Preliminaries

(1) Set Theory:

① Limit of Sets:

Def: i) Infinitely often (i.o) : For $(A_n) \subset P(\Omega)$

$$\overline{\lim_n} A_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n = \{w \mid \forall k \geq 1.\ \exists n \geq k.\ w \in A_n\}$$

$$= \{w \mid w \in A_n \text{ for infinite many } n\}.$$

$$\triangleq \{A_n\ \text{i.o}\}.$$

ii) Ultimately (ult.) : For $(A_n) \subset P(\Omega)$

$$\underline{\lim_n} A_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n = \{w \mid \exists k \geq 1.\ \forall n \geq k.\ w \in A_n\}.$$

$$= \{w \mid w \in A_n \text{ for all but finite } n\}.$$

$$= \{A_n\ \text{ult}\}.$$

iii) $A_n \longrightarrow A$. i.e. $A = \lim_n A_n$. if:

$$\overline{\lim} A_n = \underline{\lim} A_n = A.$$

Remark: $\underline{\lim} A_n \subset \overline{\lim} A_n$. It may be strict:

e.g. $A_{2n} = B$. $A_{2n+1} = C$.

Thm. i) $A_1 \subset A_2 \subset \cdots \subset A_n \cdots$ Then $A_n \uparrow A = \bigcup A_n$.

ii) $A_1 \supset A_2 \cdots \supset A_n \cdots$ Then $A_n \downarrow A = \bigcap A_n$

Pf: For i): $\underline{\lim} A_n = \cup \cap A_n = \cup A_k = A$.

$\overline{\lim} A_n = \cap \cup A_n = \cap A = A$.

cor. $\overline{\lim_n} A_n = \lim_k \bigcup_{n\ge k} A_n$. $\underline{\lim_n} A_n = \lim_k \bigcap_{n\ge k} A_n$.

② Indicators:

prop. i) $I_{A\cap B} = \min\{I_A, I_B\} = I_A I_B$

ii) $I_{A\cup B} = \max\{I_A, I_B\} = I_A + I_B - I_{A\cap B} \le I_A + I_B$

iii) $I_{A\Delta B} = |I_A - I_B|$

iv) $I_{\overline{\lim} A_n} = \overline{\lim} I_{A_n}$. $I_{\underline{\lim} A_n} = \underline{\lim} I_{A_n}$

Pf: For iii): consider $I_A = 0, 1$. $I_B = 0, 1$. four cases

For i). ii). iv). consider LHS $= 0, 1$.

Convert to discuss the relation of set.

Cor. $(A \Delta B) \Delta C = A \Delta (B \Delta C)$.

pf: By iii). consider $I_B = 0, 1$. case.

Thm. $I_{\bigcup_1^n A_k} = \sum_1^n I_{A_k} - \sum_{1\le k_1<k_2\le n} I_{A_{k_1}\cap A_{k_2}} + \sum_{1\le k_1<k_2<k_3\le n} I_{A_{k_1}\cap A_{k_2}\cap A_{k_3}}$

$\cdots + (-1)^{n-1} I_{A_1\cap A_2 \cdots \cap A_n}$

Pf: If $I_{\bigcup_1^n A_k}(w) = 0$. $\therefore w \notin \bigcup_1^n A_k$. Obvious.

If $I_{\bigcup_1^n A_k}(w) = 1$. suppose $w \in A_{k_i}$ $1 \le i \le m$.

$$\therefore RHS = \binom{m}{1} - \binom{m}{2} + \binom{m}{3} + \cdots + (-1)^{m-1}\binom{m}{m}$$

$$= \binom{m}{0} - (1-1)^m = 1.$$

Cor. By taking expectation:

$$P(\bigcup_1^n A_k) = \sum_1^n P(A_k) - \sum P(A_{k_1} \cap A_{k_2}) + \cdots + (-1)^{n-1} P(\bigcap_1^n A_k)$$

