Brownian Motion

<u>Def</u>: $(W(t))_{t\geq 0}$ is SBM if $W(t)$ is conti. a.s.

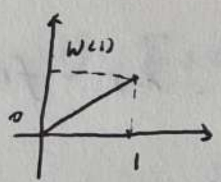
has stationary indept increment. $W(t) \sim N(t)$.

(1) <u>Lévy's Construction:</u>

We will play connect-the-dot on $[0,1]$.

Suppose $Z_k \overset{i.i.d}{\sim} N(0,1)$.

i)

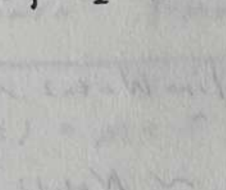


Note : $W(1) \sim N(0,1)$. Set $X^{(0)}(1) = W(1)$.

Connect $(0,0)$ to $(1, W(1))$.

$\Rightarrow X^{(0)}(t) = Z_1 t$.

ii)



Note: $W(\frac{1}{2}) \sim N(0, \frac{1}{2})$. $X^{(0)}(\frac{1}{2}) = \frac{Z_1}{2} \sim N(0, \frac{1}{4})$

Set $X^{(1)}(\frac{1}{2}) = X^{(0)}(\frac{1}{2}) + \frac{Z_2}{2} \sim N(0, \frac{1}{2})$

Connect $(0,0)$, $(\frac{1}{2}, X^{(1)}(\frac{1}{2}))$, $(1, X^{(1)}(1))$

where $X^{(1)}(1) = X^{(0)}(1) = Z_1$.

iii) Analogously, at step n, from $X^{(n)}$ to $X^{(n+1)}$:

$$\text{Set} \begin{cases} X^{(n+1)}(2k/2^{n+1}) = X^{(n)}(2k/2^{n+1}) \\ X^{(n+1)}(2k+1/2^{n+1}) = \frac{1}{2}\left(X^{(n)}(\frac{k+1}{2^n}) - X^{(n)}(\frac{k}{2^n})\right) + \frac{Z_{2^n+k}}{2^{\frac{n}{2}+1}} \end{cases}$$

By induction : $\left(X^{(n+1)}(k/2^{n+1}) - X^{(n+1)}(k+1/2^{n+1})\right)_k$

are indept increments.

iv) Prove: $X^{(n)}(t) \xrightarrow{u} W(t)$ $\forall t \in [0,1]$. a.s.

Lemma $G_n \overset{\text{iid}}{\sim} N(0,1)$. Then $P(|G_n| \le \sqrt{c\log n}$ for large $n) = 1$. for some $c > 2$

Pf: $P(G_n \ge x) = \int_x^\infty e^{-\frac{t^2}{2}} dt \le \frac{e^{-\frac{x^2}{2}}}{x}$

$\Rightarrow$ By Borel-Cantelli. Lemma:

$P(|G_n| \ge \sqrt{c\log n}$. i.o.$) = 0$

Set $M_n = \max_{t\in[0,1]} |X^{(n)}(t) - X^{(n+1)}(t)| = 2^{-\frac{n}{2}+1} \max_{2^n+1 \le k \le 2^{n+1}} |Z_k|$

By Lemma. $M_n \le 2^{-(\frac{n}{2})-1} \sqrt{c\log 2^{n+1}}$. a.s. for large n.

$\Rightarrow \sum_{n=1}^{\infty} (X^{(n)}(t) - X^{(n+1)}(t)) \le \sum M_n < \infty$. a.s.

v) $W(t) \sim N(0,t)$. $\forall t \in [0,1]$.

Pf: $\exists\ r_n = k/2^m \to t$. $W(t_n) \sim N(0, t_n)$

By conti of W: $W(t) = \lim_n W(t_n) \sim N(0,t)$

(Write $W(t_n) \sim \sqrt{t_n}\, Z_n$. $Z_n \sim N(0,1)$)

vi) W has indept. Stationary increments.

Pf: For $s<t<u$. if $s_n \to s$. $t_n \to t$. $u_n \to u$.

$(W(t_n) - W(s_n),\ W(u_n) - W(t_n)) \sim (\sqrt{t_n - s_n}\, Z_1,\ \sqrt{u_n - s_n}\, Z_2)$

$Z_1, Z_2, \overset{\text{iid}}{\sim} N(0,1)$. Let $n \to \infty$.

(2) Conditional Dist.:

prop. $W_t - tW_u/u$ is indept with W_u. $\forall u > t \geq 0$.

