Renewal Theory

(1) Recurrent Times:

① Forward:

Def. For $\varphi = (t_n)$ renewal process. $t_{N(t)} \le t \le t_{N(t)+1}$

the forward recurrent time is $A(t) =:$

$t_{N(t)+1} - t$. ($\overset{t_n}{\vdash} \underset{t}{\longrightarrow} \overset{t_{n+1}}{|} \to A(t)$)

Rmk: If $\varphi \sim POI(\lambda)$. Then $A(t) \sim Exp(\lambda)$: $P(A(t) \le x)$

$$= 1 - P(T_{N(t)+1} > x+t) = 1 - P(N(t, t+x] = 0) = 1 - e^{-\lambda x}.$$

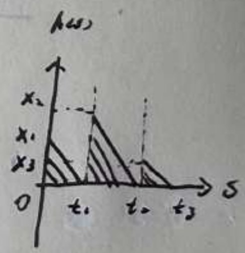
Denote: $X = (X_n)$ i.i.d interarrival times. $X \sim F$

prop. i) $\lim_{t\to\infty} \frac{1}{t} \int_0^t A(s)\, ds = E(X^2)/2E(X)$. a.s.

ii) $\lim_{t\to\infty} \frac{1}{t} \int_0^t I_{\{A(s)>a\}}\, ds = \lambda E(X-a)^+$. a.s.

iii) $\lim_{t\to\infty} \frac{1}{t} \int_0^t P(A(s)>a)\, ds = \lambda E(X-a)^+$.

Pf: 1') Intuitively. $\int_0^t A(s)\, ds \approx \sum_1^{N(t)} \frac{X_k^2}{2}$.



Actually, we have:

$$\frac{1}{t} \sum_1^{N(t)} \frac{X_j^2}{2} \le \frac{1}{t} \int_0^t A(s)\, ds \le \frac{1}{t} \sum_1^{N(t)+1} \frac{X_j^2}{2}.$$

Then apply ERT. as before

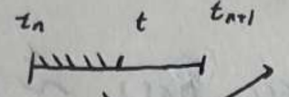
2') $\int_0^t I_{\{A(s)>a\}}\, ds \approx \sum_1^{N(t)} (X_k - a)^+$.

3') By DCT. in the expectation.

② **Backward:**

Def: For $\varphi = \{t_n\}$ renewal process. $t_{N(t)} \le t \le t_{N(t)+1}$.

$B(t) = t - t_{N(t)}$ is backward recurrence time.



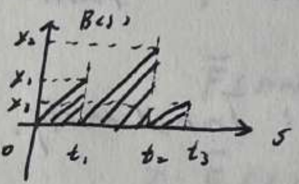
$\left(B(t) \sim F(x) = \begin{cases} 1, & x > t \\ 1 - e^{-\lambda x} & 0 \le x \le t \end{cases} \text{ if } \varphi \sim PoI \right)$

prop. i) $\lim_{t\to\infty} \frac{1}{t} \int_0^t B(s)\,ds = E(X^2)/2E(X)$. a.s.

ii) $\lim_{t\to\infty} \frac{1}{t} \int_0^t I_{\{B(s) > a\}}\,ds = \lambda E(X-a)^+$. a.s.

iii) $\lim_{t\to\infty} \frac{1}{t} \int_0^t p(B(s) > a)\,ds = \lambda E(X-a)^+$.

Pf: Similar as before :



(2) **Distribution:**

① **Equilibrim dist.:**

Denote: $\bar{F}(x) = 1 - F(x)$. $= p(X > x)$. tail prob.

Rmk: $\lambda E(X-a)^+ = \lambda \int_a^\infty p(X > y)\,dy$

$= \lambda \int_a^\infty \bar{F}(y)\,dy$.

Def: As $a \ge 0$ varies. Rmk defines a tail prob. of a dist. We denote cdf of it by F_e. which is called equilibrim dist. of F.

$F_e(x) =: \lambda \int_0^x \bar{F}(y)\,dy$. Set $X_e \sim F_e$. r.v.

Rmk: X_e is conti. since density $= F_e' = \lambda \bar{F}$. exists.

