Poisson Process

(1) Renewal Process:

Def. i) $\psi = \{t_n\}_{n\geq 1}$ is simple point process. st.

$0 < t_1 < t_2 \cdots < t_n < \cdots$, $t_n \to \infty$ $(n \to \infty)$.

ii) $\{N(t)\}_{t\geq 0}$ is counting process. st.

$N(t) = \max\{n \mid t_n \leq t\}$.

Rmk: $X_n = t_n - t_{n-1}$ is n^{th} interval time.

Def: A point process $\psi = \{t_n\}_{n\in\mathbb{Z}^+}$ is renewal process

if $(X_n)_n = (t_n - t_{n-1})_n$ is i.i.d.

Thm. For renewal process $\psi = \{t_n\}_{n\in\mathbb{Z}^+}$. We have:

$$\begin{cases} \lim\limits_{t\to\infty} N(t)/t = \lambda.\ \text{w.p.1} \\ \lim\limits_{t\to\infty} E(N(t))/t = \lambda. \end{cases} \quad \text{where } \lambda = 1/E(X_n).$$

Pf: 1°) By SLLN: $\sum_1^{N(t)} X_k / N(t) \leq t/N(t) \leq \sum^{N(t)+1} X_k / N(t)$.

2°) prove: $N(t)/t$ is u.i.

Truncate: $\hat{X}_n = a I_{\{X_n \geq a\}} \leq X_n$. $\therefore \hat{N}(t) \geq N(t)$

where a is choosen: $P(X_n \geq a) \triangleq p \in (0,1)$

Note: Arrival occur only at na. $n \in \mathbb{Z}^+$.

Denote k_n is number of arrivals at na.

(i.e. contain the arrival spent time $< a$)

$\therefore k_n \sim Geo(p)$.

(Times spent for success = arrival times)

$\therefore N(t) \leq \hat{N}(t) \leq \sum^{\lceil t/a \rceil} k_n = S(t)$. $E(S(t)^2) = O(t^2)$.

$\therefore E(N^2(t)/t^2 I_{\{N(t) \geq tc\}}) \leq E(S^2(t)/t^2) P(\frac{N(t)}{t} \geq c)$

$\leq \frac{A}{c^2} \to 0$. (Chebyshev)

(2) Poisson Point Process:

Def: Poisson process with rate $\lambda \in (0, \infty)$ is a renewal process $\varphi = \{t_n\}_{n \in \mathbb{Z}^+}$. s.t. $X_n \sim Exp(\lambda)$.

① Exponential Dist.:

i) It's memoryless: $P(X > a+b \mid X > a) = P(X > b)$ for $a, b \geq 0$.

Thm. Nonnegative, nondegenerated r.v. X is memoryless $\Leftrightarrow X \sim Exp(\lambda)$.

Pf: Set $g(x) = P(X > x)$. $\therefore g(x+y) = g(x)g(y)$

$\therefore g(n) = g^n(1)$. $g(\frac{n}{m}) = g(\frac{1}{m})^n = g(1)^{\frac{n}{m}}$

Approxi. $\mathbb{R}'$ by $\mathbb{Q}$: $g(x) = g(1)^x$

Rmk: For discrete case:

$P(X > n+k \mid X > k) = P(X > n)$. $\forall n, k \in \mathbb{Z}^+$.

$\Leftrightarrow X \sim Geo(p)$. for $p \in (0,1)$.

Pf. $P(X > n) = P(X > 1)^n \stackrel{\Delta}{=} p^n$

$\Rightarrow P(X = n) = p^{n-1}(1-p)$

ii) Approxi:

Thm. $X_n \sim Geo(p_n)$. $\lim_n np_n = \lambda$. $\Rightarrow X_n/n \rightarrow_d X \sim Exp(\lambda)$.

Pf: $$\varphi_{X_n/n}(t) = \frac{p_n}{1-(1-p_n)e^{it/n}}$$

$$= \frac{np_n}{(1-e^{it/n})/\frac{1}{n} + np_n e^{it/n}} \longrightarrow \frac{\lambda}{\lambda - it}$$

iii) n-folds:

Thm. $t_n = \sum_1^n X_k \sim Gamma(n, \lambda)$. if $X_k \sim Exp(\lambda)$. i.i.d.

