Brownian Motion

(1) Pre-Brownian Motion:

Def: G is Gaussian white noise on $(\mathbb{R}^+, \mathcal{B}_{\mathbb{R}^+})$ with intensity: μ. Lebesgue measure. Set:

$(B_t)_{t\in\mathbb{R}^+}$ is pre-Brownian motion if $B_t = G(I_{[0,t]})$

Rmk: Covariance $(k(s,t))_{\mathbb{R}^+\times\mathbb{R}^+}$. $k(s,t) = s\wedge t$.

Thm. (Characterization)

$(X_t)_{t\geq 0}$ is real-valued random process. Follows equi.:

i) It's pre-Brownian Motion

ii) It's centered Gaussian process with covariance k. st. $k(s,t) = s\wedge t$.

iii) $X_0 = 0$. a.s. $\forall\, 0\leq s<t$. $X_t - X_s \sim N(0, t-s)$. indept of $\sigma(X_r,\ r\leq s)$.

iv) $X_0 = 0$. a.s. For $0 = t_0 < t_1 \cdots < t_p$. $(X_{t_i} - X_{t_{i-1}})$ is seq of indept r.v. $X_{t_i} - X_{t_{i-1}} \sim N(0, t_i - t_{i-1})$.

Pf: i) $\Rightarrow$ ii) is trivial. For ii) $\Rightarrow$ iii):

Set H is Gaussian span generated by $(X_t)_{t\geq 0}$.

H_s is spaned by $(X_r)_{0\leq r\leq s}$. $\tilde{H}_s$ is by $(X_{s+u} - X_s)_{u\geq 0}$

Check: $H_s \perp \tilde{H}_s$. $\Rightarrow$ $\sigma(H_s)$ indept with $\sigma(\tilde{H}_s)$.

iii) $\Rightarrow$ iv) is straight. For iv) $\Rightarrow$ i):

Def: $G: f = \sum \lambda_i I_{(t_{i-1}, t_i]} \longmapsto \sum \lambda_i (X_{t_i} - X_{t_{i-1}})$. isometry

check it's well-def. Extend G on $L^2(\mathbb{R}_+, \mathcal{B}_{\mathbb{R}_+}, m)$

Since step. func's. is dense in $L^2(\mathbb{R}_+, \mathcal{B}_{\mathbb{R}^+}, m)$.

<u>Cor.</u> $(B_t)_{t\geq 0}$ is pre-Brownian motion. Then: $0 < t_1 < \cdots < t_n$. $(B_{t_1} \cdots B_{t_n}) \sim \frac{1}{(2\pi)^{\frac{n}{2}} \sqrt{t_1(t_2-t_1)\cdots(t_n-t_{n-1})}} \exp(-\sum \frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})^2})$,

<u>Pf:</u> Consider $(B_{t_1}, B_{t_2} - B_{t_1}, \cdots B_{t_n} - B_{t_{n-1}})$.

<u>prop.</u> For B pre-Brownian motion. Then:

i) (symmetry) So $-B$ is

ii) (Scaling Variance) $\forall \lambda > 0$. $B_t^\lambda = \frac{1}{\lambda} B_{\lambda^2 t}$ is also pre-Brown.

iii) (Markov Property) $\forall s \geq 0$. $B_t^{(s)} = B_{t+s} - B_s$ is pre-Brown indept with $\sigma(B_r, r\leq s)$.

<u>Pf:</u> i). ii) trivial. iii) Consider Gaussian span. $\Rightarrow$ orthogonal.

<u>Rmk:</u> We often write: $G(f) = \int_0^\infty f(s)\, dB_s$. for $f \in L^2(\mathbb{R}^+, \mathcal{B}_{\mathbb{R}^+}, dt)$. But since Gaussian white noise isn't real measure depend on w. So: $\int_0^\infty f(s)\, dB_s$ isn't real integral. Later. we will find way to extend it!

(2) <u>Construction:</u>

① <u>Def:</u> For E metric space with Borel σ-algebra

i) $(X_t)_{t\in T}$ is random process with values in E. Sample path of X is: $t \longmapsto X_t(w)$

for every fix $w \in \Omega$.

<u>Rmk</u>: We can't ever assert the path is measurable.

ii) $(X_t)_{t\in T}$. $(\tilde{X}_t)_{t\in T}$ random processes with values in E.

$\tilde{X}$ is modification of X. if: $\forall t \in T$.

We have: $P(X_t = \tilde{X}_t) = 1$.

<u>Rmk</u>: But sample path of $\tilde{X}_t$ may be very different from X_t

iii) $\tilde{X}$ is indistinguishable from X if $\exists N \subset \Omega$.

$P(N) = 0$. st. $\forall w \in \Omega / N$. $\tilde{X}_t(w) = X_t(w)$. $\forall t \in T$.

<u>Rmk</u>: i) Indistinguishable $\Rightarrow$ modification.

ii) Two indistinguishable process have a.s. same sample paths.

<u>Lemma</u> If X. $\tilde{X}$ are both left/right-conti. P-a.s.

Then: $\tilde{X}$ is modification of X $\Leftrightarrow$ indistinguishable.

<u>Pf</u>: For $(\Rightarrow)$ Consider $t \in T \cap \mathbb{Q}$. $X_t = X_t$ except N_t.

Then $N = N_0 \cup (\bigcup_{t\in\mathbb{Q}} N_t)$ is P-null.