Pf: $Cov(W_t - tW_u/u, W_u) = t \wedge u - t/u \cdot u = 0$

Cor. i) $E(W_t | W_u) = t/u \cdot W_u$. $u > t \geq 0$

ii) $W_t | W_u \sim N(tW_u/u, t(u-t)/u)$

prop. i) $W_t | W_s, W_u \sim N(W_s + \frac{t-s}{u-s}(W_u - W_s), \frac{(t-s)(u-t)}{u-s})$

ii) $E(W_s W_t | W_u) = \frac{s}{t} E(W_t^2 | W_u)$

for $\forall\ 0 \leq s < t < u$.

Pf: i) $E(W_t | W_s, W_u) = E(W_t - W_s | W_u - W_s) + W_s$

$= E(\widetilde{W}_{t-s} | \widetilde{W}_{u-s}) + W_s$

ii) Check: $E(W_t(W_s - \frac{s}{t}W_t) | W_u) = 0$

(3) Sojourn Time Problem:

$X(t) = \mu t + \sigma W(t)$. $\mu > 0$. W is SBM.

$T = \int_0^\infty I\{X(t) \in (0,\delta)\} dt$ is sojourn time in $(0,\delta)$.

$E(T) = \int_0^\infty P(-\frac{\mu\sqrt{t}}{\sigma} \leq \xi \leq \frac{\delta - \mu t}{\sigma\sqrt{t}}) dt$. $(\xi \sim N(0,1))$

With $\{\square \leq \xi \leq \square\} = \{t_1(\xi) \leq t \leq t_2(\xi)\}$. By Fubini:

$\Rightarrow E(T) = \delta/\mu$ by $E(\xi\sqrt{\xi^2\sigma^2 + 4\mu\sigma}) = 0$ (odd. sym)

Rmk: Alternatively, by occupation time Formula from local time. Then:

i) $\forall B \in \mathcal{B}_{\mathbb{R}^+}$. $E(T_B) = m(B)/\mu$. m is Lebesgue

ii) $E(T_{(-\infty,0]}) = \sigma^2/2\mu^2$.

(4) Shift Hitting Time:

Consider $X_t = -\mu t + W_t$. $\mu > 0$. W_t is SBM.

Next. we find: $P_{-a}(\tau_b < \tau_a)$. where $\tau_a = \inf\{t \geq 0 \mid X_t = a\}$.

set $T = \tau_a \wedge \tau_b$. $P_{-\mu}(\tau_b < \tau_a) = P_{-\mu}(X_T = b)$

Denote: $u(x) \triangleq P_{-\mu}(X_T = b \mid X_0 = x) \triangleq P_x(X_T = b \mid X_0 = x)$

Lemma. (Little $o(h)$)

$$P(\max_{[0,h]} X_t > \varepsilon) = o(h) \ (h \to 0). \ \forall \varepsilon > 0.$$

Pf. $LHS = P_0(\max_{0 \leq t \leq h} W(t) + \mu t \geq \varepsilon)$

$\leq P_0(\tau'_{\varepsilon - \mu h} \leq h) = o(h)$.

$\tau'_a = \inf\{t \mid W_t = a\}$. $P_x(\tau'_a \leq t) = e^{\frac{-2a(a-x)}{t}}$

By first step analysis:

$u(x) \overset{mp}{=} E_x(P(X_T = b \mid X_h)) \overset{Lemma}{=} E(u(X_h)) + o(h)$

$= u(x) + u'(x) E(X_h - x) + \frac{u''(x)}{2} E((X_h - x)^2) + \cdots$

$= u(x) + u'(x)(-\mu h) + \frac{1}{2} u''(x) h + o(h)$

by Taylor expansion at x.

Divide h at both sides. Let $h \to 0$. Then:

$\Rightarrow \frac{1}{2} u''(x) - m u'(x) = 0$. $u(a) = 0$. $u(b) = 1$.

Solve $u(x) = (e^{2mx} - e^{2ma}) / (e^{2mb} - e^{2ma})$

Set $x = 0$. $\therefore P_{-m}(Z_a > Z_b) = \dfrac{1 - e^{2ma}}{e^{2mb} - e^{2ma}}$

Rmk: i) Set $a \to \infty$. $P_{-m}(Z_b < +\infty) = e^{-2mb}$.

ii) Alternative. use Wald mart.:

$M_\lambda(t) = e^{\lambda W(t) - \frac{\lambda^2}{2} t}$

Set $M(t) = e^{2m X(t)}$. $T = Z_a \wedge Z_b$.

$E(M_{T \wedge n}) = 1$. by optional sampling

set $n \to \infty$. by DCT.