Pf: $f = \lambda e^{-\lambda t}$. $f^{*n} = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}$

iV) Combination:

prop. For $X_1 \sim Exp(\lambda_1)$. $X_2 \sim Exp(\lambda_2)$. $Z = \min\{X_1, X_2\}$.

(a) $Z \sim Exp(\lambda_1 + \lambda_2)$

(b) $Z \mid X_1 < X_2 \sim Z \mid X_1 > X_2 \sim Z$.

(c) $P(X_1 > X_2) = \lambda_1/(\lambda_1+\lambda_2)$. $P(X_1 < X_2) = \lambda_2/(\lambda_1+\lambda_2)$

Rmk: Z means X_1. X_2 work simutanuously.

② Poisson Approxi.:

Thm. $\{X_{n,k}\}_{1 \le k \le n}$ indept. $P(X_{n,k} = 1) = p_{n,k}$. $P(X_{n,k} = 0) = 1 - p_{n,k}$

$S_n = \sum_{k=1}^n X_{n,k}$. If $\sum_{k=1}^n p_{n,k} \longrightarrow \lambda \in (0, \infty)$ $(n \to \infty)$. and

$\max_{1 \le k \le n} p_{n,k} \to 0$ Then $S_n \longrightarrow_d Poisson(\lambda)$.

Pf: $\varphi_{S_n}(t) = \prod_1^n (1 + P_{n,k}(e^{it}-1))$

$\ln(1+P_{n,k}(e^{it}-1)) = P_{nk}(e^{it}-1) + \theta_k |P_{nk}(e^{it}-1)|^2$

where $|\theta_k| \le 1$. Since for large n: $|P_{nk}(e^{it}-1)| \le \frac{1}{2}$

Check $\sum_1^n \ln(1+P_{nk}(e^{it}-1)) \longrightarrow \lambda(e^{it}-1)$.

Cor. For $\{X_{n,k}\}$ indept. nonnegative. $P(X_{n,k}=1) = P_{nk}$.

$P(X_{n,k} \ge 2) = \varepsilon_{nk}$. $S_n = \sum_k^n X_{n,k}$.

If $\sum P_{n,k} \to \lambda$. $\max_k P_{n,k} \to 0$. $\sum_k^n \varepsilon_{n,k} \to 0$. Then

$S_n \xrightarrow{d}$ Poisson (λ).

Pf: Truncate: $X_{nk}' = X_{nk} I\{|X_{nk}| \le 1\} \le X_{n,k}$

$\therefore S_n' = \sum X_{nk}' \xrightarrow{d}$ Poisson (λ).

Check $\forall \varepsilon > 0$. $P(|S_n - S_n'| \ge \varepsilon) \to 0 \ (n\to\infty)$

(3) Characterization:

Def: i) Point process $\varphi = \{t_n\}$ has stationary increment if $N(t+s) - N(t)$ indept with t. but on s.

ii) Point process $\varphi = \{t_n\}$ has indept increment if $\forall I_1 \cap I_2 = \emptyset$. $N(I_1)$ indept with $N(I_2)$.

① **Thm.** For $\varphi = \{t_n\}$ point process with $N(t)$:

i) It's indept increment

ii) It's stationary increment.

iii) It's sparse: $P(N(0,h)=1) = \lambda h + o(h)$. $P(N(0,h) \ge 2) = o(h)$

Then $N(0,t) \sim \text{Poisson}(\lambda t)$

<u>Pf</u>: Denote $X_t = N(0,t) = \sum_{k=1}^{r} (X_{\frac{kt}{n}} - X_{\frac{(k-1)t}{n}}) \triangleq \sum_k X_{nk}$

Check P_{nk}, ε_{nk} satisfies approxi. condition.

Then Let $n \to \infty$. $\therefore X_t \sim \text{Poisson}(\lambda t)$.

<u>Rmk</u>: Directly. $P(N(t) \leq k) = P(t_k \geq t)$

$$= \int_t^{+\infty} \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} \lambda t = e^{-\lambda t}(1 + \lambda t + \cdots)$$

if $\{t_n\}$ is poisson process with rate λ.

<u>Thm.</u> (little $o(t)$)

If $\{t_n\}$ is Poisson process. Then $P(N(t) > 0) = \lambda t + o(t)$.

$P(N(t) > 1) = o(t)$. $\lambda = E(N(1))$. $(t \to 0)$

<u>Pf</u>. $P(N(t) > 0) = 1 - e^{-\lambda t} = \lambda t + o(t)$

$P(N(t) > 1) = e^{-\lambda t}(\frac{(\lambda t)^2}{2} + \cdots) = o(t)$.