Remark: By the proof: (the last equation)

$$\begin{aligned} P(\bigcup A_k) &\le \sum_1^n P(A_k) \\ &\ge \sum_1^n P(A_k) - \sum P(A_{k_1} \cap A_{k_2}) \\ &\le \sum_1^n P(A_k) - \sum P(A_{k_1} \cap A_{k_2}) + \sum P(A_{k_1} \cap A_{k_2} \cap A_{k_3}) \\ &\vdots \end{aligned}$$

③ Algebra and Class:

Ω is a space.

i) Algebra:

Def. (a) S is a semi-algebra. $S \subset \Omega$. if

$$\begin{cases} A, B \in S \Rightarrow A \cap B \in S \\ A \in S \Rightarrow A^c = \sum_1^n A_i.\ A_i \in S. \end{cases}$$

(b) $\mathcal{A}$ is an algebra. $\mathcal{A} \subset \Omega$. if

$$\begin{cases} A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A} \\ A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A} \end{cases}$$

(c) $\mathcal{A}$ is σ-algebra. $\mathcal{A} \in \Omega$. if

$$\begin{cases} A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}. \\ \{A_k\}_1^\infty \subset \mathcal{A} \Rightarrow \bigcup_1^\infty A_k \in \mathcal{A}. \end{cases}$$

Remark: (a). $\mathcal{A}$ is algebra $\Rightarrow \emptyset, \Omega \in \mathcal{A}$.

It's not true for semialgebra:

e.g. $S = \{\emptyset, A, A^c\}$.

(b) $\mathcal{A}$ is an algebra $\Leftrightarrow \begin{cases} A, B \in \mathcal{A} \Rightarrow A - B \in \mathcal{A} \\ \Omega \in \mathcal{A}. \end{cases}$

Relationship:

(a) Semialgebra $\supsetneq$ algebra.

e.g. $\Omega = (-\infty, +\infty]$. $S = \{(a, b] \mid -\infty \le a \le b \le \infty\}$.

(b) algebra $\supsetneq$ σ-algebra.

e.g. $\Omega = (-\infty, +\infty]$. $\bar{S} = \{\bigcup_1^m (a_i, b_i] \mid m \in \mathbb{Z}^+\}$.

prop. (generating from wider family)

(a) If S is semialgebra. Then $\bar{S} = \{\sum_1^n A_i \mid A_i \in S,$ $n \in \mathbb{Z}^+\}$. is algebra. Denote $\mathcal{A}(S)$.

(b) If $\mathcal{A}$ is algebra. Then $\bar{\mathcal{A}} = \{\sum_1^\infty A_n \mid A_n \in \mathcal{A}\}$ is σ-algebra.

ii) Classes:

Def: (a) $\mathcal{A}$ is monotone class if

$$\begin{cases} A_n \uparrow A. \ A_n \in \mathcal{A} \Rightarrow A \in \mathcal{A}. \\ A_n \downarrow A. \ A_n \in \mathcal{A} \Rightarrow A \in \mathcal{A}. \end{cases}$$

(b) $\mathcal{A}$ is π-class if $A, B \in \mathcal{A} \Rightarrow A \cap B \in \mathcal{A}$.

($\pi \approx \Pi$, i.e. "$\cap$").

(c) $\mathcal{A}$ is λ-class if $\begin{cases} \Omega \in \mathcal{A} \\ A, B \in \mathcal{A},\ A \supset B \Rightarrow A - B \in \mathcal{A}. \\ A_n \uparrow A,\ A_n \in \mathcal{A} \Rightarrow A \in \mathcal{A}. \end{cases}$

(λ = lam + da $\approx$ lim + diff)

<u>Remark</u>: $\mathcal{A}$ is λ-class $\Rightarrow$ $\mathcal{A}$ is m-class.

Since it's closed under complement.

<u>Thm</u>. For algebra $\mathcal{A}$, $\mathcal{A}$ is σ-algebra $\Leftrightarrow$ m-class.