② <u>Sample path</u>:

<u>Def</u>: $(B_t)_{t\geq 0}$ is Brownian motion if:

i) $(B_t)_{t\geq 0}$ is pre-Brownian.

ii) All sample paths are conti.

Next, we prove such process exactly exists:

Fix $x \in \mathbb{R}$. $0 < t_1 < \cdots < t_n$. Define measure on $\mathbb{R}^n$:

$$\mu_{x,t_1\cdots t_n}(A_1 \times \cdots A_n) = \int_{A_1} dx_1 \cdots \int_{A_n} dx_n \prod_1^n P_{t_m - t_{m-1}}(x_{m-1}, x_m)$$

for $A_i \in \mathcal{B}_{\mathbb{R}}$, $x_0 = x$, $t_0 = 0$. $P_t(a,b) = (2\pi t)^{-\frac{1}{2}} \exp\left(-\frac{(b-a)^2}{2t}\right)$

<u>Thm.</u> $\Omega_0 = \{\text{Func}: w: \mathbb{R}_{\geq 0} \to \mathbb{R}\}$, $\mathcal{F}_0 = \sigma(\{w \mid w(t_i) \in A_i,$ $1 \leq i \leq n\}$, $(A_i)_1^n \subseteq \mathcal{B}_{\mathbb{R}^1})$. Then for each $x \in \mathbb{R}^1$, $\exists$ unique p.m. on $(\Omega_0, \mathcal{F}_0)$, s.t. $V_x(w \mid w(0) = x) = 1$ and $0 < t_1 \cdots < t_n$. $V_x\{w \mid w(t_i) \in A_i\} = \mu_{x,t_1\cdots t_n}(A_1 \times \cdots A_n)$

<u>Pf</u>: Check Consistency condition for $(\mu_\square)_{\square \in \mathbb{R}^n}$:

Show: $\int P_{t_j - t_{j-1}}(x, \eta) P_{t_{j+1} - t_j}(\eta, z)\, d\eta = P_{t_{j+1} - t_{j-1}}(x, z)$

$\Rightarrow (s_i)_1^{n-1} \subset (t_j)_1^n$. $\mu_{x,s_1\cdots s_{n-1}}(A_1 \times \cdots A_{n-1}) = \mu_{x,t_1\cdots t_n}(A_1 \times \cdots A_{j-1} \times \mathbb{R} \times A_{j+1} \cdots \times A_n)$

<u>prop.</u> $A \in \mathcal{F}_0 \iff \exists$ seq $(t_i)_1^\infty \subseteq [0, +\infty)$. $B \in \mathcal{B}_{\mathbb{R}^{\mathbb{N}}}$. s.t.

$A = \{w \mid (w(t_1), w(t_2) \cdots) \in B\}$.

<u>Pf</u>: Set: $\mathcal{A} = \{A \mid A = \{w \mid (w(t_i) \cdots) \in B$. $(t_i) \subseteq [0, \infty)$, $B \in \mathcal{B}_{\mathbb{R}^{\mathbb{N}}}\}$.

prove: $\mathcal{A} = \mathcal{F}_0$. $\iff$ prove: $\mathcal{A}$ is σ-algebra.

(Since $\forall A \in \mathcal{A}$. $A \in \mathcal{F}_0$. And generator of $\mathcal{F}_0 \subseteq \mathcal{A}$.)

For $A_n = \{w \mid (w(t_1^n), w(t_2^n), \cdots) \in B_n\}$.

Reorder $\{t_n\} = \{t_j^i\}_{i,j} \Rightarrow A_n = \{w \mid (w(t_i) \cdots) \in E_n\}$.

$\therefore A = \cup A_n = \{w \mid (w(t_i) \cdots) \in \cup E_n\} \in \mathcal{A}$.

<u>Rmk</u>: It means: $\mathcal{F}_0$ only depends on countably many coordinates. So, we know:

for $C = \{w \mid t \mapsto w(t)$ conti$\} \notin \mathcal{F}_0$. i.e. it's not measurable. So ν_x isn't p.m. of BM.

To solve this problem:

Denote: $Q_2 = \{m/2^n \mid m, n \in \mathbb{Z}_{\geq 0}\}$. $\Omega_2 = \{w : Q_2 \to \mathbb{R}^1\}$.

$\mathcal{F}_2$ is σ-algebra generated by finite dimensional sets in Ω_2. Restrict ν_x on $(\Omega_2, \mathcal{F}_2)$

Thm. For $T < \infty$, $x \in \mathbb{R}^1$. $\nu_x(\{w : Q_2 \to \mathbb{R} \mid w$ is uniformly conti on $Q_2 \cap [0,T]\}) =: \nu_x(\Omega_{2,c}) = 1$

Rmk: After proving this Thm. Then to reconstruct:

Consider $C = \{w : \mathbb{R}_{\geq 0} \to \mathbb{R}^1$, conti$\}$. $\mathcal{C} = \sigma(C)$ (*)

(*): It depends on countable coordinates as $\mathcal{F}_0$ (cylinder sets)

$\varphi : \Omega_{2,c} \longrightarrow C$ $\quad \tilde{w}(t)$ is conti extension of w.