<u>Rmk</u>. Check: $N(t+s) - N(t) \sim N(s)$. And by indept of $\{X_n = t_n - t_{n-1}\}$. So a Poisson process also satisfies characterization i), ii), iii).

(Patition by arrival time $\{t_n\}$. $I \cap \{t_n\} = \emptyset$?)

② <u>Generating Func.</u>

<u>Denote</u>: $G_{N(t)}(z) = E(z^{N(t)})$, $|z| \leq 1$. g.f of $N(t)$ counting process.

Note: $\frac{1}{\Delta t}(G_{N(t+\Delta t)} - G_{N(t)}) =$

$$\frac{1}{\Delta t} E(z^{N(t)}(z^{N(t+\Delta t)-N(t)} - 1))$$

$$\overset{\text{indept}}{=} \frac{1}{\Delta t} E(z^{N(t+\Delta t)-N(t)} - 1) E(z^{N(t)})$$

$$= \frac{1}{\Delta t} E(z^{N(\Delta t)} - 1) E(z^{N(t)}).$$

$$E(z^{N(\Delta t)} - 1) = p(N(\Delta t)=0) - 1 + p(N(\Delta t)=1)z + \sum_{k\geq 2} p(N(\Delta t)=k)z^k$$

1°) $p(N(\Delta t)=0) = p(N(\Delta s)=0)\, p(N(\Delta t-\Delta s)=0)$

$$\Rightarrow \exists \lambda.\ p(N(\Delta t)=0) = e^{-\lambda\Delta t}.$$

2°) $p(N(\Delta t)=1)z + \sum_{k\geq 2} p(N(\Delta t)=k) z^k$

$$= p(N(\Delta t)=1)\left(z + \sum_{k\geq 2} \frac{p_k}{p_1} z^k\right)$$

$$\left|\sum_{k\geq 2} \frac{p_k}{p_1} z^k\right| \leq \sum_{k\geq 2}^{\infty} \frac{p_k}{p_1} |z|^k \leq \frac{p(N(\Delta t)\geq 2)}{p(N(\Delta t)=1)}$$

$$\rightarrow 0\ (\Delta t \rightarrow 0) \text{ by sparsity (little } o(t))$$

$$\Rightarrow LHS = \frac{1}{\Delta t}(e^{-\lambda\Delta t} - 1) + G_{N(t)} \frac{(z + o(\Delta t))\, p_1}{\Delta t}$$

$$\rightarrow (\lambda z - \lambda) G_{N(t)}(z).\quad (\Delta t \rightarrow 0)$$

i.e. we obtain: $G'_{N(t)}(z) = (\lambda z - \lambda) G_{N(t)}(z)$.

$$\Rightarrow G_{N(t)}(z) = G_{N(0)}(z)\, e^{-(\lambda+\lambda z)t} = e^{-(\lambda+\lambda z)t}.$$

<u>Rmk</u>: Commonly. Set $\lim_{\Delta t\to 0} p(N(\Delta t)=1)/\Delta t = \lambda$.

<u>Modification</u>:

i) <u>Ignore "Stationary increment"</u>:

Then: $E(z^{N(t+\Delta t)-N(t)} - 1) \neq E(z^{N(\Delta t)} - 1)$

$E(z^{N(t+\Delta t)-N(t)} - 1) = P(\Delta N = 0) - 1 + z P(\Delta N = 1) + \sum_{k\geq 2} P(\Delta N = k) z^k.$

Add assumption: $\lim_{\Delta t \to 0} \frac{P(N(t+\Delta t) - N(t) = 0) - 1}{\Delta t} = -\lambda(t).$

$\Rightarrow \lim_{\Delta t \to 0} P(\Delta N = 1)/\Delta t = \lambda(t)$. So: $G_{N(t)}(z) = e^{(z-1)\int_0^t \lambda(s)ds}$.

ii) Ignore "Sparsity":

Assume: $P(N(\Delta t) = k)/P(N(\Delta t) = 1) \xrightarrow{\Delta t \to 0} p_k$. $P(N(\Delta t) \geq 1)/\Delta t \xrightarrow{\Delta t \to 0} \lambda$.

