eta_{-(N+1)}, \eta_{N+1})$. we're done.

since $P_{2N+2}(\eta_{-(N+1)}, \eta_{N+1}) \overset{\Delta}{=} \alpha_N \to \pi(\eta_{N+1}) = \frac{1}{2}$

$$\Rightarrow \mu^{\eta}_{\beta,N}(\sigma_0 = -1) = \frac{1}{Z^{\eta}_N(\beta)} \sum_{\substack{\sigma_0=-1 \\ \sigma\in\Lambda_N}} e^{-\beta H^{\eta}_N(\sigma)}$$

$$= \frac{1}{\alpha_N} \sum_{\sigma_{-N},\cdots\sigma_{-1}} P(\eta_{-(N+1)}, \sigma_{-N}) \prod_{-N}^{-2} P(\sigma_z, \sigma_{z+1})$$

$$\cdot P(\sigma_{-1}, -1) \sum_{\sigma_1,\cdots\sigma_N} P(-1, \sigma_1) \prod_{1}^{N-1} P(\sigma_z, \sigma_{z+1}) P(\sigma_N, \eta_{N+1})$$

$$= \frac{1}{\alpha_N} P^{N+1}(\eta_{-(N+1)}, -1)\, P^{N+1}(-1, \eta_{N+1}) \to \frac{1}{\frac{1}{2}}\left(\frac{1}{2}\right)^2 = \frac{1}{2}$$

(3) <u>In dimension two:</u>

① <u>At low temperature:</u>

<u>Thm.</u> $\exists \beta_0$, s.t. $\forall \beta > \beta_0$, $\lim_{N\to\infty} \mu^{+}_{N,\beta}(\sigma_0 = -1) < \frac{1}{2}$

<u>Rmk:</u> There is a phase transition at low temperature. Since the boundary action will affect the prob of original spin.

<u>Pf:</u> Denote: $\mathcal{E}^2 = \{\{x,y\} \subseteq \mathbb{Z}^2 : |x-y| = 1\}$

$\mathcal{E}^2_N = \{\{x,y\} \in \mathcal{E}^2 : \{x,y\} \cap \Lambda_N \neq \emptyset\}$

$\sigma_z = +1$, $z \notin \Lambda_N$

$$\therefore H^{+}_N(\sigma) = -\sum_{\{x,y\}\in\mathcal{E}^2_N} \sigma_x \sigma_y$$

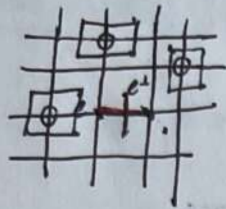
$$= \sum_{\{x,y\}\in\mathcal{E}^2_N} (1 - \sigma_x\sigma_y) - |\mathcal{E}^2_N|$$

$$\overset{\Delta}{=} \hat{H}^{+}_N(\sigma) - \underbrace{|\mathcal{E}^2_N|}$$ will cancel in $\mu^{+}_{N,\beta}$

$\therefore \mu^+_{N,\beta}(\sigma) = \dfrac{e^{-\beta \hat{H}^+_N(\sigma)}}{\sum_{\sigma\in\Omega_N} e^{-\beta \hat{H}^+_N(\sigma)}}$. $\mathbb{Z}^2_* = \mathbb{Z}^2 + (\frac{1}{2},\frac{1}{2})$

$\mathcal{E}^2_*$ is the set of edges in $\mathbb{Z}^2_*$.

o: minus spin
others are +1



$\mathcal{E}^2 \longrightarrow \mathcal{E}^2_*$. one-to-one map.

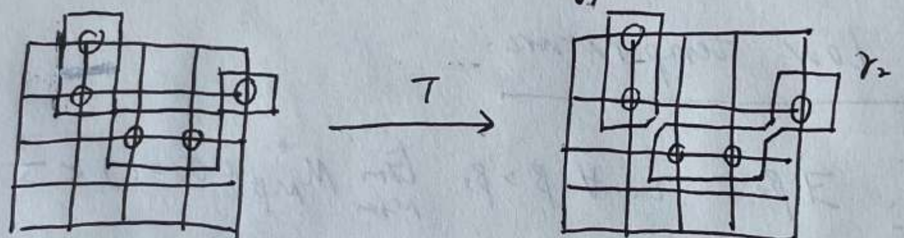
$e \longmapsto e\perp$

$\Omega_N \longrightarrow \mathcal{E}^2_*$

$\sigma \longmapsto B(\sigma)$. $e_\perp \in B(\sigma) \iff \sigma_x \neq \sigma_y$ for $e=\{x,y\}$

Actually $B(\sigma)$ is the closed line encircle the different spins. ("-1" in the figure)

$Q = (-\frac{1}{2},\frac{1}{2})^2$. $Q_x = x+Q$. $M(\sigma) = \bigcup_{\substack{x\in\Lambda_N \\ \sigma_x=-1}} Q_x$



by "  in $\mathbb{Z}^2_*$.

to give the connected area.

