Regenerative Process.

(1) Conti. Time:

Def: A stochastic process $\tilde{X} = (\tilde{X}_t)_{t\ge 0}$ is regenerative process if $\exists\ (Z_k)_{k\ge 1}$ regenerative time. s.t. $X_k = Z_k - Z_{k-1}$ is i.i.d cycle length. $C_k = (\{\tilde{X}(t+Z_{k-1}) \mid 0 \le t < X_k\}, X_k)$ are i.i.d cycles. Besides $\tilde{X}(Z_k)$ is indept of Z_k.

Rmk: i) $\Psi = (Z_k)$ is renewal process

ii) We say it's positive recurrent if $E(X) < \infty$.

e.g. i) Recurrent CTMC: $X_0 = i$. $Z_k = T_i^k$.

ii) $(B(t))_{t\ge 0}$. $(A(t))_{t\ge 0}$. $(S(t))_{t\ge 0}$ for renewal process.

① Apply of Renewal Reward:

It's easy to see: $R(t) = \int_0^t \tilde{X}(s)\,ds$. $R_i = \int_{Z_{i-1}}^{Z_i} \tilde{X}(s)\,ds$.

$\Rightarrow \lim \frac{1}{t}\int_0^t \tilde{X}(s)\,ds \overset{a.s.}{=} E(R)/E(X)$. Next, we extend it to $Y(t) = f(\tilde{X}(t))$.

Thm. $\tilde{X}$ is positive recurrent regenerative process. f is measurable. s.t. $E(\int_0^{X_1} |f(\tilde{X}(s))|\,ds) < \infty$. Then:

i) $\lim_{t\to\infty} \frac{1}{t}\int_0^t f(\tilde{X}(s))\,ds = E(R)/E(X)$. a.s.

ii) $\lim_{t\to\infty} \frac{1}{t} E(\int_0^t f(\tilde{X}(s))\,ds) = E(R)/E(X)$

Pf: 1) $\frac{1}{t}\int_0^t f(\tilde{X}(s))ds = \frac{1}{t}\sum_1^{N(t)} R_i + \frac{1}{t}\int_{t_{N(t)}}^t f(\tilde{X}(s))ds$.

$R_i = \int_{z_{i-1}}^{z_i} f(\tilde{X}(s))ds$. the second term $\to 0$.

2) $|R(t)/t| \le Y(t) \xrightarrow{a.s} E(R^*)/E(X)$. $R^* = \int_0^{z_1} |f(\tilde{X}(s))|ds$.

$Y_t = \frac{1}{t}\sum_1^{N(t)+1} R_i^*$. By Wald's equation: $E(Y(t)) \to \frac{E(R^*)}{E(X)}$

So $Y(t)$ is u.i. $\Rightarrow R(t)/t$ is u.i.

Cor. A positive recurrent regenerative process have a limiting dist. denoted by X^*. Then. We have: $\forall b$. $p(X^* \le b) = \lim_{t\to\infty}\frac{1}{t}\int_0^t p(\tilde{X}(s)\le b)ds = E(R)/E(X)$. $R = \int_0^{X_1} I_{\{\tilde{X}(s)\le b\}}ds$.

Thm. A positive recurrent regenerative process with a nonlattice cycle length dist. $F(x) = p(X \le x)$. Then $\tilde{X}(t) \xrightarrow[d]{t\to\infty} X^*$.

Pf. It is direct application of KRT.

Fix $f \in C_B$. Construct renewal equation:

$p(f(\tilde{X}(t)) > x) = p(\square, z_1 > t) + \int_0^t p(f(\tilde{X}(t-s)) > x)dF(s)$

follows from $(f(\tilde{X}(t)))_{t\ge 0}$ is still regenerative.

Integrate on x. We have:

$E(f(\tilde{X}(t))) = E(f(\tilde{X}(t)) I_{\{z_1>t\}}) + \int_0^t E(f(\tilde{X}(t-s)))dF$

$=: Q(t) + H * F$. $H(t) = E(f(\tilde{X}_t))$

$Q(t)$ is DRI. right-conti.

$$\Rightarrow E(f(\tilde{X}(t))) \xrightarrow[KRT]{t\to\infty} \int_0^\infty E(f(\tilde{X}(t)) I_{\{Z_1 > t\}}) dt \,/\, E(Z_1)$$

$$\overset{\text{Fubini}}{=} E\left(\int_0^{Z_1} f(\tilde{X}(t)) dt\right) / E(Z_1)$$

$$= E(f(X^*)). \text{ by Thm above.}$$

③ Delayed Version:

Def: A delayed regenerative process has a initial cycle C_0 with length Z_0. $C_0 = (\{\tilde{X}(t) \mid 0 \le t \le Z_0\})$. Z_0) indept with $(C_k)_{k\ge1}$. has different dist.

