Q - Process.

Def: We say a matrix Q is a Q-matrix if $0 \le q_{ij} < \infty$. $i \neq j$. $\sum_{i \neq j} q_{ji} \le -q_{jj} =: q_j$

Moreover, if $\sum_{i \in E} q_{ji} = 0$. Then we say Q is conservative.

Rmk: If generator of CTMC is a Q-matrix. and state space of CTMC is finite. then Q is conservative.

Pf: $\lim_{t \to 0} \frac{1 - \sum_j P_{ij}(t)}{t} = \sum_{j \in E} q_{ij} = 0$.

Rmk: Note that generator of every CTMCs is conservative. (By MCT)

Given a Q-matrix. A CTMC has Q as a generator is called Q-process.

Rmk: It's well-def:

Lemma. $\lim_{t \downarrow 0^+} \frac{1 - P_{ii}(t)}{t} = \sup_{t > 0} \frac{1 - P_{ii}(t)}{t} =: q_i$.

$\lim_{t \to 0+} \frac{P_{ij}(t)}{t} = q_{ij}$ exists. and

$\sum_{j \neq i} q_{ij} \le q_i$ for trans. func. $(P_{ij}(t))$.

Pf: It follows from Frkete Lemma.

Note that $p_{ij}(t+h) \geq p_{ii}(t)p_{ii}(h)$

$\Rightarrow$ So generator matrix of CTMC exists.

With Fatou's Lemma. It's Q-matrix.

Recall definition of transition functions:

i) $p_{ij}(t) \geq 0$. ii) $\sum_{j \in E} p_{ij}(t) = 1$. iii) $\lim_{t \downarrow 0^+} p_{ii}(t) = 1$.

Thm. $\sum_{j \in E} | p_{ij}(t+h) - p_{ij}(t) | \leq 2 | 1 - p_{ii}(h) |$.

Pf: $p_{ij}(t+h) - p_{ij}(t) = \sum_{k \neq i} p_{ik}(h) p_{kj}(t) - p_{ij}(t)(1 - p_{ii}(h))$

$$\begin{cases} (p_{ij}(t+h) - p_{ij}(t))^+ \leq \sum_{k \neq i} p_{ik}(h) p_{kj}(t) \\ (p_{ij}(t+h) - p_{ij}(t))^- \leq p_{ij}(t)(1 - p_{ii}(h)) \end{cases}$$

Rmk: $(p_{ij}(\cdot))_j$ is "uniformly" uniform conti.

Thm $p_{ii}(t) > 0$. $\forall t > 0$. If $\exists t_0$. St. $p_{ij}(t_0) > 0$. $i \neq j$.

Then $p_{ij}(t) > 0$. $\forall t > t_0$.

Pf: $p_{ii}(t) \geq (p_{ii}(\frac{t}{n}))^n$. By def iii).

Thm. $\lim_{t \to \infty} p_{ij}(t) = \pi_{ij}$ exists.

Pf: Consider $(Y_n) = (X(nh))_{n\geq 0}$. fix $h>0$.

which has stat. dist. $\pi(h)$. $\forall h>0$.

Then by uniform conti. of $(P_{ij}(t))$.

Check it satisfies Cauchy seq. $(t\to\infty)$.

Thm. For $(X_t)_{t\geq 0}$ is conservative right-conti Q-process. s.t. $0\leq q_i<\infty$. $T_n = \inf\{t\geq T_{n-1} \mid X_t \neq X_{T_{n-1}}\}$

$T_0 = 0$. $Z_n = T_n - T_{n-1} I_{\{T_{n-1}<\infty\}}$. $Y_n = X_{T_n} I_{\{T_n<\infty\}} +$

$X_{T_{n-1}} I_{\{T_n=\infty\}}$. If $\mathbb{P}(\lim_{n\to\infty} T_n = \infty) = 1$.

Then: i) $X_t = \sum_{n=0}^{\infty} Y_n I_{\{T_n\leq t<T_{n+1}\}}$.

ii) Y_n is embedded DTMC. with

trans. prob. $r_{ij} = \begin{cases} \delta_{ij} & q_i = 0. \\ (1-\delta_{ij})\, q_{ij}/q_i. & q_i\neq 0 \end{cases}$

iii) $P(Z_1>t_1 \cdots Z_n>t_n \mid Y_0=i_0, \cdots Y_{n-1}=i_{n-1})$

$= \prod_{1}^{n} e^{-q_{i_{j-1}} t_j}$.

Pf: ii) By strong Markov prop. of CTMC.

$\Rightarrow (Y_n)$ is DTMC.

