Conti. Time Markov Chain

(1) Definitions:

① Def: A stochastic process $(X_t)_{t\geq 0}$ with discrete state space S $(|S|=S)$ is CTMC if $p(X(s+t)=j \mid X(s)=i, (X(u))_{0\leq u<s}) = p(X(s+t)=j \mid X(s)=i)$ $= P_{ij}(t)$. indept with s. $\forall i, j \in S$.

Denote: i) Trans prob. matrix $P(t) = (P_{ij}(t))_{S\times S}$. $P(0)=I$.

ii) H_i is the holding time for i when it enter state i.

Rmk: H_i has memoryless property. So it's exponential dist. $\exists (\lambda_i)$. st. $H_i \sim Exp(\lambda_i)$.

A CTMC can be described by:

i) Transition Matrix $P=(P_{ij})_{S\times S}$. describes how chain changes at transition epoches. $(P_{ii}=0)$

ii) Set of transition rates $(\lambda_i)_{i\in S}$

e.g. If $X(t)=i$. Then, next time it changes has prob. $\lambda_i dt$

Def: $Q = P'(0)$. infestimal generator of $(X_t)_{t\geq 0}$.

prop. $Q = P'(0) = (P'_{ij}(0))_{S\times S}$. satisfies: $\forall i \neq j \in S$.

$$P'_{ij}(0) = \lambda_i P_{ij}. \quad P'_{ii}(0) = -\lambda_i. \quad \text{So}: \sum_{j\in S} P'_{ij}(0) = 0$$

Pf: $(N_i(t))_{t\geq 0}$ is Poisson counting process with rate λ_i.

recall it has "little $o(t)$" property.

$$P'_{ij}(0) = \lim_{h\downarrow 0} P_{ij}(h)/h = \lim_{h} P_{ij}\, P(N_i(h)=1)/h$$

$$= P_{ij} \lim_{h\downarrow 0} (\lambda_i h + o(h))/h = \lambda_i P_{ij}.$$

$$P'_{ii}(0) = \lim_{h} (P_{ii}(h) - 1)/h = -\lim_{h} \frac{P_i(X(h)\neq i)}{h}$$

$$= \lim_{h} -\frac{P(N_i(h)=1)}{h} = -\lambda_i.$$

② Def: Set (Z_n) is seq of times that transition occurs.

$(X_n) =: (X(Z_n t)) \hookrightarrow (X(t))$ is embedded DTMC

from $(X_t)_{t\geq 0}$. with transition matrix $P = (P_{ij})$

prop. (Chapman-Kolmogorov Equation)

$$P_{ij}(t+s) = \sum_{k\in S} P_{ik}(t) P_{kj}(s). \text{ i.e. } P(t+s) = P(t)P(s)$$

Def: A CTMC is explosive if the transition in a finite time is infinite.

e.g. $P_{i,i+1} = 1.\ \forall i \geq 0.\ \lambda_i = 2^i.\ i\geq 0.\ S = \mathbb{Z}^+ \cup \{0\}$.

$$E(\sum_i M_i) = \sum 1/\lambda_i < \infty.$$

Rmk: $\sup_S |\lambda_i| < \infty \Rightarrow$ It's nonexplosive.

③ Queue Models:

i) FIFO $M/M/1$:

"FIFO" means "first in queue first out of queue".

Suppose Arrivals to the queue are Poisson at rate λ. (S_n) service times $\overset{i.i.d}{\sim}$ $Exp(\mu)$.

$X(t)$ is number of customers in the system.

1°) Holding time:

$H_0 \sim Exp(\lambda)$. $H_i = \min\{S_r, X\} \sim Exp(\lambda+\mu)$.

X is time untill arival. (Note: $p(S_r = X) = 0$)

2°) Transition prob.:

$p(0,1) = 1$. $p(i,i+1) = p(X < S_r) = \lambda/\lambda+\mu$.

$p(i,i-1) = \mu/\lambda+\mu$.

From: $a_i = \begin{cases} \lambda, & i=0 \\ \lambda+\mu, & i\geq 1 \end{cases}$ $\Rightarrow$ Obtain $Q = P'(0)$.

ii) $M/M/c$:

1°) Holding times:

$H_0 \sim Exp(\lambda)$. $H_1 = \min\{X, S_r\} \sim Exp(\lambda+\mu)$

$H_2 = \min\{X, S_{r_1}, S_{r_2}\} \sim Exp(\lambda+2\mu)$. ...

$\Rightarrow H_i \sim Exp(\lambda+i\mu)$. $0\leq i\leq c$. $H_i \sim Exp(\lambda+c\mu)$. $i\geq c$.

2°) Transition prob.:

$p(i,i+1) = p(X < \min_{1\leq k\leq i}\{S_{jk}\}) = \lambda/\lambda+i\mu$. $0\leq i\leq c$.