Note that in a long term:

$$\lim_{t\to\infty}\int_0^t \frac{P(A(s)<x)}{t}\,ds = \lim_{t\to\infty}\int_0^t \frac{P(B(s)<x)}{t} = 1-\lambda E(X-x)^+$$

$$= \lambda\int_0^x \bar F(y)\,dy = F_e(x).$$

$\Rightarrow$ The stationary dist of $A(s)$, $B(s) \overset{d}{\sim} F_e$

② Spread:

Def: Spread as length of interarrival time is

$$S(t) =: t_{N(t)+1} - t_{N(t)} = A(t)+B(t) = X_{N(t)+1}$$

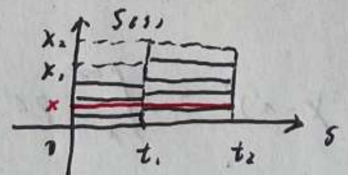
Rmk: Immediately, $\lim_{t\to\infty}\frac{1}{t}\int_0^t S(s)\,ds = E(X^2)/E(X)$.

Prop. i) $\lim_{t\to\infty}\frac{1}{t}\int_0^t I_{\{S(s)>x\}}\,ds = \lambda E(X I_{\{X>x\}})$. a.s.

ii) $\lim_{t\to\infty}\frac{1}{t}\int_0^t P(S(s)>x)\,ds = \lambda E(X I_{\{X>x\}})$

Pf: $\frac{1}{t}\int_0^t I_{\{S(s)>x\}}\,ds \approx \frac{1}{t}\sum_1^{N(t)} X_j I_{\{X_j>x\}}$

by:



Rmk: In a long term:

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t P(S(s)<x)\,ds = \lambda E(X I_{\{X<x\}})$$

Denote: $\bar F_s(x) = \lambda E(X I_{\{X>x\}})$

$$= \lambda\int_x^\infty (t-x)^+ + x\,dF$$

$$= \lambda E(X-x)^+ + \lambda x\bar F(x).$$

$$= \bar F_e(x) + \lambda x \bar F(x).$$

by prop above. the stationary dist. of

$S(x) \sim 1-\bar{F}_s(x) = F_s(x)$. denote r.v. X_s.

$\Rightarrow$ X_s may not be conti. generally. (F' doesn't exist). $E(X_s) = E(X^2)/E(X)$.

(3) <u>Inspection Paradox</u>:

<u>prop</u>. $S(t) \geq_{stoch} X$. i.e. $p(X>x) \leq p(S(t)>x)$. $\forall t, x \geq 0$.

Moreover. $p(X_s \geq x) \geq p(X \geq x)$. $X_s \geq_{stoch} X$.

<u>Pf</u>: 1') $p(S(t)>x \mid N(t)=n.\ t_n=s) = p(X_{n+1}>x \mid X_{n+1}>t-s)$

$= \bar{F}(\max\{x, t-s\}) / \bar{F}(t-s)$

$\geq \bar{F}(x)$.

$\Rightarrow E_{N, t_N}(p(S(t)>x \mid \square)) = p(S(t)>x) \geq \bar{F}(x)$

2') By $p(X_s \geq x) = \lim \frac{1}{t}\int_0^t p(S(s)>x)\,ds \geq p(X \geq x)$

<u>Rmk</u>: Note that : $E(S(t)) \geq E(X)$

$E(X_s) \geq E(X)$

$\Rightarrow$ The prop. above implies a paradox : Observing makes the expectation of lifetime be longer than usual. That's because if we observe at time t. then the part of lifetime < t will be ignored.

<u>e.g.</u> A extreme case :

X is r.v. of lifetime of bulbs. $\begin{cases} p(X=0)=0.9 \\ p(X=1)=0.1 \end{cases}$

Then. all bulbs we observed has lifetime 1.

The right way to estimate lifetime of them is SLLN: $\frac{1}{n}\sum_1^n X_k \approx E(X)$. for large n.

(4) Renewal Reward Thm:

Suppose R_i is r.v. of reward between t_i, t_{i+1}.