$w(t) \longmapsto \tilde{w}(t)$

Set $P_x = \nu_x \circ \varphi^{-1}$. $(\varphi^{-1}\{\tilde{w}(t_i) \in A_i\} =: \{w(t_i) \in A_i\})$

Pf. WLOG. Suppose $B_0 = 0$. $T = 1$. Set $c = E|B_1|^4$

Then: $E_0|B_t - B_s|^4 = E_0|B_{t-s}|^4 = (t-s)^2 c$

Thm. If $E|X_s - X_t|^\beta \leq k|t-s|^{1+\alpha}$, $\alpha, \beta > 0$.

for $\gamma < \frac{\alpha}{\beta}$. $\exists\, C(w)$, s.t.

$P(\{w \mid |X_q - X_r| \leq C|q-r|^\gamma, \forall q, r \in Q_2 \cap [0,1]\}) = 1$.

Pf. $G_n = \bigcap_{0 \leq i \leq 2^n} \{|X_{i/2^n} - X_{i-1/2^n}| \leq 2^{-\gamma n}\}$.

By chebyshev:

$P(G_n^c) \leq 2^n \cdot 2^{n\gamma\beta} \cdot 2^{-n(1+\alpha)} \cdot k = k \cdot 2^{-n(\alpha - \beta\gamma)}$

Lemma. On $H_N = \bigcap_{n\geq N} G_n$. Then: $|X_q - X_r| \leq \frac{3}{1-2^{-\gamma}} |q-r|^\gamma$.

$\forall q, r \in Q_2 \cap [0,1]$. $|q-r| < 2^{-N}$.

Since $P(H_N^c) \leq \sum_N^\infty P(G_n^c) \leq K \sum_N 2^{-n\lambda} = \frac{K 2^{-N\lambda}}{1-2^{-\lambda}}$. $\lambda = \alpha - \beta\gamma$.

$\Rightarrow \sum_N^\infty P(H_N^c) < \infty$. So $P(H_N. \text{ult}) = 1$.

For $w \in \{H_N . \text{ult}\}$. by Lemma: $|X_q - X_r| \leq A|q-r|^\gamma$. $\forall q, r \in Q_2$.

and $|q-r| \leq \delta(w)$.

Extend to $\forall q, r \in Q_2 \cap [0,1]$. Set $0 = s_0 < s_1 \cdots < s_n = T$

s.t. $|s_i - s_{i-1}| \leq \delta(w)$. Then: by triangle inequality.

Thm. Brownian path is γ-Hölder conti. for $\gamma < \frac{1}{2}$.

Pf: $E(|B_t - B_s|^{2m}) = C_m |t-s|^m$. $C_m = E|B_1|^{2m}$.

By Thm. above. Set $\beta = 2m$. $\alpha = m-1$. Let $m \to \infty$.

So B_t is γ-Hölder. a.s. (m is along $\mathbb{Z}^+$).

Set Brownian motion is modification of B_t.

Rmk: i) For $I = \mathbb{R}^+$. Use Thm successively on $[n, n+1]$

ii) Alternative method to construct BM:

Begin with Pre-Brownian Motion: B_t.

Then by Kolmogorov's Lemma: $\exists \tilde{B}_t$ the modification of B_t. with γ-Hölder conti path. for $\forall\, 0 < \gamma < \frac{1}{2}$. $\forall w \in \Omega$.

Consider: $\varphi: \Omega \to C(\mathbb{R}^+, \mathbb{R}^1)$. $\varphi(w) = (t \mapsto \tilde{B}_t(w))$

$W = P \circ \varphi^{-1}$ is Wiener measure on $(C, \mathcal{B}(C))$.

P is p.m. of Pre-Brownian Motion. $P(B_0 = 0) = 1$.

$W(dw)$ doesn't depend on choice of BM. And:

$W(\{w \mid w(t_i) \in A_i,\ 1 \leq i \leq n\}) = P(B_{t_i} \in A_i.\ 1 \leq i \leq n)$

Thm. Brownian Motion paths are not Lipschitz conti. at any point. P_x-a.s. (So BM is not differentiable P_x-a.s)

Pf: Fix $c<\infty$. Set $A_n = \{w \mid \exists s \in [0,1]. \text{ st. } |B_t - B_s| \leq c|t-s|. \text{ when } |t-s| < \frac{3}{n}\}$. $A_{n+1} \subseteq A_n$

$Y_{k,n} = \max_{j=0,1,2} |B(\frac{k+j}{n}) - B(\frac{k+j-1}{n})|$. $1 \leq k \leq n-2$

$B_n = \{\exists 1 \leq k \leq n-2, \text{ st. } Y_{k,n} \leq 5c/n\}$. $\Rightarrow A_n \subseteq B_n$

$\Rightarrow p(A_n) \leq p(B_n) \leq n\, p^3(|B(\frac{1}{n})| \leq 5c/n) \xrightarrow{n \to \infty} 0$

follows from $B_n = \bigcup \{Y_{k,n} \leq 5c/n\}$. and

$Y_{k,n} = \bigcap_{j=0,1,2} \{|B(\frac{k+j}{n}) - B(\frac{k+j-1}{n})| \leq 5c/n\}$

Rmk: Denote $H_\gamma(w)$ is set of times at which path $w \in C$ is γ-Hölder.