$p(i,i+1) = \lambda/\lambda+c\mu$. $i\geq c$.

iii) M/M/∞:

$H_i \sim Exp(\lambda + i\mu)$. $P(i, i+1) = \lambda / \lambda + i\mu$. $\forall i \geq 0$.

iv) Birth and Death Process:

Its $P = (P_{ij})$ satisfies: $P_{i,i+1} + P_{i,i-1} = 1$. $S = \mathbb{Z}^+ \cup \{0\}$.

$P_{i,i+1} = P(B_i < D_i) = \lambda_i / \lambda_i + \mu_i$. B_i. D_i are time until Birth or death when there i population. $B_i \sim Exp(\lambda_i)$. $D_i \sim Exp(\mu_i)$. It's alike M/M/1 model.

(2) Limit Theory:

Def: i) $i \in S$ is recurrent / commutative with $j \in S$ / accessible from $j \in S$ if it holds in embedded DTMC.

ii) $i \in S$ is positive recurrent if $E(T_{ii}) < \infty$.

Rmk: i) CTMC is positive recurrent $\not\Leftrightarrow$ Embedded DTMC is positive recurrent.

ii) Positive recurrent is still a class property.

iii) $P_j = \lim_{t \to \infty} \frac{1}{t} \int_0^t P_{ij}(s) ds \in \overline{\mathbb{R}^+}$ exists. by RRT.

If it's finite. Then, set $\vec{P} = (P_1 \cdots P_n)$ stationary dist. for CTMC. $P^* = \begin{pmatrix} \vec{P} \\ \vdots \\ \vec{P} \end{pmatrix} = \lim_{t \to \infty} \frac{1}{t} \int_0^t P(s) ds$.

is called limitting prob. dist.

prop. If $(X(t))_{t\geq 0}$ is positive recurrent CTMC.

Then p^* exists. unique. $P_j = 1/\lambda_j E(T_{jj})$

Pf: Apply RRT:

$(Z_n) = (T_{jj}^n)$. $R_n = \int_{Z_{n-1}}^{Z_n} I\{X(s)=j\} ds$.

$N_j(t)$ is counting process for (Z_n).

$N_j(t)/t \to 1/E(T_{jj})$. a.s. by ERT.

Combined with $R_n \overset{d}{\sim} H_j$.

Cor. Under the condition above: $P_j = \lim_{t\to\infty} P_{ij}(t)$.

Cor. If X is null recurrent. Then p^* doesn't exist. and $p_i = 0$. $\forall i \in S$

Rmk: If $X(0) \overset{d}{\sim} \vec{v}$. initial dist. Then:

$X(t) \overset{d}{\sim} \vec{v} P(t)$. From above. We have:

$\lim_{t\to\infty} \frac{1}{t}\int_0^t \vec{v} P(s) ds = \vec{v} p^* = \vec{P}$. So it

means $\vec{P}$ (or say p^*) indept with $\vec{v}$.

① Stationary Version:

prop. For a positive recurrent CTMC with limiting dist. P. If $X(0) \sim \vec{P}$. Then $X(t) \sim \vec{P}$. $\forall t$.

i.e. $\vec{P}\cdot P(t) = \vec{P}$. ($\sum_i P_i P_{ij}(t) = P_j$. $\forall j \in S$.)

Rmk: $\vec{P}$ is unique: if $\vec{v}\cdot P(t) = \vec{v}$. then:

$\lim_{t\to\infty} \frac{1}{t}\int_0^t \vec{v}\cdot P(s) ds = \vec{P} = \lim_{t} \frac{1}{t}\int_0^t \vec{v} ds = \vec{v}$.

Pf: $p^* p(t) = \lim_{n\to\infty} \int_0^n \frac{1}{n} p(s)\, p(t)\, ds$

$= \lim_{n\to\infty} \int_0^n \frac{1}{n} p(s+t)\, ds$ (by C-K equation)

$= p^*$.

Denote: Set $X^* = \{X^*(t)\}_{t\geq 0}$ is the CTMC with $X^*(0) \sim p$.

the limiting dist. stationary version.

Rmk: $X^*_s \sim X^*$ is easy to see. $\forall s \geq 0$ shift.

③ Equations:

i) Thm. (Kolmogorov Backward Equation)

For CTMC with infestimal generator $Q = p'(0)$.

We have: $p'(t) = Q \cdot p(t)$ holds. $\forall t \geq 0$. Besides,

the unique solution is: $p(t) = e^{Qt} =: \sum_0^\infty \frac{t^n Q^n}{n!}$

Rmk: i) It means $p(t)$ is determined by Q.

ii) For forward equation: $p'(t) = p(t) Q$.

it will cause some problems on inter-

change the sum and limit.

But e^{Qt} is the common solution.

Pf: $p(t+h) - p(t) = (p(h) - I)\, p(t)$

$= (p(h) - p(0))\, p(t)$.

ii) Balance Equations:

Note that every time $X(t)$ want to enter state i, then it must leave i state first. $\Rightarrow$ The number of entering i diff the number of leaving i at most one.