Rmk: It's not required: $E(|Z_0|) < \infty$. but need: $P(Z_0 < \infty) = 1$. So C_0 can end.

prop. For delayed positive recurrent regenerative process

i) Denote X^* is its limiting dist. it's same as nondelayed version. $P(X^* \le b) = \lim_{t\to\infty} \frac{1}{t} \int_0^t P(\tilde{X}(s) \le b)$ $= E(R)/E(X)$. $R = \int_{Z_0}^{Z_0+X_1} I\{X(s) \le b\} ds$.

ii) For f. $E\left(\int_{Z_0}^{Z_0+X_1} |f(\tilde{X}(s))| ds\right) < \infty$. then:

$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(\tilde{X}(s)) ds = E(R)/E(X)$. a.s.

$\lim_{t\to\infty} \frac{1}{t} \int_0^t E(f(\tilde{X}(s))) ds = E(R)/E(X)$. where.

$R = \int_{Z_0}^{Z_0+X_1} f(\tilde{X}(s)) ds$.

iii) Cycle length dist. $F(x) = P(X_1 \le x)$ is nonlattice.

Then: $\tilde{X}(t) \xrightarrow[d]{t\to\infty} X^*$

Pf: $R_0^* = \int_0^{Z_0} |f(\tilde{X}(s))| ds < \infty$. a.s. can be ignored.

③ Stationary Version:

For positive recurrent regenerative process $\tilde{X} = (\tilde{X}(t))_{t\ge0}$

Note $X(t)$ has limit dist. Consider $\tilde{X}_s = (\tilde{X}(t+s))_{t\ge0}$

shift Version. $\tilde{X}_s \xrightarrow{s\to\infty} \tilde{X}^* = (\tilde{X}^*(t))_{t\ge0}$

Def: $\tilde{X}^*$ is stationary version of $\tilde{X}$.

Rmk: $\tilde{X}^*_s \overset{d}{\sim} \tilde{X}^*$. $\forall s\ge0$. it's stationary.

Denote: C_0^* is limit of delay cycle $C_0(s)$ of $\tilde{X}_s$.

with cycle length Z_0^*. $C_0^* = (\{X_t \mid 0\le t\le Z_0^*\},$

$Z_0^*)$.

prop. i) $Z_0^* \overset{d}{\sim} F_e$

ii) $P(C_0^* \in B) = \lim_{t\to\infty} \frac{1}{t}\int_0^t P(C_0^*(s)\in B)\,ds = \frac{E(R)}{E(X)}$

$R = \int_0^{X_1} I_{\{C_0(s)\in B\}}\,ds$. B is measurable.

iii) $C_0^* \overset{d}{\sim} C_0^*(n)$. $\forall n\ge0$.

Pf: i) Note $Z_0(s) = A(s)$

ii) By RRT. iii) Ignore the finite shift.

Thm. (Time averges as expectation of stationary dist.)

If f is measurable. s.t. $E(\int_0^{X_1}|f(\tilde{X}(s))|\,ds) < \infty$

Then: $\lim_{t\to\infty}\frac{1}{t}\int_0^t f(X(s))\,ds = E(f(X^*(0)))$. a.s.

$\lim_{t\to\infty}\frac{1}{t}\int_0^t E(f(X(s)))\,ds = E(f(X^*(0)))$.

Pf: From the delayed form: $LHS \to E(\int_{z_0}^{z_0+Y_0} f(\tilde{X}(s))ds)/E(X)$.

Then let $u = z_0 + s$. Let $z_0 \to \infty$.

$\Rightarrow RHS = E(\int_0^{X_1} f(X^*(s))ds)/E(X_1) = E(f(X^*(0)))$

(2) Discrete Times:

Def: DTRP is $(\tilde{Y}_n)_{n\in\mathbb{Z}^+}$. associated with (Z_n). st.

$\tilde{Y}$ regenerate at time Z_n. $Y_n = Z_n - Z_{n-1}$. i.i.d.

$C_n = (\{\tilde{Y}_k \mid Z_{n-1} \le k \le Z_n\}, Z_n)$ is i.i.d. cycle.

Thm. (Renewal)

For $\tilde{Y}$ positive recurrent DTRP. If f is measurable. st. $E(\sum_1^{X_1-1} |f(\tilde{Y}_k)|) < \infty$. Then:

i) $\lim_{n\to\infty} \frac{1}{n} \sum_1^n f(\tilde{Y}_k) = E(R)/E(Y)$. a.s.

ii) $\lim_{n\to\infty} \frac{1}{n} \sum_1^n E(f(\tilde{Y}_k)) = E(R)/E(Y)$. $R = \sum_1^{X_1-1} f(\tilde{Y}_n)$.

e.g. (Busloads of passengers)

i^{th} Bus carries M_i passengers. M_k i.i.d. $P(M=k) =: p_k$

passengers $\tilde{Y} = (\tilde{Y}_n)$ index by i if sitting at i^{th} position in bus. It's a DTRP with cycle lengths $(M_k)_{k\ge1}$

i) Denote $\hat{Y}$ is a random chosen passenger.

$P(\hat{Y} = j) = \lim_{n\to\infty} \frac{1}{n} \sum_1^n I_{\{\tilde{Y}_k = j\}} = E(I_{\{M \ge j\}})/E(M)$.

ii) Dist. of size of bus.

$P(M=j) = \lim_{n\to\infty} \frac{1}{n} \sum_1^n I_{\{M_k = j\}} = E(I_{\{M=j\}})/E(M)$