Infinitely Divisable

(1) Definition:

Consider for seq $\{a_n\}$ const. $S_n - a_n \xrightarrow{d} F$.

where $S = \sum_i^{k_n} X_{ni}$. $\{X_{ni}\}_i$ indept. Then F is d.f

of r.v. added by infinite number of small r.v.'s.

Def: ch.f φ is infinitely divisable if $\forall n \in Z^+$.

$\exists$ ch.f φ_n. st. $\varphi = \varphi_n^n$.

Rmk: i) It mean $X \sim$ ch.f: φ. Then $\exists \{X_{ni}\}_{i=1}^n$

i.i.d. $\sim$ ch.f φ_n. st. $X = \sum_i^n X_{ni}$.

ii) Written in d.f: $F = F_n^{*n}$. $\forall n$. $\exists$ d.f. F_n.

Thm. φ is infinitely divisable $\Leftrightarrow \exists (n_k) \to \infty$. and

seq of ch.f $\{\varphi_k\}$. st. $\varphi = \varphi_k^{n_k}$.

Pf: ($\Leftarrow$) $\forall n_k \in Z^+$. $\exists (m_k) \subset Z^+$. st. $\frac{m_k}{n_k} \to \frac{1}{n}$.

We want to make approxi.:

$\varphi_{n_k}^{m_k} \to \varphi^{\frac{1}{n}}$. Apply Ascoli on $\{\varphi_{n_k}^{m_k}\}$.

Since $\varphi \neq 0$. suppose $\varphi = r(x) e^{i\theta(x)}$.

$\varphi_{n_k} = r^{\frac{1}{n_k}} e^{\frac{i\theta}{n_k}}$. check $\{\varphi_{n_k}\}$ is equiconti.

$\therefore \exists \tilde{\varphi}_{n_k}^{m_k} \xrightarrow{u.c.c} \varphi^{\frac{1}{n}}$. conti at 0 $\therefore$ ch.f.

(2) Properties:

i) φ is i.d $\Rightarrow$ $\varphi \neq 0$. $\forall t$.

Pf: By conti. for $\varepsilon > 0$. $\exists\, \delta > 0$. st.

$\forall\, |t| \leq \delta$. $|\varphi| > 1-\varepsilon$.

$\therefore$ $|\varphi_n| > (1-\varepsilon)^{\frac{1}{n}} \to 1$. $(n \to \infty)$

$\therefore \exists\, N_0$. st. $|\varphi_{N_0}| \geq 1 - \frac{\varepsilon}{8}$.

By $(1-|\varphi_{N_0}(2t)|) \leq 8(1-|\varphi_{N_0}(t)|)$

$\therefore$ $|\varphi| \neq 0$ on $|t| \leq 2\delta$.

Repeat to choose large N. then for $|t| \leq \delta_N$

Let $N \to \infty$. $\therefore \varphi \neq 0$. $\forall t \in \mathbb{R}'$ $(\delta_N \to \infty)$

Thm. For i.d. ch.f. $\exists ! \lambda(t)$. $\lambda(0) = 0$. st.

$\varphi = e^{\lambda(t)}$.

Cor. If $\varphi = \varphi_n^n$. Then $\varphi_n = \varphi^{\frac{1}{n}} \to 1$.

ii) If φ is i.d. ch.f. Then so is $|\varphi|$.

Pf: $\forall n \in \mathbb{Z}^+$. $\varphi = \varphi_{2n}^{2n}$ $\therefore |\varphi| = (|\varphi_{2n}|^2)^n$

where $|\varphi_{2n}|^2$ is ch.f.

iii) $\{\varphi^{(m)}(t)\}$ seq of i.d. ch.f $\to$ some ch.f $\varphi(t)$.

Then φ is i.d. ch.f as well.

Pf: $\forall n \in Z^+$. $\lim_m \varphi^{(m)} = \lim_m (\varphi_n^{(m)})^n = (\lim_m \varphi_n^{(m)})^n$

$= \varphi$. $\therefore \varphi^{\frac{1}{n}} = \lim_m \varphi_n^{(m)}$.

since $\varphi^{\frac{1}{n}}$ conti at 0 $\Rightarrow$ $\lim_m \varphi_n^{(m)}$ is ch.f.

Cor. Under the condition:

$\lambda_k \to \lambda$. where $\varphi^{(k)} = e^{\lambda_k(t)}$. $\varphi = e^{\lambda(t)}$.

(3). Representation:

① Denote: $B(t; a, u) = e^{a(e^{iut}-1)}$. a is a parameter.

i.e. ch.f of uX. $X \sim$ Poisson (a).

Rmk: $\prod_1^k B(t; a_i, u_i)$ is i.d. ch.f. since $B(t; a, u)$ is.

Thm. The class of i.d. ch.f's is the closure of Poisson ch.f's w.r.t. vague converge.

Pf: $\forall f(t)$ is i.d. suppose $f(t) = e^{\lambda(t)}$. $f = f_n^n$

Note that $n(f_n - 1) = n(e^{\frac{\lambda(t)}{n}} - 1) \longrightarrow \lambda(t)$.

$\therefore e^{n(f_n-1)} \longrightarrow e^{\lambda(t)}$. Approxi. $n(f_n - 1) = \int (e^{itu} - 1) n\, dF_n(u)$.

by Riemann Sum: $\sum_1^{k_n} (e^{itu_{nk}} - 1) a_{nk}$

$\therefore \sup_{|t| \le n} |e^{n(f_n-1)} - \prod_1^{k_n} B(t, a_{nk}, u_{nk})| = O(\frac{1}{n})$.

② Levy's Thm:

$\forall$ i.d. ch.f $\varphi(t) = \exp(iat + \int_{\mathbb{R}} (e^{itu} - 1 - \frac{itu}{1+u^2}) \frac{1+u^2}{u^2} dG)$

where G is bounded increasing. a is some const $\in \mathbb{R}^1$

Cor. φ is i.d. ch.f. So is φ^λ for some $\lambda \ge 0$.