Likelihood Ratios

It's from Radon-Nikodym Thm. — change measure.

e.g. in $(\Omega, \mathcal{F})$. P_1, P_2 are prob. measure s.t. $P_1 \ll P_2$.

Then for r.v. X: $E_{P_1}(X) = E_{P_2}(XL)$. $L = \frac{dP_1}{dP_2}$

Applications:

① Monte Carlo:

To estimate the prob of an event. The accuracy depends on the frequency of the event.

e.g. For $X \overset{P_0}{\sim} N(0,1)$: $P_0(X > 2.5)$ is small. So $\{X > 2.5\}$ seldom occurs under P_0. but if consider:

$E_0(I_{\{X>2.5\}}) = E_{2.5}(L(X) I_{\{X>2.5\}})$.

$$L(x) = f_0(x) / f_{2.5}(x) = \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}}{\frac{1}{\sqrt{2\pi}} e^{-(x-2.5)^2/2}} = e^{-2.5x + \frac{25}{8}}$$

$X \sim N(2.5, 1)$ under $dP_{2.5} = f_{2.5}\, dx$.

$\Rightarrow$ Estimated by: $\sum_1^n L(X_i) I_{\{X>2.5\}}(X_i)/n$.

② Gamble's Ruin:

Suppose $X_k \overset{i.i.d}{\sim} N(\mu_1, 1)$. $S_n = \sum_1^n X_k$. $\mu_1 < 0$

$\tau_b = \min\{n \mid S_n > b\}$.

prop. $P_{\mu_1}(\tau_b < \infty) \le e^{-2\mu_1 b}$.

Pf: Note RHS $= P_{\mu_1}(\tilde{\tau}_b < \infty)$. $\tilde{\tau}_b = \inf\{t \mid X(t) = b\}$.

where $X(t) = \mu_1 t + W(t)$.

Next, prove: $P_{\mu_1}(\tau_b = \infty) \ge P_{\mu_1}(\tilde{\tau}_b = \infty)$

Note $X(n) \sim S_n \sim N(n\mu_1, n)$ $n \in Z^+$.

$$\text{RHS} = P(\forall t \ge 0,\ X(t) < b) \le P_{\mu_1}(\forall n \in Z^+,\ X(n) < b)$$
$$= P_{\mu_1}(\forall n \in Z^+,\ S_n < b)$$
$$= P_{\mu}(\tau_b = \infty)$$

Note: $X_k \overset{P_1}{\sim} N(\mu_1, 1)$, $X_k \overset{P_2}{\sim} N(\mu_2, 1)$. Then: we have,

$$L(\vec{x}) = dP_1/dP_2 = e^{-\frac{1}{2}\sum_1^n (x_i - \mu_1)^2} / e^{-\frac{1}{2}\sum_1^n (x_i - \mu_2)^2} \quad \text{for } \vec{x} = (x_1 \cdots x_n)$$

$$P_{\mu_1}(B(X_1 \cdots X_n)) = E_{\mu_2}(L(\vec{x}) I_B).$$

Next, we will extend "n" to stopping time τ:

Def. i) $\mathcal{F}_n = \{Y \mid Y = h(X_1, \cdots X_n)\}$.

ii) $\mathcal{F}_\tau = \{Y \mid Y I_{\{\tau = n\}} \in \mathcal{F}_n, \forall n\}$. τ is a stopping time.

Thm. (Wald's LR Identity)

$Y \in \mathcal{F}_\tau$, bdd. If $(X_1 \cdots X_n) \overset{P_i}{\sim} f_i^n$, $i = 1, 2$.

$L_n = f_1^n / f_2^n$. Then: $E_1(Y I_{\{\tau < \infty\}}) = E_2(Y L_\tau I_{\{\tau < \infty\}})$

Pf:
$$\text{LHS} = \sum E_1(Y I_{\{\tau = n\}})$$
$$= \sum E_2(Y I_{\{\tau = n\}} L_n)$$
$$= \sum E_2(Y I_{\{\tau = n\}} L_\tau)$$
$$= E_2(Y L_\tau I_{\{\tau < \infty\}})$$

i) Apply on τ_b. Set $\mu_2 = -\mu_1$. $B = \{\tau_b < \infty\}$.

Then: $P_{\mu_1}(\tau_b < \infty) = E_2(e^{2\mu_1 S_{\tau_b}} I_{\{\tau_b < \infty\}})$

$\leq e^{2\mu_1 b}$ directly.

ii) Consider $T = \tau_a \wedge \tau_b$. $B = \{S_T > b\}$. $\mu_2 = -\mu_1$

Then: $P_{\mu_1}(S_T > b) = E_{\mu_2}(e^{2\mu_1 S_T} I_{\{S_T > b\}})$

$= e^{2\mu_1 b} E_{\mu_2}(e^{2\mu_1 R_b} I_{\{S_T > b\}})$

$R_b = S_{\tau_b} - b$. $e^{2\mu_1 b}$ is small which will influence accurracy if we want to simulate S_T.