Then: $p(H_\gamma = \varnothing) = 1$. $\forall \gamma > \frac{1}{2}$

$p(t \in H_{\frac{1}{2}}) = 0$. $\forall t \geq 0$. But:

$p(H_{\frac{1}{2}} \neq \varnothing) = 1$. (measure 0. not empty)

(3) Property:

① Markov Property:

i) Denote: $\mathcal{F}_s^0 = \sigma(B_r, r \leq s)$. $\mathcal{F}_s^+ = \bigcap_{t>s} \mathcal{F}_t^0$

Rmk: i) $\mathcal{F}_s^+$ is right-conti: $\bigcap_{t>s} \mathcal{F}_t^+ = \mathcal{F}_s^+$

ii) $\mathcal{F}_s^+$ allow "Infinitesimal peek at future. i.e. $A \in \mathcal{F}_s^+ \iff A \in \mathcal{F}_{s+\varepsilon}^0$. $\forall \varepsilon > 0$.

iii) $\mathcal{F}_s^+ \neq \mathcal{F}_s^\circ$. e.g. $\overline{\lim_{t\downarrow s}} \frac{B_t - B_s}{f(t-s)} \in \mathcal{F}_s^+$. but not $\mathcal{F}_s^\circ$.

Def: n-dimension Brownian Motion $B_t = (B_t^1 \cdots B_t^n)$.

where $\{B_t^i\}$. indept. Brownian Motions

Rmk: It's easy to generalize the construction of measure for multidimension: $P_x(dw) = \bigotimes_{i=1}^n P_x^i(dw_i)$

on $(C, \mathcal{C})$. $C = \{w : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$. conti$\}$. $\mathcal{C} = \sigma(C)$.

Thm. (Simple Markov)

If $s \geq 0$. Y bdd : $C \to \mathbb{R}^1$. measurable. Then. $\forall x \in \mathbb{R}^d$.

$E_x(Y \circ \theta_s \mid \mathcal{F}_s^+) = E_{B_s}(Y)$

Pf: Show : $E_x(Y \circ \theta_s I_A) = E_x(E_{B_s}(Y) I_A)$. $\forall A \in \mathcal{F}_s^+$.

Suppose $Y = \prod_1^n f_m(w(t_m))$. $0 < t_1 < \cdots < t_n$. f_m. bdd. measurable.

Let: h: $0 < h < t_1$. $0 < s_1 \cdots < s_k \leq s+h$. $A = \{w(s_i) \in A_i.\ A_i \in \mathcal{B}_{\mathbb{R}^d}.\ 1 \leq i \leq k\}$.

$\Rightarrow E_x(Y \circ \theta_s I_A) = \int_{A_1} dx_1\, p_{s_1}(x, x_1) \cdots \int_{A_k} dx_k\, p_{s_k - s_{k-1}}(x_{k-1}, x_k)$

$$\int dy\, p_{s+h-s_k}(x_k, y)\, \varphi(y, h).$$

$\varphi(y, h) = \int dy_1\, p_{t_1 - h}(y, y_1) f_1(y_1) \cdots \int dy_n\, p_{t_n - t_{n-1}}(y_{n-1}, y_n) f_n(y_n)$.

So: $E_x(Y \circ \theta_s I_A) = E_x(\varphi(B_{s+h}, h) I_A)$. $A \in \mathcal{F}_{s+h}^\circ$.

By π-λ $\Rightarrow \forall A \in \mathcal{F}_{s+h}$. it holds!

Note $\varphi(B_{s+h}, h)$ is bdd (By induction) $\Rightarrow$ Let $h \to 0$. by DCT.

So: $E_x(Y \circ \theta_s I_A) = E_x(\varphi(B_s, 0) I_A)$. $\forall A \in \mathcal{F}_s^+$.

Apply MCT to extend $\prod_1^n f_m$ to general $Y \in \mathcal{C}$.

Cor. $E_x(Y\circ\theta_s \mid \mathcal{F}_s^+) = E_x(Y\circ\theta_s \mid \mathcal{F}_s^\circ) \in \mathcal{F}_s^\circ$

Pf: $\mathcal{F}_s^\circ \subseteq \mathcal{F}_s^+$. $E(Y\circ\theta_s \mid \mathcal{F}_s^+) = E_{B_s}(Y) \in \mathcal{F}_s^\circ$.

Cor. $E_x(Z \mid \mathcal{F}_s^+) = E_x(Z \mid \mathcal{F}_s^\circ)$. for $Z \in C$. bdd.

$\forall s \geq 0$. $x \in \mathbb{R}^n$.

Pf: By MCT. Prove for $Z = \prod_i^n f_m(B_{t_m}(w))$.

$\Rightarrow Z = X \cdot (Y\circ\theta_s)$. $X \in \mathcal{F}_s^\circ$. $Y \in C$.

So: $E_x(Z \mid \mathcal{F}_s^+) = X\, E_{B_s}(Y) \in \mathcal{F}_s^\circ$.

Rmk: $Z \in \mathcal{F}_s^+ \Rightarrow Z = E_x(Z \mid \mathcal{F}_s^+) = E_x(Z \mid \mathcal{F}_s)$

$\in \mathcal{F}_s^\circ$. So $\mathcal{F}_s^+ = \mathcal{F}_s^\circ$. up to null-sets.

Thm. (Blumental's 0-1 Law)

If $A \in \mathcal{F}_0^+$. $\forall x \in \mathbb{R}^n$. Then: $P_x(A) \in \{0,1\}$.

Pf: $\mathcal{F}_0^\circ = \{\emptyset, \Omega\}$. trivial. By corollary above.

Rmk: We say: $\mathcal{F}_0^+$ (germ σ-field) is trivial as well.