Then: $\frac{1}{\Delta t} E(Z^{N(\Delta t)} - 1) = \frac{P(N(\Delta t) = 0) - 1}{\Delta t} + \frac{P(N(\Delta t) \geq 1)}{\Delta t} \left(\sum_{k\geq 2} \frac{P(N(\Delta t) = k)}{P(N(\Delta t) \geq 1)} Z^k \right)$

$$\xrightarrow{\Delta t \to 0} -\lambda + \lambda \sum_{k\geq 1} p_k z^k$$

$\Rightarrow G_{N(t)}(z) = e^{\lambda t (\sum_{k\geq 1} p_k z^k - 1)}$.

iii) Ignore "Indept increment":

Consider compound Poisson process $Y_t = \sum_1^{N(t)} X_k$.

1') If $X_k \overset{i.i.d}{\sim} F_X$. indept with $N(t)$:

$G_{Y(t)}(z) = E(E(Z^{\sum_1^n X_k} \mid N(t) = n)) = E(G_X(z)^{N(t)})$

$= e^{\lambda(G_X(z) - 1)}$.

2') To remove "indept". Consider $Y_t = \sum_1^{N(t)} X_k(t, S_k)$

S_k is r.v. the k^{th} arrival time. which is called filtering Poisson process. Calculate its ch.f:

$\phi_{Y(t)}(z) = E(e^{izY(t)})$,

$= E(E(e^{iz\sum_1^n X_k(t,S_k)} \mid S_1, S_2 \cdots S_n, N(t) = n))$

Denote: $B_k(t, S_k) = E_{X_k}(e^{izX_k(t,S_k)})$.

$$\Rightarrow LHS = E_N(E_S (\prod_1^n B_k(t, S_k) \mid N(t) = n))$$

$$= E_N (\int_{0 \le s_1 \le \cdots \le s_n \le t} \frac{n!}{t^n} \prod_1^n B_k(t, S_k) \, ds_1 \cdots ds_n)$$

$$\overset{(Sym)}{=} E_N (\frac{1}{n!} \int_{[0,t]^n} \frac{n!}{t^n} \prod_1^n B_k(t, S_k) \, d\vec{S})$$

$$= E_N ((\frac{1}{t} \int_0^t B(t,s))^{N(t)}) = e^{\lambda (\int_0^t B(t,s) ds - t)}$$

follows from $(S_k)_1^n \mid N(t) = n \overset{d}{\sim} \{ X_{(k)} \}_1^n$ order stat.

(4) <u>Partition:</u>

① <u>Bernoulli Trails:</u>

Consider at time t_n. Do a "Bernoulli Trail", which generates object of type 1 with prob. p type 2 with prob $1-p$ indeptly in $\varphi = \{t_n\}$ Poisson Process.

<u>Thm.</u> If $X \sim$ Poisson (α). X_1 is number of objects of type 1. X_2 is number of type 2. Then $X_1 + X_2 = X$. $X_1 \sim$ Poisson $(p\alpha)$. $X_2 \sim$ Poisson $((1-p)\alpha)$. X_1, X_2 indept.

<u>Pf.</u> To show: X_1, X_2 are Poisson dist.

Note: $p(X_1 = k, X_2 = m) = p(X_1 = k, X = k+m)$

$$= p(X_1 = k \mid X = k+m) \, p(X = k+m)$$

$$= \frac{(p\alpha)^k}{k!} e^{-p\alpha} \frac{(q\alpha)^m}{m!} e^{-q\alpha}$$

where $q \triangleq 1-p$.

Thm. $\Psi = \{t_n\}$. λ-Poisson process. Ψ_i is point process of type i arrivals. $i=1,2$. Then Ψ_1 is $p\lambda$-Poisson process. Ψ_2 is $q\lambda$-Poisson process. indept each other.

Pf: Ψ_i are poisson process by Charac. and the parameter is from $X = X_1 + X_2$. Thm above.

For indept. consider $(A_i)_1^n$. $(B_j)_1^m$ two collections of disjoint intervals.

Show: $(X_1(A_1) \cdots X_1(A_n))$ indept of $(X_2(B_1) \cdots X_2(B_m))$

For A_i: $X_1(A_i \cap B_j)$ indept of X_2 by partition indeptly

$X_1(A_i - B_j)$ indept of X_2 by indept increment

$\Rightarrow X_1(A_i) = X_1(A_i \cap B_j) + X_1(A_i - B_j)$ indept with $X_2(B_j)$

for $\forall j \in \{1, 2, \cdots m\}$.