Denote $I(\sigma) = \{\gamma_1, \gamma_2, \cdots \gamma_p\}$. γ_i is the closed contour encircle "o". $B(\sigma) = \bigcup_i \gamma_i$

$\Rightarrow$ Note that $1-\sigma_x\sigma_y = \begin{cases} 0, & \{x,y\}_\perp \notin B(\sigma) \\ 2, & \{x,y\}_\perp \in B(\sigma) \end{cases}$

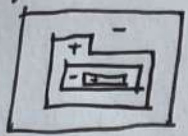
$$\therefore \hat{H}^+_N(\sigma) = \sum_{\{x,y\}\in \mathcal{E}^+_N} 2 I_{\{x,y\}_\perp \in B(\sigma)} = 2|B(\sigma)|$$

$$= 2\sum_{\gamma_j\in I(\sigma)} |\gamma_j|$$, $|\gamma_j|$ is the length of contour path

$|\gamma_1| = 2+2+1+1 = 6$, $|\gamma_2| = 10$ in figure.

observe that the contour seperate different spin:

e.g.



. Actually, the spins contained by γ_i form a connected component:

$\{\sigma_x : \exists y.\ |x-y|=1.\ \sigma_x=\sigma_y\}$

Note that $\exists\ \gamma \in I(\sigma)$. s.t. int γ contains σ_0. if $\sigma_0=-1$

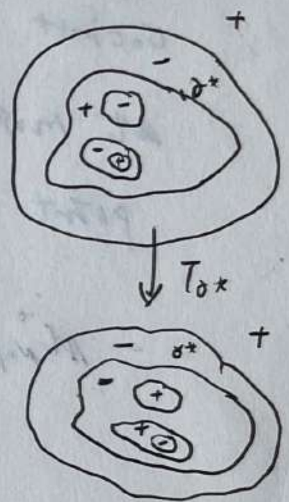
since exists "+" spins!

Choose $\gamma^* \subseteq \mathbb{Z}^2_*$. a special closed curve

$N_N(\gamma^*) = \{\sigma \in N_N : \gamma^* \in I(\sigma)\}$.

A transform: $T_{\gamma^*} : N_N(\gamma^*) \longrightarrow I_N(\gamma^*) \subseteq \Lambda_N$

$\sigma \longmapsto T(\sigma)$



where $T(\sigma)$ is the configuration reverse the spins included in γ^*. e.g.

$\because T^2_{\gamma^*} = Id$ $\therefore T_{\gamma^*}$ is bijection.

Obeserve: $I(T_{\gamma^*}(\sigma)) = I(\sigma)/\gamma^*$

Lemma. For $\gamma^* \subseteq B(\sigma)$. $\mu^+_{N,\beta}(\{\sigma : \gamma^* \in I(\sigma)\}) \le e^{-2\beta|\gamma^*|}$

It says if contour $|\gamma^*|$ is long $\longrightarrow$ prob. of $\{\cdots\}$ will be small.

Pf: $\mu^+_{N,\beta}(N_N(\gamma^*)) = e^{-2\beta|\gamma^*|} \sum_{\sigma \in N_N(\gamma^*)} \mu^+_{N,\beta}(T_{\gamma^*}(\sigma))$

by expressing $\hat{H}^+_{N,\beta}(\sigma)$ in $\{\gamma_i\}$!

$\because \sum_{\sigma \in N_N(\gamma^*)} \mu^+_{N,\beta}(T_{\gamma^*}(\sigma)) = \mu^+_{N,\beta}(I_N(\gamma^*)) \le 1$

Let γ^* be the closed curve includes $\sigma_0 = -1$!

$$\mu^+_{N,\beta}(\sigma_0=-1) \le \mu^+_{N,\beta}\Big\{\bigcup_{\substack{\gamma^*: \\ \sigma_0 \in \mathrm{int}\,\gamma^*}} \{\sigma : \gamma^* \in \Gamma(\sigma)\}\Big\}$$

$$\le \sum_{\substack{\gamma^*: \\ \sigma_0 \in \mathrm{int}\,\gamma^*}} \mu^+_{N,\beta}(\Omega_N(\gamma^*)) = \sum_{k\ge 4} \sum_{\substack{|\gamma^*|=k \\ \sigma_0\in \mathrm{int}\,\gamma^*}} e^{-2\beta k}$$

Since a closed curve is with length at least 4 units. e.g. $\boxed{+}$

$$\sum_{\substack{|\gamma^*|=k \\ \sigma_0\in \mathrm{int}\,\gamma^*}} e^{-2\beta k} \le e^{-2\beta k}\cdot \frac{k\cdot 3^{k-1}}{2},$$ since the contour at most includes $[\frac{k}{2}]$ points. At each point we have 3 choice for direction

$$\therefore \mu^+_{N,\beta}(\sigma_0=-1) \le \sum_{k\ge 4} \frac{k\cdot 3^{k-1}}{2} e^{-2\beta k} < \frac{1}{2} \quad \square$$

② <u>At high temperature:</u>

<u>Thm.</u> $\exists \beta_1,\ \forall \beta<\beta_1,\ \lim\limits_{N\to\infty} \langle\sigma_0\rangle^+_{N,\beta} = 0.$

where $\langle\sigma_0\rangle^+_{N,\beta}$ is the expectation of value at origin in each configuration

i.e. $$\langle\sigma_0\rangle^+_{N,\beta} = \sum_{\sigma\in\Omega_N} \sigma_0 \frac{e^{-\beta H^+_N(\sigma)}}{Z^+_N(\beta)}$$

<u>Pf:</u> Step. 1. $\langle\sigma_0\rangle^+_{N,\beta} \ge 0$

Step. 2 $\overline{\lim}\, \langle\sigma_0\rangle^+_{N,\beta} \le 0$

It's more complicated to argue that the case ①.