Gambler's Ruin Problem.

Consider a gambler starts with i fortune. On each successive gamble, either win 1 or lose 1 fortune. Denote X_n is total fortune the gambler have after n^{th} game.

$X_n = i + \sum_1^n \Delta_k$. $P(\Delta_k = 1) = p$. $P(\Delta_k = -1) = q = 1-p$.

(Δ_k) i.i.d. r.v's.

<u>Denote</u>: $\tau_i^N = \{n \geq 1 \mid X_n \in \{0, N\}\}$. $S = \{0, 1 \cdots N\}$.

<u>prop.</u>

$$P_i(N) =: P(X_{\tau_i^N} = N) = \begin{cases} (1-(q/p)^i)/(1-(q/p)^N). & p \neq q. \\ i/N & . p = q. \end{cases}$$

<u>Pf.</u> $P_i(N) = P(X_{\tau_i^N} = N \mid \Delta_1 = 1) \cdot p + P(X_{\tau_i^N} = N \mid \Delta_1 = -1) q$

$= p \cdot P_{i+1}(N) + q \cdot P_{i-1}(N)$.

<u>Cor.</u> $P_i(\infty) = \lim_{N \to \infty} P_i(N) = \begin{cases} 0. & p \leq \frac{1}{2} \\ 1-(q/p)^i. & p > \frac{1}{2} \end{cases}$

Consider $R_n = \sum_1^n \Delta_k$. $R_0 = 0$. i.e. the random walk starts at origin initially. For $a, b \in \mathbb{Z}^+$.

$p(a,b) =: P(R_n$ hits level a before hitting level $-b)$

Note it's equi. with a gambler starts at b fortune and wish to get target $N = a+b$.

$$\Rightarrow \quad P(a,b) = \begin{cases} b/a+b. & p=q \\ (1-(q/p)^b)/(1-(q/p)^{a+b}) & p \neq q. \end{cases}$$

Let $b \to \infty$. $\Rightarrow P(a,\infty) = P(\max_n R_n \geq a) = \begin{cases} 1. & p \geq \frac{1}{2} \\ (q/p)^a. & p < \frac{1}{2} \end{cases}$

Rmk: i) Dist. of $\max_n R_n$: $P(\max_n R_n = k) = (q/p)^k (1 - q/p)$.

ii) Symmetrically : $P(\min_n R_n \leq -b) = (p/q)^b$.

if $q < \frac{1}{2}$. and 1 if $q \geq \frac{1}{2}$.

② Denote : $\sigma_a = \inf\{n \geq 1 \mid X_n = a\}$.

Rmk: We have find $P(\sigma_a < \sigma_b)$ above.

Next, we will find dist. of σ_a :

Thm. (Reflection Principle of RW).

For (X_n) defined as above. $X_0 = 0$. We have:

$\{\sigma_a < n.\ X_n = a+1\} \longrightarrow \{\sigma_a < n.\ X_n = a-1\}$

$w \longmapsto w'$

is one-to-one correspond.

Pf: Set $A = \{f : \{0, \cdots n\} \to \mathbb{Z}.\ f(0)=0.\ f(n)=a-1.\ \exists m \in (0,n).\ s.t.\ f(m) = a\}$.

$B = \{ f : \{0, \cdots n\} \to \mathbb{Z}.\ f(0) = 0.\ f(n-1) = a+1.$

$\exists\, m \in (0, n).\ f(m) = a \}.$

Let : $\sigma_f = \inf \{ j \geq 1 \mid f(j) = a \}$

$$\phi : A \longrightarrow B$$

$$f \longmapsto 2 f(m \wedge \sigma_f) - f(m)$$

<u>Cor.</u> $\{ \sigma_a < n.\ X_n > a \} \xrightarrow{\phi} \{ \sigma_a < n.\ X_n < a \}$

is one-to-one correspond

<u>Thm.</u> $P_0(\sigma_a = n) = \begin{cases} 0 & \text{. otherwise} \\ \binom{n}{(a+n)/2} \frac{|a|}{n} p^{\frac{n+a}{2}} q^{\frac{n-a}{2}} & \text{. } |a| \leq n.\ a+n = 2k. \end{cases}$

<u>Pf</u>: If $X_n = a$. Then suppose $(A_k)_1^n$ has x

" -1 ". y. " $+1$ ". $\Rightarrow \begin{cases} x + y = n \\ y - x = a \end{cases} \therefore \begin{cases} x = \frac{n-a}{2} \\ y = \frac{n+a}{2} \end{cases}$

(i.e. if $a+n$ is odd. $P_0(\sigma_a = n) = 0$).