<u>Thm</u>. $\mathcal{A}$ is σ-algebra $\Leftrightarrow$ $\mathcal{A}$ is π-class and λ-class

<u>Pf</u>: ($\Rightarrow$) trivial ($\Leftarrow$) By $\bigcup_1^n A_k = (\bigcap_1^n A_k^c)^c$

<u>Graphical illustration</u>:

$$\begin{tikzcd}
\sigma\text{-algebra} \arrow[d, Rightarrow] \arrow[drr, leftrightarrow] & \Longleftrightarrow & \lambda\text{-class} + \pi\text{-class} \arrow[d, Rightarrow] \\
\text{algebra} \arrow[d, Rightarrow] & & m\text{-class} \\
\text{semi-algebra} \arrow[d, Rightarrow] & & \\
\pi\text{-class} & &
\end{tikzcd}$$

"$\longleftrightarrow$": For algebra.

"$\Rightarrow$": For class.

iii) <u>Minimal class</u>:

<u>Lemma</u>. For $\{\mathcal{A}_\gamma \mid \gamma \in I\}$ collection of algebra / σ-algebra / λ / m / π-class. Then so $\bigcap_{\gamma \in I} \mathcal{A}_\gamma$ is.

Remark: It fails when replace by "$\bigcup$".

e.g. $\Omega = \mathbb{N}$. $\mathcal{A}_1 = \sigma(\{1\})$. $\mathcal{A}_n = \sigma(\mathcal{A}_{n-1}, \{n\})$

$\{2n\} \in \mathcal{A}_{2n}$. But $\bigcup_1^\infty \{2n\} \notin \bigcup_1^\infty \mathcal{A}_n$

It's true $\bigcup \mathcal{A}_n$ is algebra. ($\mathcal{A}_k \subset \mathcal{A}_{k+1}$)

Thm. For any class $\mathcal{A}$. There exists a unique minimal σ-algebra / algebra / m / λ / π-class contains $\mathcal{A}$.

Pf: Intersects those containing $\mathcal{A}$.

Thm. $\mathcal{S}$ is semi-algebra. $\bar{\mathcal{S}} = \mathcal{A}(\mathcal{S})$. Then $\sigma(\mathcal{S}) = \sigma(\bar{\mathcal{S}})$.

Pf. $\mathcal{S} \subseteq \sigma(\bar{\mathcal{S}})$. $\bar{\mathcal{S}} \subseteq \sigma(\mathcal{S})$.

e.g. Borel σ-algebra generates on different lines.

(a) $R_0 = (-\infty, +\infty)$. $\mathcal{S}_0 = \{R_0, (a,b], (-\infty, a], (b, +\infty)\}$

(b) $R_1 = (-\infty, +\infty]$. $\mathcal{S}_0 = \{(a,b]\}$

(c) $R_2 = [-\infty, +\infty]$. $\mathcal{S}_0 = \{(a,b], R_2, \{-\infty\}, \{+\infty\}\}$.

Then we obtain $\sigma(\mathcal{S}_0)$.

Remark: Introduction of $\pm\infty$ can simplify the structure of semi-algebra $\mathcal{S}_0$.

iv) Monotone Class Theorem:

Thm. If A is an algebra. Then $\sigma(A) = m(A)$.

So for B is m-class. $A \subset B \Rightarrow \sigma(A) \subset B$.

Pf: 1°) $A \subset \sigma(A)$. $\therefore m(A) \subset \sigma(A)$ as $\sigma(A)$ is m-class.

2°) Prove: $m(A)$ is σ-algebra. (for converse)

Define:
$$C_1 = \{A \mid A \in m(A).\ A \cap B \in m(A).\ \forall B \in A\}.$$
$$C_2 = \{B \mid B \in m(A).\ A \cap B \in m(A).\ \forall A \in A\}.$$
$$C_3 = \{A \mid A \in m(A).\ A^c \in m(A)\}.$$

Check: $C_1.\ C_2.\ C_3$ are m-class. $A \subset C_i.\ \forall 1 \le i \le 3$.

$\therefore m(A) \subset C_i. \Rightarrow$ we obtain: $m(A) = C_i.\ \forall 1 \le i \le 3$.

It suffices to show it's σ-algebra.

Thm. If A is a π-class. Then $\lambda(A) = \sigma(A)$.

So $\forall B.\ \lambda$-class. $A \subset B. \Rightarrow \sigma(A) \subset B$.