Set $Z_n = \dfrac{[2^n Z_1]+1}{2^n} \downarrow Z_1$

$P_i(X_{Z_1}=j) = \lim_{n\to\infty} P_i(X_{Z_n}=j)$

$= \lim_{n\to\infty} \sum_{k\geq 1} P_i(X_{\ell/2^n}=i,\ \ell<k,\ X_{k/2^n}=j)$

$$\overset{(MP)}{=} \lim \sum_{k\geq 0} (P_{ii}(\tfrac{1}{2^n}))^{k-1} P_{ij}(1/2^n) \overset{(q_i>0)}{=} \frac{q_{ij}}{q_i}$$

iii) By Strong Markov Property.

<u>Thm</u>: (Converse)

(Y_n) is DTMC with prob. trans. $R=(r_{ij})$.

(Z_n) seq of r.v.'s. s.t. $P(Z_1>t_1, \dots Z_n>t_n \mid Y_0=i_0, \dots Y_{n-1}=i_{n-1}) = \prod_1^n e^{-q_{i_{j-1}} t_j}$. $q: E \to \mathbb{R}^1$. set $T_0=0$.

$T_n = T_{n-1} + Z_n$. $X_t = \sum_{n\geq 0} Y_n I\{T_n \leq t < T_{n+1}\}$.

If $P(\lim_{n\to\infty} T_n = \infty) = 1$. Then $(X_t)_{t\geq 0}$ is

a Q-process. satisfies Kolomogorov. Back/

Forward equation:

$$\begin{cases} p_{ij}'(t) = \sum_{k\in E} q_{ik} P_{kj}(t). & P'(t) = QP(t). \\ p_{ij}'(t) = \sum_{k\in E} p_{ik}(t) q_{kj}. & P'(t) = P(t)Q. \end{cases}$$

i.e. $$\begin{cases} p_{ij}(t) = \delta_{ij} e^{-q_i t} + \sum_{k\neq i} \int_0^t q_{ik} e^{-q_i v} p_{kj}(t-v) dv \\ p_{ij}(t) = \delta_{ij} e^{-q_j t} + \sum_{k\neq j} \int_0^t p_{ik}(v) q_{kj} e^{-q_j(t-v)} dv \end{cases}$$

<u>Rmk</u>: $q_{ij} = -\delta_{ij} q_i + (1-\delta_{ij}) q_i r_{ij}$.

(1) <u>Regular Q-process</u>:

Def: A conservative Q-process (X_t) is regular if $\sum_{n\ge 1} q_{Y_n}^{-1} = \infty$ a.s. (Y_n) is its embed DTMC.

Rmk: i) $q = \sup_E q_i < \infty \Rightarrow (X_t)$ is regular

ii) (Y_n) is recurrent $\Rightarrow (X_t)$ is regular.

Thm: (X_t) is right-conti. conservation Q-process $q_i \in [0, \infty)$. Then (X_t) is regular $\Leftrightarrow$ $P(\lim_{n\to\infty} T_n = \infty) = 1$.

Lemma. $(Z_n)_{n\ge 0} \overset{\text{indept}}{\sim} Exp(\lambda_n)$. Then:

$\sum Z_n < \infty$. a.s. $\Leftrightarrow$ $\sum \frac{1}{\lambda_n} < \infty$.

Pf: By ch.f's: $\mathbb{E}(e^{-\sum_0^\infty Z_n}) = \frac{1}{\prod(1+\frac{1}{\lambda_n})}$

Pf: Consider $\mathbb{E}(e^{-\sum Z_n} \mid Y_n = i_n, n \ge 0)$.

Thm: (One-to-One Correspondence)

Q is regular Q-matrix. $(\mu_i)_E$ is dist.

Then. $\exists$ unique Q-process. (right-conti.) s.t. initial dist is $(\mu_i)_E$. has Q as its generator. satisfies Kolomogorov. Back/

Forward equation. (w.r.t. $P_{ij}(t)$).

<u>Pf</u>: Construct (Y_n). DTMC and (Z_n). indept. each other. $X_t = \sum Y_n I_{\{\cdots\}}$:

$Z_n = V_{nn} / q_{Y_{n}}$. $(V_n) \overset{\text{i.i.d}}{\sim} \text{Exp}(1)$.

Uniqueness is from Kolmogorov equation.

(2) <u>Recurrence</u>:

<u>prop.</u> $i \xrightarrow{X_t} j \iff i \xrightarrow{Y_n} j$

<u>pf</u>: $(\Rightarrow)$ $\exists t > 0$. $P_{ij}(t) > 0 \Rightarrow \exists n \in \mathbb{Z}^+$. s.t.