$\Rightarrow R(t) = \sum_1^{N(t)} R_j$ Assume $(X_i, R_i)_{i\in Z^+}$ i.i.d. but R_i can depend on X_i. (X,R) denote a cycle.

Thm. (RRT)

For a positive recurrent renewal process where a reward R_i is earned during a cycle with length X_i. If $E(|R_i|) < \infty$.

Then: $\lim_{t\to\infty} \frac{R(t)}{t} = E(R)/E(X)$. a.s.

$\lim_{t\to\infty} \frac{E(R(t))}{t} = E(R)/E(X)$.

Pf: 1°) Suppose $R_i \geq 0$. or set $R_i = R_i^+ - R_i^-$.

Note: $\frac{1}{t}\sum_1^{N(t)} R_i \leq \frac{R(t)}{t} \leq \frac{1}{t}\sum_1^{N(t)+1} R_i$.

Apply ERT.

2°) $|R(t)|/t \leq Y(t) = \frac{1}{t}\sum_1^{N(t)+1} |R_i| \xrightarrow{a.s.} E|R|/E(X)$

$\Rightarrow |R(t)|/t$ is bdd a.s. By DCT.

<u>Application:</u> We can see some byproduct as "reward" to calculate its long time rate by RRT.

i) Set $R(t) = \int_0^t A(s)ds$. conti reward.

$$R_j = \int_{t_{j-1}}^{t_j} A(s)ds. = \int_{t_{j-1}}^{t_j} (t_j - s) ds$$

$$\overset{s=t_{j-1}+u}{=} \int_0^{X_j} (X_j - u)\, du = \frac{X_j^2}{2}$$

$\Rightarrow$ By RRT: $\frac{1}{t}\int_0^t A(s)ds \xrightarrow{a.s.} E(\frac{X^2}{2})/E(X)$.

ii) Similarly. $R(t) = \int_0^t B(s)ds$.

iii) $R(t) = \int_0^t I_{\{A(s)>x\}} ds$. $R_i = \int_0^{X_i} I_{\{X_i - s > x\}} ds$

(5) <u>Central Limit Thm</u>:

consider renewal process $\{t_n\}$ with interarrival times $X_n = t_n - t_{n-1}$. s.t. $E(X) = \frac{1}{\lambda}$. $Var(X) = \sigma^2$.

<u>Thm.</u> (CLT for counting process)

$$Z(t) = \frac{N(t) - \lambda t}{\sigma\sqrt{\lambda^3 t}} \xrightarrow[d]{t\to\infty} Z \sim N(0,1).$$

<u>Rmk:</u> It implies: $E(N(t)) \sim \lambda t$. $Var(N(t)) \sim \sigma^2\lambda^3 t$

<u>Pf:</u> The key is : $p(N(t) < n) = p(t_n > t)$.

set $n(t,x) = [\lambda t + x\sqrt{\sigma^2\lambda^3 t}]$.

$\Rightarrow p(Z(t) < x) = p(N(t) < n(t,x))$

$= p(t_{n(t,x)} > t)$

$$= P\left(\frac{t_{N(t,x)} - N(t,x)/\lambda}{\sigma\sqrt{N(t,x)}} > \frac{t - N(t,x)/\lambda}{\sigma\sqrt{N(t,x)}} \right)$$

Note by CLT. $t_n - n/\lambda \,/\, \sqrt{n}\,\sigma \xrightarrow{d} Z \sim N(0,1)$.

So : $P(Z(t) < x) \xrightarrow{t\to\infty} P(Z > -x) = P(Z < x)$

Rmk: We don't need "Renewal" actually, but require : $\psi = \{t_n\}$ satisfies CLT.

(6) Delayed Renewal Process:

Note that the counting process $N(t)$ of renewal process $\psi = \{t_n\}$ generally doesn't have stationary increment like poision process.

Def: ψ_s is a shift by time s version of point process $\psi = \{t_n\}$. which is $\{t_{N(s)}\}$. moving the origin to be $t = s$. with counting process:

$N_s(t) = N(s+t) - N(s)$.

Rmk: $\psi_s \overset{d}{\sim} \psi$. $\forall s \geq 0 \iff N(t)$ has stationary increment.