Thm. If $\tau = \inf\{t > 0 \mid B_t > 0\}$. Then $P_0(\tau = 0) = 1$.

Pf: $P_0(\tau \leq t) \geq P_0(B_t > 0) = \frac{1}{2} \Rightarrow P_0(\tau = 0) = \lim_{t \downarrow 0} P(\tau \leq t) \geq \frac{1}{2}$

$\{\tau = 0\} = \bigcap_{\varepsilon > 0} \{\tau \leq \varepsilon\} \in \mathcal{F}_0^+$. By 0-1 Law.

Cor. If $T_0 = \inf\{t > 0 \mid B_t = 0\}$. Then $P(T_0 = 0) = 1$.

Pf: By symmetric of Thm. above. BM hits

$\mathbb{R}^+$. $\mathbb{R}^-$ both immediately.

Cor. If $a<b$. Then with prob. one. a is limit of points $t\in(a,b)$. s.t. B_t is local maximum.

Pf: WLOG. consider B_t in $(0,c)$. $B_0=0$.

$\exists\ (t_n).\ (s_n)\downarrow 0$. s.t. $B(t_n)>0$. $B(s_n)<0$. a.s.

Select subseq: $s_{n_1}>t_{n_1}>s_{n_2}>t_{n_2}\cdots>t_{n_k}\cdots>0$

So on each $[t_{n_k}, t_{n_{k-1}}]$. local max exists.

Rmk: It means local maximum/minimum points form a countably dense set.

Thm: If B_t is Brownian motion starts at 0. Then for $X_0=0$. $X_t=tB(\frac{1}{t})$. $t>0$. is also BM.

Pf: $E(X_tX_s)=t\wedge s$ easy to check. and conti. $\forall t>0$

$(X(t_1)\cdots X(t_n))$ is multinormal dist.

For conti. at. $t=0$:

By SLLN: $B_n/n\to 0$. a.s. For values between Z^+:

By Kolmogorov Inequi: $P(\max\limits_{0\le k\le 2^m}|B(n+\frac{k}{2^m})-B(n)|\ge n^{\frac{2}{3}})\le n^{-\frac{4}{3}}E|B_1|^2$.

Let $m\to\infty$. $\Rightarrow \sup\limits_{[n,n+1]}|B(u)-B(n)|\le n^{\frac{2}{3}}$. u/t. a.s. $\Rightarrow B_t/t\to 0$. a.s.

Denote: $\mathcal{F}_t'=\sigma(B_s,\ s\ge t)=$ future at time t. $\mathcal{T}=\bigcap\limits_{t\ge 0}\mathcal{F}_t'$.

Thm. If $A\in\mathcal{T}$. Then: $P_x(A)\in\{0,1\}$. indept with x.

Rmk: In Blumenthal's Thm. $P_x(A)$ may depend on x.

e.g. $A=\{w\mid w(0)\in B\}\in\mathcal{F}_0^+$.

Pf: Tail σ-field of B_t is germ σ-field of X_t in the Thm above. So $P_0(A) \in \{0,1\}$.

Note $A \in \mathcal{F}_1'$. $I_A = I_D \circ \theta_1$.

$\Rightarrow P_x(A) = E_x(I_D \circ \theta_1) = E_x(E_{B_1}(I_D))$

$$= \int (2\pi)^{-\frac{n}{2}} e^{-\frac{(y-x)^T(y-x)}{2}} P_y(D)\, dy$$

Set $x=0 \Rightarrow P_y(D) = 0$. a.s. $\forall y$. if $P_0(A)=0$

Let $\tilde{A} = A^c$. if $P_0(A)=1$. so $P_y(D)=1$. a.s

Replace in the equation above!

Cor. B_t is one-dimension BM. starts at 0.

Then: $\overline{\lim\limits_{t\to\infty}}\, B_t/\sqrt{t} = +\infty$. $\underline{\lim\limits_{t\to\infty}}\, B_t/\sqrt{t} = -\infty$. P_0-a.s.

Pf: By Fatou's: $P_0(B_n/\sqrt{n} \geq k. \text{ i.o}) \geq \overline{\lim\limits_{n}}\, P_0(B_n \geq k\sqrt{n})$

$$= P_0(B_1 \geq k) > 0.$$

Note: $\{B_n/\sqrt{n} \geq k. \text{ i.o}\} \in \mathcal{T}$. Second by symmetry.

Cor. B_t is one-dimension BM. $A = \bigcap\limits_{n\in\mathbb{Z}^+} \{B_t = 0.\ \exists t \geq n\}$

Then $P_x(A) = 1$. $\forall x$.

Pf: By translation invariant. Continity.

Rmk: It means: one-dimension Brownian motion is recurrent. Actually. B_t will hit zero infinite times in $(0,\varepsilon)$. $\forall \varepsilon > 0$. (Consider $X_t = t B(\frac{1}{t})$)

Thm. $t \longmapsto B_t$ is not monotone on any interval. a.s.

Pf: Note $P(\sup_{0\le t\le \varepsilon} B_t > 0,\ \forall \varepsilon > 0) = P(\inf_{0\le t\le \varepsilon} B_t < 0,\ \forall \varepsilon > 0) = 1$.