$P_i(Y_n = j,\ T_n \le t < T_{n+1}) > 0$

So: $r_{ij}^{n} > 0$.

$(\Leftarrow)$ $\exists\ i = k_0, k_1 \cdots k_n = j$. $r_{k_n k_{n+1}} > 0$

So: $q_{k_i k_{i+1}} > 0 \Rightarrow P_{k_i k_{i+1}}(t) > 0$. $\exists t > 0$

<u>Def</u>: $\tau_X(i) = \inf\{t > T_1 \mid X_t = i\}$

$\tau_Y(i) = \inf\{n > 0 \mid Y_n = i\}$

<u>Rmk</u>: $\tau_X(i) = \sum_{k \ge 1}^{\tau_Y(i)} Z_k$. So:

$\tau_X(i) < \infty \iff \tau_Y(i) < \infty$. a.s.

So: i is recurrent in $(X_t) \iff$

So i does in (Y_n)

Thm. $\sigma_j =: \inf\{t \geq 0 \mid X_t = j\}$. If $\forall j$. $P_i(\sigma_j < \infty) = 1$.

Then. $\{\eta_i\} = \{\mathbb{E}_i(\sigma_j)\}_E$ satisfies equation:

$$z_j = 0. \quad z_i = \frac{1}{q_i} + \sum_{k \neq j} r_{ik} z_k. \quad i \neq j.$$

Besides, it's the min nonnegative solution

of : $z_i \geq \frac{1}{q_i} + \sum_{k \neq j} r_{ik} z_k. \quad i \neq j.$

Pf: By inductively iteration:

$$\mathbb{E}_i(\sigma_j) = \mathbb{E}_i(\tau_j(X)) = \mathbb{E}_i(T_1) + \sum_{k \notin \{i,j\}} P_i(X_{T_1} = k)$$

$$\cdot \mathbb{E}_k(\sigma_j) = 1/q_i + \sum_{k \notin \{i,j\}} q_{ik} \mathbb{E}_k(\sigma_j)/q_i$$

$$\geq 1/q_i.$$

Lemma. $\forall i.j \in E.$ $\int_0^\infty p_{ij}(t)dt = \delta_{ij}/q_i + \frac{1}{q_j}\sum_{n \geq 1} r_{ij}^n$

Pf: $LHS = \mathbb{E}_i(\int_0^\infty I\{X_t = j\}dt)$

$$= \mathbb{E}_i(\sum_{n \geq 1} \int_{T_{n-1}}^{T_n} I\{X_{T_{n-1}} = j\} dt)$$

$$= \mathbb{E}_i(\sum_{n \geq 1} \tau_n I\{X_{T_{n-1}} = j\})$$

$$= \sum_{n \geq 1} \mathbb{E}_i(\tau_n \mid X_{T_{n-1}} = j) P_i(X_{T_{n-1}} = j)$$

Rmk: It's conti version of $\sum p_{ij}^n = \mathbb{E}_i(N_j)$

Thm. i) i is recurrent in (X_t). ii) $\int_0^\infty p_{ij}(t)dt = \infty$

iii) i is recurrent in (Y_n). All're equi.

Thm. If (X_t) is irred. $\exists j \in E$. $V: E \to \mathbb{R}^+$.

s.t. $QV(i) = \sum q_{ij} V(j) \le 0$. $\forall i \ne j$.

and $\{V(i) < r\} \subset E$ is finite. $\forall r < \infty$.

Then: (X_t) is recurrent.

Thm. (Foster-Lyapunov Criteria)

(X_t) is irred. recurrent. Then it's

positive recurrent $\Leftrightarrow$ $\exists (\eta_i)_E \subset \mathbb{R}_{\ge 0}$.

and $j \in E$. s.t. $\sum_{k \ne j} q_{jk} \eta_k < \infty$. $\forall i \ne j$

$\sum_{k \in E} q_{ik} \eta_k \le -1$.

Pf: It's identical as DTMC case.

(3) Stationary Dist.

Thm. For (X_t). regular Q-process.

i) $(\pi_i)_E$ is stat. dist. $\Leftrightarrow$ $\sum_{k \in E} \pi_k q_{ki} = 0$

ii) $(\pi_i)_E$ is reversible $\Leftrightarrow$ $\pi_k q_{ki} = \pi_i q_{ik}$.

Pf: i) Balance Equation. As DTMC. case.

ii) $(\Leftarrow)$ Set $\tilde{p}_{ij}(t) = \pi_j p_{ji}(t) / \pi_i$.

Check it's satisfies Kolmogorov equation

By uniqueness $\Rightarrow$ $\tilde{p}_{ij}(t) = p_{ij}(t)$.