② Superposition:

We can put $\Psi_1 \sim Poi(\lambda_1)$. $\Psi_2 \sim Poi(\lambda_2)$ indept processes together to obtain $\Psi = \Psi_1 + \Psi_2$. So $N_t = N_1(t) + N_2(t)$.

$\Rightarrow \Psi \sim Poisson(\lambda_1 + \lambda_2)$. by Charac.

Besides, denote $Y_i \sim exp(\lambda_i)$. the arrival time of type i. $i=1,2$. Then the partition prob of type 1 is: $p(Y_1 < Y_2) = \lambda_1/(\lambda_1+\lambda_2)$. of type 2 is $\lambda_2/(\lambda_1+\lambda_2)$.

(5) Construct a Poisson Process:

Fix $t>0$. Consider $u_k \overset{i.i.d}{\sim} U[0,t]$. $1 \le k \le n$.

1°) Note: φ is Poisson(λ) Process. for $s \le t$:

$$P(t_1 \le s \mid N(t)=1) = \frac{P(N(s)=1,\ N(t)-N(s)=0)}{P(N(t)=1)} = \frac{s}{t}$$

2) Recall: $(u_{(1)}, \cdots u_{(n)}) \sim g = n!/t^n$. order. stat.

<u>prop.</u> $(t_1, t_2 \cdots t_n) \mid N(t)=n \overset{d}{\sim} (u_{(1)}, \cdots u_{(n)})$

for $\varphi = \{t_n\}$ poisson process.

<u>Pf:</u> Note: $\{ t_i = s_i .\ 1 \le i \le n.\ N(t)=n \} =$

$\{ X_1 = s_1,\ X_2 = s_2 - s_1 \cdots X_n = s_n - s_{n-1}.\ X_{n+1} > t - s_n \}$

where $X_i \overset{i.i.d}{\sim} \exp(\lambda)$.

$\Rightarrow$ To simulate a Poisson process up to time t:

First simulate the value of $N(t)$. if $N(t) = n$. then: generate n i.i.d $U[0,t]$. r.v's: $(u_1 \cdots u_n)$. place them in $(u_{(1)}, \cdots u_{(n)})$.

(6) <u>Application:</u>

<u>Def:</u> $M/G/k$ is a queue model:

i) "M" means Markovinan. i.e. modulated by Poisson Process.

ii) "G" means service time has general distribution.

iii) "k" means there're k servers.

Next, consider $M/G/\infty$: Arrivals time $\{t_n\} \sim \text{Poi}(\lambda)$.

Service times $S_n \overset{i.i.d}{\sim} G(x)$ with mean μ^{-1}

<u>Denote:</u> $X(t)$ is number of customers in system at time t.

prop. $X(t) \sim \text{Poisson}(\alpha(t))$. $\alpha(t) = \int_0^t \lambda P(S > x)\,dx$.

Rmk: Since $\alpha(t) \xrightarrow{t\to\infty} \int_0^\infty \lambda P(S>x)\,dx = \lambda/\mu =: \ell$.

$\Rightarrow \lim_{t\to\infty} P(X(t)=n) = e^{-\ell}\,\ell^n/n!$. $\forall n \geq 0$.

Pf: n^{th} customer arrives at t_n. departs at $t_n + S_n$.

Denote $D(t)$ is number of departs at time t.

$\Rightarrow N(t) = X(t) + D(t)$ partitioned in 2 types.

Suppose U is r.v. of arrival time. S is service time.

A customer still in system at time t $\Leftrightarrow$

$U + S > t$. $\Leftrightarrow$ $S > t - U$.

So: $p(t) = P(S > t-U)$ is partition prob.

$U \sim U(0,t)$ since we know he arrives in $[0,t]$.

$\Rightarrow t - U \sim U(0,t)$. $\therefore p(t) = P(S > U)$

$= \int_0^t \frac{1}{t} P(S>x)\,dx$.

So: $\alpha(t) = \lambda t\, p(t) = \int_0^t \lambda P(S>x)\,dx$.

RMK: Partition in binomial trails: Under $N(t) = n$

$X(t) = \sum_1^n I_{\{S_i > t - U(i)\}} = \sum_1^n I_{\{S_i > t - U_i\}}$.

Consider "stay" is "1". "depart" is "0".

Then $X(t) \mid N(t) = n \sim B(n, P(U+S>t))$