$\Rightarrow \forall q \in \mathbb{Q}^+,\ \forall \varepsilon > 0 \quad \sup_{q\le t\le q+\varepsilon} B_t > B_q,\ \inf_{q\le t\le q+\varepsilon} B_t < B_q$ a.s.

prop. $0 < t_0^n < t_1^n \cdots < t_{p_n}^n = t$. seq of subdivision of $[0,t]$.

whose mesh $\to 0$. i.e. $\sup_{1\le i\le p_n} |t_i^n - t_{i-1}^n| \to 0$. Then:

$\lim_{n\to\infty} \sum_1^{p_n} (B_{t_i^n} - B_{t_{i-1}^n})^2 = t$. in L^2.

Pf: $B_{t_i^n} - B_{t_{i-1}^n} = G(\mu(t_{i-1}^n, t_i^n]))$, with μ. Lebesgue measure

Cor. $t \longmapsto B_t$ has infinite variation on any interval

with probability one.

Pf: $\sum_1^{p_n} (B_{t_i^n} - B_{t_{i-1}^n})^2 \le \sup_i |B_{t_i^n} - B_{t_{i-1}^n}| \sum_1^{p_n} |B_{t_i^n} - B_{t_{i-1}^n}|$

By conti. $\sup_i |B_{t_i^n} - B_{t_{i-1}^n}| \to 0$.

Rmk: It shows that it's impossible to define

$\int_0^T f(t)\, dB_t$ as case of Stieltjes integral

w.r.t functions of finite variation.

ii) Strong Markov:

For convention, $N_x = \{A \mid A \subset D,\ P_x(D) = 0\}$. $\mathcal{F}_s^x = \sigma(\mathcal{G}_s^+ \cup N_x)$

Then set: $\mathcal{F}_s = \bigcap_x \mathcal{F}_s^x$. (Not want a filtration depends

on the initial state)

Denote: $\mathcal{G}_\infty = \sigma(B_r,\ r\ge 0)$. $\mathcal{F}_T = \{A \in \mathcal{G}_\infty \mid A \cap \{T \le t\} \in \mathcal{F}_t\}$.

Thm. (Strong Markov)

$(t,w) \longmapsto Y_t(w) \in \mathbb{R}'$ is bdd and $\in \mathcal{B}_{\mathbb{R}'} \times \mathcal{C}$

If S is stopping time. Then $\forall x \in \mathbb{R}^n$:

$E_x(Y_S \circ \theta_S \mid \mathcal{F}_S) = E_{B(S)}(Y_S)$ on $\{S<\infty\}$.

Pf. 1°) Assume: $\exists (t_n) \uparrow \infty$. $P_x(S<\infty) = \sum_n P_x(S=t_n)$

Then: $E_x(Y_S \circ \theta_S I_{A\cap\{S<\infty\}}) = \sum_n E_x(Y_{t_n} \circ \theta_{t_n} I_{A\cap\{S=t_n\}})$

Note: $A\cap\{S=t_n\} = A\cap(\{S\le t_n\} - \{S\le t_{n-1}\}) \in \mathcal{F}_{t_n}$.

It reduced to simple Markov case.

2°) To remove assumption:

Set $S_n = ([2^n S]+1)/2^n$. stopping time.

First consider: $Y_t(w) = f_0(t) \prod_{m=1}^{k} f_m(w(t_m))$

$0<t_1 \cdots < t_k$. $f_0 \cdots f_k$ bdd conti

By induction: $\varphi(x,t) = E_x(Y_t) = f_0(t) \int d\eta_1 P_{t_1}(x,\eta_1) f(\eta_1)$

$\cdots \int d\eta_k P_{t_k - t_{k-1}}(\eta_{k-1},\eta_k) f_k(\eta_k) \in C_B$

For $\forall A \in \mathcal{F}_S \subseteq \mathcal{F}_{S_n}$. Note: $\{S_n<\infty\} = \{S<\infty\}$.

By 1°): $E_x(Y_{S_n} \circ \theta_{S_n} I_{A\cap\{S<\infty\}}) = E_x(\varphi(B(S_n), S_n) I_{A\cap\{S<\infty\}})$

Apply BCT $\Rightarrow n\to\infty$. We obtain conclusion.

3°) For general form of Y:

By MCT: Set $\mathcal{H} = \{Y \mid Y$ satisfies $\cdots\}$.

Consider $A = G_0 \times \{w(s_i) \in G_i,\ 1\le i\le k\}$. cylinder set.

$f_i^n(x) = 1 \wedge n\, d(x, G_i^c) \uparrow I_{G_i}$. $(n\to\infty)$. f_i^n conti.

$Y_s^n(w) = f_0^n(s) \prod_1^k f_i^n(w(s_i)) \in \mathcal{H}$.

Note $\mathcal{H}$ satisfies ii). iii) $\Rightarrow Y_s^n \uparrow I_A \in \mathcal{H}$.

Cor. T is stopping time. $P(T<\infty)>0$. Then:

$\forall x \in \mathbb{R}^n$. For $B_t^{(T)} = I_{\{T<\infty\}}(B_{T+t} - B_T)$.

it's BM indept with $\mathcal{F}_T$. under $P(\cdot \mid T<\infty)$

Pf. Set $Y_T = I\{B_{T+t} - B_T \in A\}$. for $A \in \mathcal{B}_{\mathbb{R}^n}$.

② Path Properties:

Next, consider one-dimension Brownian Motion B_t, $t \geq 0$.

Denote: $R_t = \inf\{u>t \mid B_u = 0\}$. $T_a = \inf\{t>0 \mid B_t = a\}$.