Pf: 1°) $\lambda(A) \subset \sigma(A)$ as $\sigma(A)$ is λ-class.

2°) prove: $\lambda(A)$ is π-class.

Define:
$$C_1 = \{A \mid A \in \lambda(A).\ A \cap B \in \lambda(A).\ \forall B \in A\}.$$
$$C_2 = \{B \mid B \in \lambda(A).\ A \cap B \in \lambda(A).\ \forall A \in \lambda(A)\}.$$

Check $C_1.\ C_2$ are λ-class. $A \subset C_1.\ C_2$.

$\therefore C_1.\ C_2 = \lambda(A)$.

Remark: procedure of using MCT:

$\mathcal{A}$ has property P $\Rightarrow$ show $\sigma(\mathcal{A})$ has P as well

(a) Define $\mathcal{B} = \{B \mid B \text{ has property } P\}$. $\therefore \mathcal{A} \subset \mathcal{B}$.

(b) Show:
$$\begin{cases} \mathcal{A} \text{ is } \pi\text{-class}. \ \mathcal{B} \text{ is } \lambda\text{-class}. & \text{or} \\ \mathcal{A} \text{ is algebra}. \ \mathcal{B} \text{ is } m\text{-class}. \end{cases}$$

(c) So $\sigma(\mathcal{A}) \subset \mathcal{B}$. $\sigma(\mathcal{A})$ has P.

e.g. Using Monotone Convergence Thm to prove Fubini Thm. ($\mathcal{B} = \{B$ satisfies F-T$\}$)

④ Product Space:

For measurable spaces $(\Omega_i, \mathcal{A}_i)$.

Def: i) Rectangles in $\prod_1^n \Omega_i$: $\prod_1^n A_i$, $A_i \in \mathcal{A}_i$, $\forall i$

ii) product σ-algebra: $\prod_1^n \mathcal{A}_i = \sigma(\{\prod_1^n A_i \mid A_i \in \mathcal{A}_i\}) \triangleq \sigma(\mathcal{A})$.

product measurable space: $(\prod_1^n \Omega_i, \prod_1^n \mathcal{A}_i)$

prop. $\bar{\mathcal{A}} = \{\sum_k^m \prod_1^n A_{kj} \mid A_{kj} \in \mathcal{A}_j\}$ is an algebra.

Pf: 1°) "$\cap$" is trivial 2°) For $A = \sum_{k=1}^m \prod_1^n A_{kj} \in \bar{\mathcal{A}}$, $m \in \mathbb{Z}^+$.

Note that $\prod_1^n \Omega_i = A_1 \times \prod_2^n \Omega_i + A_1^c \times \prod_2^n \Omega_i = \cdots$

$$= \prod_1^n A_i + A_1 \times \cdots A_{n-1} \times A_n^c + A_1 \times \cdots A_{n-2} \times \Omega_n \ldots$$
$$+ A_1^c \times \Omega_2 \times \cdots \Omega_n.$$

$\therefore (\prod_1^n A_i)^c$ is union of rectangles in $\prod_1^n \Omega_i$

(2) Measure Theory:

① **Def:** i) Conti from above: $A_n \downarrow A \Rightarrow \mu(A_n) \to \mu(A)$.

ii) Conti from below: $A_n \uparrow A \Rightarrow \mu(A_n) \to \mu(A)$.

iii) Conti: $A_n \to A \Rightarrow \mu(A_n) \to \mu(A)$.

Remark: i) For p.m. i), ii), iii) all satisfied.

For general measure, i) should condition:

$\exists N$. s.t. $\mu(A_N) < \infty$.

ii) For additive set Func. μ:

continuity $\Leftrightarrow$ σ-additive

Pf: 1') $\sum A_k = \sum_1^n A_k + \sum_{n+1}^{\infty} A_k \overset{\Delta}{=} \sum_1^n A_k + B_n$.

$\therefore \mu(\sum A_k) = \sum_1^n \mu(A_k) + \mu(B_n)$.

$B_n \downarrow \varnothing$. Let $n \to \infty$.

2') For $A_n \to A$. $\therefore A = \bigcup_n \bigcap_{k \geq n} A_k = \bigcap_n \bigcup_{k \geq n} A_k$.