$P_0(\sigma_a = n) = P_0(X_n = a.\ X_k \neq a.\ 0 \leq k \leq n-1)$

$= P_0(X_n = a) - P_0(X_n = a.\ \sigma_a \leq n-1)$

$P_0(X_n = a) = \binom{n}{\frac{n-a}{2}} p^{\frac{n+a}{2}} q^{\frac{n-a}{2}}$

$P_0(X_n = a.\ \sigma_a \leq n-1) = P_0(X_n = a.\ \sigma_a \leq n-2)$

$= P_0(X_n = a.\ X_{n-1} = a+1.\ \sigma_a \leq n-2)$

$+ P_0(X_n = a.\ X_{n-1} = a-1.\ \sigma_a \leq n-2)$

Note $\forall \omega \in \{X_n = a,\ X_{n-1} = a+1,\ Z_n \le n-2\}$

or $\{X_n = a,\ X_{n-1} = a-1,\ Z_n \le n-2\}$.

$$P_0(\omega) = p^{\frac{n+a}{2}} q^{\frac{n-a}{2}}$$

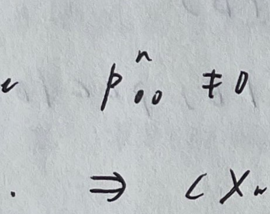
with reflection principle. $\#\{\cdots\} = \#\{\cdots\}$.

$$\Rightarrow P_0(\sigma_a = n) = \binom{n}{(n+a)/2} p^{\frac{n+a}{2}} q^{\frac{n-a}{2}} -$$

$$2\binom{n-1}{(n+a)/2} p^{\frac{n+a}{2}} q^{\frac{n-a}{2}} = \frac{|a|}{n}\binom{n}{\frac{n+a}{2}} p^{\square} q^{\square}.$$

(3) For recurrent: Note (X_n) is irred. consider state 0. $E(N_0) = \sum p_{00}^{n} = \sum p_{00}^{2n} = \sum \binom{2n}{n} p^n q^n \sim \frac{(4pq)^n}{\sqrt{n\pi}}$ (Note $p_{00}^{n} \neq 0$ iff n is even) by Stirling Formular. $\Rightarrow (X_n)$ recurrent $\Leftrightarrow p = q = \frac{1}{2}$.

<u>Rmk</u>: For simple Random Walk on $\mathbb{Z}^d$. i.e. $X_n = \sum_{1}^{n} A_k$. $P(A_k = z) = \frac{I_{\{|z|=1\}}}{2d}$. We can decompose it into d simple random walks on $\mathbb{Z}^1$. e.g. $d = 2$:



It's product of 2 indept SRWs. each moves $1/\sqrt{2}$ units vertically of parallelly along $y=x$.

So: $p^{2m}(\vec{0}, \vec{0}) = p^{2m}(0,0)\, p^{2m}(0,0) \sim 1/m\pi$.

For $d = n$. $p^{2m}(\vec{0}, \vec{0}) \sim (m\pi)^{-\frac{n}{2}}$.

<u>Thm</u>: More precisely. $p^{n}(0,x) = 2\left(\frac{d}{2\pi n}\right)^{\frac{d}{2}} e^{-\frac{d|x|^2}{2n}} + O(n^{-\frac{d+2}{2}})$

④ Conti. Type RSW:

- RSW on $\mathbb{Z}^d$ is a Q-process. with generator $Qf = \frac{1}{2d} \Delta f$. Δ is discrete Laplace operator.

Rmk: We can find its prob. transition function: Note $P_t(x, y) = P_t(0, x-y)$.

$$P_t(0, x) = e^{-t} \sum_{n \geq 0} \frac{t^n}{n!} P^{(n)}(0, x).$$

where $P^{(n)}$ is p.m of Discrete type.

Thm. (Local Limit)

$$P_t(0, x) = \left(\frac{d}{2\pi t} \right)^{\frac{d}{2}} e^{-d|x|^2/2t} + O(t^{-\frac{dt^2}{2}})$$

Rmk: Note that $\int_0^\infty P_t(0,0)\, dt = \infty$

$\Leftrightarrow d = 1, 2.$

So: it's recurrent only in 1.2-dim.