Claim: the long run rates of these two will coincide.

Def: The balance equation for positive recurrent CTMC is $\vec{P} Q = 0$. $Q = P'(0)$.

Rmk: $\vec{P} Q = 0 \Rightarrow a_i p_i = \sum_{j \neq i} p_j a_j P_{ji}$.

LHS is rate of leaving i.

RHS is rate of entering i.

Thm. A nonexplosive irreducible CTMC is positive recurrent $\Leftrightarrow$ $\exists$ unique probability solution $\vec{P}$ for $\vec{P} \cdot Q = 0$. i.e. $p_i > 0$. and it's limiting dist. for CTMC.

Pf: ($\Rightarrow$) $\exists \vec{P}$ is stationary dist.:

$$\vec{P} \cdot p(t) = \vec{P} \Leftrightarrow \vec{P} \cdot \frac{p(t) - I}{t} = 0. \ t \to 0$$

($\Leftarrow$) Show: $\vec{P} \cdot p(t) = \vec{P}$.

by backward equation:

$$\vec{P} Q p(t) = \vec{P} \cdot p'(t) = 0 = \frac{d}{dt} (\vec{P} \cdot p(t))$$

$$\therefore \vec{P} \cdot p(t) = \vec{P} \cdot p(0) = \vec{P}$$

Then it follows from a Lemma:

<u>Lemma.</u> For a CTMC. If it has stationary dist. $\vec{p}$. s.t. $\vec{p} \cdot p(t) = \vec{p}$. Then it's positive recurrent.

<u>Pf:</u> $p_j = \lim_{t \to \infty} \frac{1}{t} \int_0^t p_{ij}(s) ds = 0 \Rightarrow \vec{p} = 0$. Contradict!

if we assume it's null recurrent.

<u>Thm.</u> An irreducible CTMC with finite state space S is always positive recurrent.

<u>Pf:</u> Note that the embedded DTMC is irred. So it's positive recurrent.

Denote: $S = \{1, 2, \cdots b\}$. Z_{11} is return time to state 1 for DTMC. T_{11} is for CTMC.

Set $Y_n \overset{\text{iid}}{\sim} Exp(\min\{a_1, \cdots a_b\})$. where the holding time $H_i \sim Exp(a_i)$.

$$\Rightarrow E(T_{11}) \le E(\sum_1^{Z_{11}} Y_k) = E(Z_{11}) / \min_i a_i < \infty$$

<u>RMK:</u> When using Balance Equations to solve (p_i) stationary dist. if S is infinite. We need use: $a_i p_i = \sum_{j \ne i} p_j a_j P_{ji}$ rather than matrix form. (use it recurrsively)

③ <u>General Case:</u>

We treat $P_{ii}=0$ of transition matrix in the discussion above. But in general, we can set $P_{ii} \in [0,1)$.

1) Set K is r.v. of total number of visiting state i before transitioning to state $j \neq i$. $K \sim Geo(p)$, $p = 1-P_{ii}$.

2) Renew holding time for i: $\widetilde{H}_i = \sum_1^K H_i^n$.

Calculation by ch.f: $(H_i^n \overset{i.i.d}{\sim} H_i)$

$\Rightarrow \widetilde{H}_i \overset{\wedge}{\sim} Exp(pa_i)$

3) Reset $\tilde{a}_i = pa_i$, $\widetilde{P}_{ii}=0$, $\widetilde{P}_{ij} = P_{ij}/p$.

Then we obtain the reduced form.

(3) PASTA:

"PASTA" refers to "Poisson Arrival See Time Average".

Consider: $M/M/1$ queue model:

Denote: π_j^a is long-run proportion of a arrival customer finding there're j customers in the system. i.e.

$\pi_j^a = \lim_{N\to\infty} \frac{1}{N} \sum_1^N I\{X(t_n-)=j\}$. (t_n) is the arrival times.

We can see: $\lambda \pi_j^{a} = \lambda P_j$.

By: LHS is long-run rate of $j \to j+1$. And RHS is long-run rate of arrival happens when j customers are in the system.

$\Rightarrow$ Proportion of Poisson Arrivals who see j customers in the system is equi. with the proportion of time when there're j customers in

Def: A Poisson Process $\psi = \{t_n\}$ satisfies LAC (Lack of Anticipation Condition) if for $N(t)$: $\{N(t+s) - N(t)\}_{s \geq 0}$ indept with $\{(N(u), X(u))\}_{0 \leq u \leq t}$.

Thm. If poisson process $\psi = \{t_n\}$ satisfies LAC for $(X(t))_{t \geq 0}$. Then we have a.s.:

$$\lim_{N \to \infty} \frac{1}{N} \sum_{1}^{N} f(X(t_n^-)) = \lim_{t \to \infty} \frac{1}{t} \int_0^t f(X(s))\, ds$$ if one of limits exists, finite.

Rmk: Set $f(x) = I_{\{x=j\}}$. Then we obtain the conclusion in M/M/1. above.