Denote $B_n = \bigcap_{k \geq n} A_k \uparrow$. $B_0 = \varnothing$.

$\therefore \mu(A) = \sum_1^{\infty} \mu(B_n - B_{n-1}) = \lim_N \sum_1^N \mu(B_n - B_{n-1})$

$= \lim_N \mu(B_N) \leq \lim_N \mu(A_N)$.

Conversely. $\mu(A) = 1 - \mu(\bigcup \bigcap A_k^c)$

$\geq 1 - \lim_N \mu(A_N^c)$.

$\therefore \mu(A) = \lim_N \mu(A_N)$.

② Properties:

i) For semialgebra:

Thm. μ is a nonnegative additive set Func. on semialgebra $\mathcal{A}$. $A, B \in \mathcal{A}$. $\{A_n, B_n\} \subseteq \mathcal{A}$.

Then (a). $A \subset B \Rightarrow \mu(A) \le \mu(B)$

(b) $\sum A_n \subset B \Rightarrow \sum \mu(A_n) \le \mu(B)$.

Pf: (a) suppose $A^c = \sum_1^N M_i$.

$\therefore B = A + B \cap A^c = A + \sum_1^N B \cap M_i$

$\mu(B) = \mu(A) + \sum_1^N \mu(B \cap M_i) \ge \mu(A)$.

(b) By extension Thm. $\bar{\mu}|_{\mathcal{A}} = \mu$.

$\bar{\mu}$ is defined on $\sigma(\mathcal{A})$.

ii) For algebra:

Thm. μ is measure on algebra $\mathcal{A}$. Then we have:

$A \subset \bigcup_1^\infty A_n$. $A, (A_n) \subset \mathcal{A}$. $\Rightarrow \mu(A) \le \sum \mu(A_n)$

Pf: $A = A \cap (\cup A_n) = \bigcup_1^\infty (A \cap A_n) \triangleq \cup B_n$.

$= B_1 + (B_2 - B_1) + (B_3 - B_1 - B_2) + \cdots$

$= \sum C_n$. $C_n \subset B_n \subset A_n$.

$\therefore \mu(A) = \sum \mu(C_n) \le \sum \mu(B_n) \le \sum \mu(A_n)$.

iii) For σ-algebra:

We're so familiar with it.

③ Extension:

i) From S to $A(S)$:

Thm. For m. nonnegative additive set func on S. ($\emptyset \in S$) semialgebra. Then exists unique extension $\bar{m}$ on. $\bar{S} = A(S)$. st. $\bar{m}|_S = m$. m is additive.

Besides. m is σ-additive $\Rightarrow$ $\bar{m}$ is σ-additive.

Pf: $A = \sum_1^N A_k \in A(S)$. Define $\bar{m}(A) = \sum_1^N m(A_k)$.

Check it's well-def.

ii) Outer Measure:

(m, S) induce $(m^*, P(\Omega))$. $(m^*|_{A^*}, A^*, \Omega)$ is a measure space.

Relation: $S \subset A(S) = \bar{S} \subset \sigma(S) \subset A^* \subset P(\Omega)$.

For m is σ-finite $\Rightarrow m^*|_{\sigma(S)}$ is unique.

Remark: $m^*|_{A^*}$ is completion of $m^*|_{\sigma(S)}$.

$$A^* = \sigma(S) + \{\text{all } M_\sigma\text{-null sets}\}.$$

prop. $A = \sum A_n$. $A_n \in A^*$. For $\forall B \in P(\Omega)$. Then.

$$m^*(A \cap B) = \sum m^*(A_n \cap B).$$

Pf: $m^*(A \cap C) + m^*(A \cap C^c) = m^*(C)$.

Set $C = A \cap B$.

④ Construction:

i) Procedure:

Define: $\mu: S \to \mathbb{R}^1$. on semialgebra. a measure.

Then by extension: $\mu^*|_{\sigma(S)}$ on $\sigma(S)$.

ii) Criteria:

μ nonnegative set Func on S. $\emptyset, \Lambda \in S$.