$Z(w) = \{t \mid B_t(w) = 0\}$. zeros of $B_t(w)$.

i) Thm. i) $Z(w)$ is closed. has no isolated point. So it's perfect set (Hence uncountable)

ii) $\mu(Z(w)) = 0$. μ is Lebesgue measure.

its Hausdorff dimension is $1/2$.

Pf: i) $P_x(R_t<\infty) = 1. \Rightarrow P_x(T_0 \circ \theta_{R_t} > 0 \mid \mathcal{F}_{R_t}) = P_0(T_0>0) = 0$

$\Rightarrow P_x(T_0 \circ R_t > 0, \exists t \in \mathbb{Q}) = 0$.

So if $u \in Z(w)$. isolated on left side. Then: it's decreasing limit point in $Z(w)$.

ii) By Fubini: $E_x(\mu(Z(w) \cap [0,T])) = \int_0^T E_x(I_{Z(w)}(t))$

$= \int_0^T E_x(I_{\{B_t=0\}}) = \int_0^T P_x(B_t=0) = 0$

ii) Hitting Time:

Thm. Under P_0. $\{T_a, a \geq 0\}$ has stationary indept increments.

Pf: 1°) Stationary:

if $0<a<b$. then: $T_b \circ \theta_{T_a} = T_b - T_a$.

$\forall f$. bdd. measurable. $E_0(f(T_b - T_a) \mid \mathcal{F}_{T_a})$

$= E_0(f(T_b) \circ \theta_{T_a} \mid \mathcal{F}_{T_a}) = E_a(f(T_b))$

By translation invariant: $E_a(f(T_b)) = E_0(f(T_{b-a}))$

$\Rightarrow E_0(f(T_b - T_a)) = E_0(f(T_{b-a}))$

2°) Indept:

Set $a_0 < a_1 \cdots < a_n$. f_i. bdd. measurable.

Set $F_i = f_i(T_{a_i} - T_{a_{i-1}})$. $1 \le i \le n$.

$E_0(\prod_1^n F_i) = E_0(E_0(F_n \mid \mathcal{F}_{T_{a_{n-1}}}) \prod_1^{n-1} F_i)$

$= E_0(F_n) E_0(\prod_1^{n-1} F_i) = \cdots = \prod_1^n E_0(F_i)$

Rmk: By Scaling: $T_n \sim n^2 T_1$ ($B_t \sim \frac{1}{\lambda} B_{\lambda^2 t}$)

So: $t_k = T_k - T_{k-1}$. i.i.d. $\sum_1^n t_k / n^2 \xrightarrow{d} T_1$

since $\sum_1^n t_k = T_n$.

Cor. For $\varphi_a(\lambda) = E_0(e^{-\lambda T_a})$. $a \ge 0$. $\Rightarrow \varphi_x(\lambda)\varphi_y(\lambda)$

$= \varphi_{x+y}(\lambda)$. So: $\varphi_a(\lambda) = e^{-a c(\lambda)}$.

Thm. (Reflection Principle)

Set $a>0$. Then: $P_0(T_a < t) = 2P_0(B_t > a)$.

Pf: Set $Y_s(w) = \begin{cases} 1. & \text{if } s<t.\ w(t-s) > a \\ 0. & \text{otherwise.} \end{cases}$

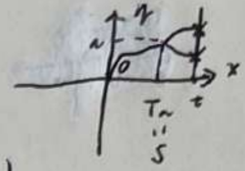
$S = \inf\{s<t \mid B_s = a\}$. $Y_S(\theta_S(w)) = I_{\{S<t.\ B_t > a\}}$

$E_0(Y_S \circ \theta_S \mid \mathcal{F}_S) = E_a(Y_S)$ on $\{S<\infty\}$.

So: $P_0(T_a<t,\ B_t>a) = E_0(Y_S\circ\theta_S\, I_{\{S<\infty\}})$

$= E_0(E_0(Y_S\circ\theta_S \mid \mathcal{F}_S)\, I_{\{S<\infty\}}) = E_0(\frac{1}{2} I_{\{T_a<t\}})$

since $\{S<\infty\} = \{T_a<t\}$. $E_a(Y_S) = \frac{1}{2}$:



$\Rightarrow P_0(T_a<t) = 2P_0(T_a<t,\ B_t>a) = 2P_0(B_t>a)$

follows from $\{B_t>a\} \subseteq \{T_a<t\}$.

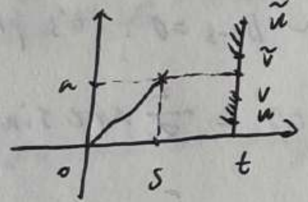
Rmk: Set $S_t = \sup_{0\le s\le t} B_s \Rightarrow \{T_a<t\} = \{S_t>a\}$.

We obtain $S_t \sim |B_t|$. (And $T_a \sim a^2/B_1^2$)

Thm. (Generalization)

If $u<v\le a$. Then: $P_0(T_a<t,\ B_t\in(u,v)) = P_0(B_t\in(2a-v,\ 2a-u))$

Pf:



Denote $\tilde{u} = 2a-u$. $\tilde{v} = 2a-v$. Set:

$$Y_s = \begin{cases} 1. & s<t.\ W(t-s)\in(u,v) \\ 0. & \text{otherwise.} \end{cases}$$

$$\tilde{Y}_s = \begin{cases} 1. & s<t.\ W(t-s)\in(\tilde{v},\tilde{u}) \\ 0, & \text{otherwise.} \end{cases}$$

$$E_0(Y_s\circ\theta_s \mid \mathcal{F}_s) = E_a(Y_s) \overset{\text{sym.}}{=} E_a(\tilde{Y}_s) = E_0(\tilde{Y}_s\circ\theta_s \mid \mathcal{F}_s)$$

Take expectation on both sides!