Def: A delayed renewal process is a renewal process where the first arrivial time $t_1 = X_1$ indeptly has a different dist. $\sim F_1$

For $(X_n)_{n\geq 2}$ interarrival times. are i.i.d. $\sim F$

Rmk: i) ERT remains valid. F_1 doesn't need to have finite first moment.

ii) Ψ is stationary renewal process $\Leftrightarrow$ $t_1(s) = A(s)$ of Ψ_s has the same dist. $\forall s \geq 0$.

As $s \to \infty$. Ψ_s has a limiting dist. of the delayed version Ψ. denoted by $\Psi^* = \{t_n\}_{n\geq 1}$. $t_1^* \sim F_e$. With its counting process $N^*(t)$. forward recurrent time process $\{A^*(t)\}_{t\geq 0}$.

<u>Def:</u> Ψ^* is stationary version of Ψ.

<u>Rmk:</u> Ψ^* is stationary renewal process:

$$p(A^*(n) \leq x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t p(A(s+n) \leq x)\,ds$$

$$= \lim_t \frac{1}{t}\int_n^{t+n} p(A(s) \leq x)\,ds = F_e(x)$$

$$\Rightarrow A^* \sim F_e. \text{ i.e. } t_1^* \sim F_e.\ \Psi_s^* = \Psi^*.$$

<u>prop.</u> Stationary version Ψ^* of renewal process with rate $\lambda = E(X)^{-1}$ satisfies: $E(N^*(t)) = \lambda t$. and $\lambda = E(N^*(1))$.

<u>Pf:</u> Denote $u(t) = E(N^*(t)) \Rightarrow u(n) = n u(1)$. $\forall n$.

So: $u(t) = t\,u(1)$. With $u(t)/t \to \lambda$. so $u(1) = \lambda$.

follows from ERT. ($\lim_{t\to\infty}\lim_{s\to\infty} \square = \lim_{s\to\infty}\lim_{t\to\infty} \square$)

(7) <u>Renewal Equations:</u>

For renewal process $\Psi = \{t_n\}$ with $p(X \leq y) = F(y)$.

Note: $p(A(t)>x) = p(A(t)>x \mid X_1>t)\, p(X_1>t) + \int_0^t p(\square \mid X_1=s)\, p(X_1=s) ds$

$$= p(X_1 - t > x) + \int_0^t p(A(t-s)>x)\, dF(s)$$

$$= \bar{F}(t+x) + \int_0^t p(A(t-s)>x)\, dF(s)$$

Denote: $\bar{F}(x+t) = a(t)$. $H(t) = p(A(t)>x)$.

$\Rightarrow H(t) = a(t) + H * F(t)$. renewal equation.

By Iteration: $H(t) = a + (a + H*F)*F$

$$= \cdots = a + \sum_{n\geq 1} a * F^{*n}.\ H \text{ is unknown}$$

prop. $m(t) = E(N(t))$ satisfies: $H(t) = a + a * m(t)$. (*)

Pf: $E(N(t)) = E(\sum_1^\infty I_{\{t\geq t_n\}}) = \sum F^{*n}$.

since $F^{*n}(t) = p(\sum_1^n X_i \leq t) = p(t_n \leq t) \xrightarrow{n\to\infty} 0$

Thm. (Key renewal Thm)

If the renewal equation holds for given non-lattice F with mean $1/\lambda$ and a is DRI (directly Riemann integrable, i.e. $\int_0^\infty a(t)dt$ exists)

Then solution for (*) holds. $\lim_{t\to\infty} H(t) = \lambda \int_0^\infty a(t) dt$.

Rmk: There remain valid for delayed case:

in sense. $F_0 = F$. F_1 is delay X_1's dist. then.

$H_1 = a_1 + H_0 * F_1$. $H_0 = a_0 + a_0 * m_0$. $m_0 = E(N(t))$

With $H_1 = a_1 + a_0 * m_1$. $m_1 = E(N_1(t))$

And $\lim_{t\to\infty} a_0 * m_1 = \lim_{t\to\infty} a_0 * m_0 = \lambda \int_0^\infty a_0(t) dt$.

Pf: It's equi. with Blackwell's Thm