If (a) μ is additive.

(b) $A \subset \sum A_n$. $A, A_n \in S \Rightarrow \mu(A) \leq \sum \mu(A_n)$.

Pf: By property: For $A = \sum A_n$.

$\mu(A) \geq \sum \mu(A_n)$. with (b). $\therefore \mu$ is σ-additive.

⑤ Radon - Nikodym Thm:

· It extends the ideal of Prob. mass. density over real number to p.m. over arbitrary sets.

e.g. Prove the existence of Condition Expectation.

Remark: σ-finite is necessary in the Thm:

On $(\mathbb{R}^1, \mathcal{B}_{\mathbb{R}^1})$: $\mu(A) = \#A$. not σ-finite.

$\therefore m \ll \mu$. m is Lebesgue measure.

Then $m(A) = \int_A f \, d\mu$.

Let $A = \{a\}$. Then $f(a) = 0$. $\forall a \in \mathbb{R}^1$.

(3) Distribution Func's:

① Different Types:

Def: i) Degenerate d.f.: $\delta_t(x) = I\{x \geq t\}$.

ii) Discrete d.f.: $F(x) = \sum_{n=1}^{\infty} p_n \delta_{a_n}(x)$.

iii) Conti d.f.: F is conti $\forall x \in \mathbb{R}'$.

Remark: The jumps of discrete d.f. can be dense: $(a_n) = \mathbb{Q}^+$. $p_n = \frac{1}{2^n}$.

Def: Support of d.f. F is: $S(F) = \{x \mid F(x+\varepsilon) - F(x-\varepsilon) > 0,\ \forall \varepsilon > 0\}$.

prop. i) points of support is isolated $\Rightarrow$ It's jump.

ii) $S(F)$ is closed.

Pf: i) $F(x+\varepsilon) - F(x-\varepsilon) > F(x) - F(x-) > 0$. $\forall \varepsilon > 0$. for jump x. $\therefore$ Jumps $\in S(F)$.

ii) $(x_n) \subset S(F) \to x$. Then we obtain:

$F(x+\varepsilon) - F(x-\varepsilon) > F(x_{n_0} + \frac{\varepsilon}{2}) - F(x_{n_0} - \frac{\varepsilon}{2}) > 0$

$\exists\, x_{n_0} \in (x_n)$. $n_0 = n(\varepsilon)$.

Remark: $S(F)$ can be $\mathbb{R}$. i.e. $\sum \delta_{q_n}(t) \cdot p_n$. $p_n = \frac{1}{2^n}$. $\mathbb{Q} = \{q_n\}$. dense.

② Decomposition:

Thm. $\forall$ d.f. F can be written as convex combination of discrete one and conti one.

$F = \alpha F_1 + (1-\alpha) F_2$. It's unique.

Pf: $F = F_c + F_d$. Then by normalization.

Def: F is absolutely conti if:

$\exists f \geq 0$. s.t. $F(x) = \int_{-\infty}^{x} f(t) dt$.

Remark: Then $F_c = F_{ac} + F_s$. where F_s is singular.

(4) Mappings:

$\cdot X : \Omega_1 \to \Omega_2$. Note that X^{-1} perserves all set operations from Ω_2 to Ω_1. So:

prop. i) $\mathcal{A}$ is σ-algebra on Ω_2. So $X^{-1}(\mathcal{A})$ is in Ω_1.

ii) $\mathcal{C}$ is class in Ω_2. Then $X^{-1}(\sigma(\mathcal{C})) = \sigma(X^{-1}(\mathcal{C}))$.

Pf: ii). Set $\mathcal{G} = \{ G : X^{-1}(G) \in \sigma(X^{-1}(\mathcal{C}))\}$.

Check: $\begin{cases} \mathcal{G} \text{ is } \sigma\text{-algebra.} \\ \mathcal{C} \subset \mathcal{G}. \end{cases}$

$\Rightarrow \sigma(\mathcal{C}) \subset \mathcal{G}$. $\therefore X^{-1}(\sigma(\mathcal{C})) \subset \sigma(X^{-1}(\mathcal{C}))$

Converse is trivial.