③ Brownian Motion:

Def: $\mathcal{F}_\tau = \{ Y \mid Y I_{\{\tau \leq t\}} \in \mathcal{F}_t . \forall t \geq 0\}$.

$\mathcal{F}_t = \{ Y \mid Y = h(\langle W \rangle_0^t)\}$. for process $(W_t)_{t \geq 0}$.

For $0 = t_0 < t_1 < \cdots < t_n = t$. if $W_t \overset{P_i}{\sim} N(\mu_i t, t)$. $i = 1, 2$.

$P_{\mu_1}(W(t_i) \in dw_i.\ 1 \leq i \leq n) / P_{\mu_2}(W(t_i) \in dw_i.\ 1 \leq i \leq n)$

$= \exp(-\frac{1}{2}\sum_1^n \frac{1}{t_i - t_{i-1}}(w_i - w_{i-1} - \mu_1(t_i - t_{i-1}))^2) / \square_{\mu_2} = e^{(\mu_1 - \mu_2)W_t - \frac{t}{2}(\mu_1^2 - \mu_2^2)}$

To obtain L_t. set $n \to \infty$ to approxi. by finite set

$\Rightarrow L_t = \exp((\mu_1 - \mu_2)W(t) - \frac{t}{2}(\mu_1^2 - \mu_2^2))$

prop. $Y \geq 0$. $Y \in \mathcal{F}_\tau$. $E_{\mu_1}(Y I_{\{\tau < \infty\}}) = E_{\mu_2}(Y L_\tau I_{\{\tau < \infty\}})$

Pf: Set $\tau_n = [2^n \tau] + 1 / 2^n$. Obtain equation

Let $n \to \infty$ by MCT. $(Y \geq 0)$

i) For $b>0$. $\mu>0$. $\mu_1=-\mu=-\mu_2$. $\forall Y\in\mathcal{F}_{\tau_b}$.

$$E_{-\mu}(Y I_{\{\tau_b<\infty\}}) = E_{\mu}(Y e^{-2\mu W(\tau_b)} I_{\{\tau_b<\infty\}})$$

$$= e^{-2\mu b} E_{\mu}(Y I_{\{\tau_b<\infty\}})$$

By $P_\mu(\tau_b<\infty)=1$ $(\mu>0)$. RHS $= e^{-2\mu b}E_\mu(Y)$.

$\Rightarrow E_{-\mu}(Y \mid \tau_b<\infty) = E_{-\mu}(Y I_0)/P_{-\mu}(\tau_b<\infty) = E_\mu(Y)$

Set $Y = \{\tau_b\le t\}$. So: $P_{-\mu}(\tau_b\le t \mid \tau_b<\infty) = P_\mu(\tau_b\le t)$

i.e. drift $-\mu$ of $\tau_b \mid \tau_b<\infty \overset{d}{\sim}$ drift μ of τ_b

ii) From i). and reflection principle. We have:

$P_{-\mu}(\tau_b\le t) = 1-\Phi\left(\frac{b+\mu t}{\sqrt t}\right) + e^{-2\mu b}\,\Phi\left(\frac{-b+\mu t}{\sqrt t}\right)$. It's called inverse Gaussian dist. ($W_t \overset{P_{-\mu}}{\sim} N(-\mu t, t)$)

LHS $= P_{-\mu}(\tau_b\le t, W_t\ge b) + P_{-\mu}(\tau_b\le t, W_t\le b)$

1°) $P_{-\mu}(\tau_b\le t, W_t\ge b) = P_{-\mu}(W_t\ge b) = 1-\Phi\left(\frac{b+\mu t}{\sqrt t}\right)$

2°) $P_{-\mu}(\tau_b\le t, W_t\le b) = \int_0^t P_{-\mu}(W_t\le b \mid \tau_b=s)\, P_{-\mu}(\tau_b\in ds)$

$$P_{-\mu}(W_t\le b \mid \tau_b=s) \overset{MP}{=} P_{-\mu}(W_t\le b \mid W_s=b)$$

$$= P(N(-\mu(t-s), t-s)\le 0)$$

$$= P(N(\mu(t-s), t-s)\ge 0)$$

$$\overset{MP}{=} P_\mu(W_t\ge b \mid \tau_b=s)$$

Combined with: $P_{-\mu}(\tau_b\in ds) = e^{-2\mu b} P_\mu(\tau_b\in ds)$

follows from i).