Rmk: We can obtain dist. of (S_t, B_t) from above.

$$(S_t, B_t) \sim g(a,b) = \frac{2(2a-b)}{\sqrt{2\pi t^3}} e^{-(2a-b)^2/2t} I_{\{a>0.\ b<a\}}.$$

Thm. (Arcsine Law)

$\forall s\in[0,1]$. $L = \sup\{t\le 1 \mid B_t = 0\}$. Then: $P_0(L\le s) = \frac{2\arcsin\sqrt{s}}{\pi}$.

Pf: $P_0(T_a<t) = 2P_0(B_t\ge a) = 2\int_a^\infty (2\pi t)^{-\frac{1}{2}} e^{-x^2/2t}\,dx$.

$$\overset{x=\frac{a\sqrt{t}}{\sqrt{s}}}{=} \int_0^t (2\pi s^3)^{-\frac{1}{2}} a\, e^{-a^2/2s}\,ds$$

$$P_0(L \le t) = E_0(I_{\{T_0 > 1-t\}} \circ \theta_t)$$

$$= E_0(E_{X_t}(I_{\{T_0>1-t\}}))$$

$$= \int_{\mathbb{R}^1} p_t(0,x) P_x(T_0 > 1-s)\,dx$$

$$= \int_{\mathbb{R}^1} p_t(0,x) P_0(T_x > 1-s)\,dx$$

By the Formular before. $= \frac{2}{\pi} \arcsin\sqrt{s}$.

<u>Rmk</u>: i) Note the density is symmetric at $s=\frac{1}{2}$

and blow up at 0.

ii) Another form of Arcsine Law:

<u>Def</u>: $T = \arg\max_{0\le t\le 1} B_t$ (well-def. since that:

$B_t = B_s \iff B_t - B_s \sim B_{t-s} = 0$. it's prob. 0)

Claim: $\forall t \in [0,1]$. $P_0(T \le t) = \frac{2}{\pi}\arcsin\sqrt{t}$.

<u>Pf</u>: $P_0(T \le t) = P_0(\max_{[0,t]} B_s > \max_{[t,1]} B_u)$

$$= P_0(\max_{[0,t]}(B_s - B_t) > \max_{[t,1]}(B_u - B_t))$$

$$= P_0(\max_{[0,t]}(B_{t-s} - B_t) > \max_{[0,1-t]}(B_{u+t} - B_t))$$

$$= P_0(\max_{[0,t]} X_s > \max_{[0,1-t]} Y_u)$$

where X_s, $0\le s\le t$. BM indept with

Y_u. $0\le u\le 1-t$. BM. both from 0.

Apply dist of $\max_{[0,t]} B_s = S_t \sim |B_t|$.

$$\Rightarrow P_0(T\le t) = P_0(\sqrt{t}|z_1| > \sqrt{1-t}|z_2|) = P_0\left(\frac{|z_2|}{\sqrt{z_1^2+z_2^2}} < t\right)$$

(4) <u>Martingales</u>:

Note: We have B_t, $B_t^2 - t$, $e^{\theta B_t - \frac{\theta^2}{2}t}$, $\theta \in \mathbb{R}^1$, are all martingales.

Thm. If $a < x < b$. Then: $P_x(T_a < T_b) = (b-x)/(b-a)$.

Pf: Set $T = T_a \wedge T_b$. $T < \infty$ a.s.

$\therefore E_x(B_{T\wedge t}) = x$. Apply BDT. Let $t \to \infty$.

Thm. Set $T = \inf\{t \mid B_t \notin (a,b)\}$. $a < 0 < b$. Then $E_0(T) = -ab$.

Pf: $E_0(B^2_{T\wedge t}) = E_0(T\wedge t)$. By BDT. Let $t \to \infty$.

Thm. $E_0(e^{-\lambda T_a}) = e^{-a\sqrt{2\lambda}}$.

Pf: $E_0(e^{\theta B_{T_a\wedge t} - \frac{\theta}{2} T_a^2 \wedge t^2}) = 1$. Set $\theta = \sqrt{2\lambda}$. Let $t \to \infty$.

Thm. If $u(t,x)$ is polynomial in t, x. s.t. $\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} = 0$

Then: $u(t, B_t)$ is martingale.

Pf: Show: $E_x(u(t,B_t)) =: \varphi(t) \equiv \text{Const.} \iff \frac{\partial \varphi}{\partial t} = 0$

Then: $\forall s < t$. Set $V(r,x) = u(s+r, x)$ satisfies:

$\partial v/\partial r = \frac{1}{2}\partial^2 v/\partial x^2$. So $E_x(V(t, B(t)))$ is const.

$\Rightarrow E_x(u(t,B_t) \mid \mathcal{F}_s) = E_x(V(t-s, B_{t-s}) \circ \theta_s \mid \mathcal{F}_s)$

$= E_{B_s}(V(t-s, B(t-s))) = V(0, B_s)$

$= u(s, B_s)$.

Rmk: It can be extended to $\forall u(t,x)$. s.t.

$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}$. and guarantee: $E_x|u(t,B_t)| < \infty$