Gassian Process.

(1) Background:

. Consider a particle in time duration τ. set: $\ell(\eta)$ is d.f of the particle moves η units on a line. s.t. $\ell(\eta) = \ell(-\eta)$. So: $\int_{\mathbb{R}^1} \eta \ell(\eta) = 0$.

Denote: $D = \int_{\mathbb{R}^1} \eta^2 \ell(\eta) d\eta$. $f(x,t)$: the number of particles at x at time t. suppose $f(0,0) = c$.

① Conservation Law: (To find $f(x,t)$)

By assumption: we have: $f(x, t+\tau) = \int_{\mathbb{R}^1} f(x-\eta, t) \ell(\eta) d\eta$

From expansion: $\begin{cases} f(x,t+\tau) = f(x,t) + \frac{\partial f}{\partial t} \tau + o(\tau) \\ f(x-\eta, t) = f(x,t) - \frac{\partial f}{\partial x} \eta + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \eta^2 + o(\eta^2). \end{cases}$

Ignore infestimal terms. Replace in the equation:

$$\frac{\partial f}{\partial t} = \frac{D}{2\tau} \frac{\partial^2 f}{\partial x^2} \Rightarrow \text{Solve } f(x,t) = \frac{1}{\sqrt{2\pi c' t}} e^{-x^2/2c't} \cdot c' = \frac{D}{2\tau}$$

② Entropy:

Actually, the process above has max entropy.

Rmk: More entropy means more randomness!

Consider $H(x) = -\int_{\mathbb{R}^1} f_X(t) \log f_X(t) dt$. $X \sim f_X$.

which defines the entropy of r.v. X.

suppose $E(X) = m$. $E(X^2) = \sigma^2$

$$G(f) = -\int f_X \log f_X + \lambda_1 (\int f_X - 1) + \lambda_2 (\int x f_X - m)$$

$$+ \lambda_3 (\int x^2 f_X - \sigma^2).$$

By variation method: If f_0 is optimal solution,

$H(t) = G(f_0 + tg)$. for func. g. Note: $H(0) \geq H(t)$.

$$\Rightarrow \frac{\partial H}{\partial t}\Big|_{t=0} = 0 \Rightarrow \int_{\mathbb{R}^1} g(-\log f_X + \tilde{\lambda}_1 + \lambda_2 x + \lambda_3 x^2) = 0 . \forall g.$$

So $\log f_X = \tilde{\lambda}_1 + \lambda_2 x + \lambda_3 x^2$. $f_X = e^{\lambda_3 x^2 + \lambda_2 x + \tilde{\lambda}_1}$. $\tilde{\lambda}_1 = \lambda_1 + 1$

which is d.f has same form in ①.

<u>Rmk</u>: i) If $X(\omega)$ take value in $\mathbb{R}^+$. ignore "$E(X^2) = \sigma^2$".

i.e. discard "$\lambda_3 \int x^2 f_X - \sigma^2$". Then: $f_X = e^{\lambda_2 x + \tilde{\lambda}_1}$

which is exponential dist.

ii) If $X(\omega) \in [a,b]$. without moment constraint.

Then f_X is d.f of uniform dist.

(2) Gaussian Vectors:

① Gaussian r.v.'s:

<u>Def</u>: $X(\omega) \in \mathbb{R}^1$ is standard Gaussian variable if

$X \sim p_X = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. Y is Gaussian with

$N(m, \sigma^2)$-dist. if $Y = \sigma X + m$.

prop. For seq of r.v.'s (X_n). $X_n \sim N(m_n, \sigma_n^2)$. X is r.v.

i) $X_n \xrightarrow{d} X \Rightarrow X \sim N(m, \sigma^2)$, $m = \lim m_n$. $\sigma^2 = \lim \sigma_n^2$.

ii) $X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{L^p} X$. $\forall\ 1 \le p < \infty$.

Pf. i) $\lim_n f_n(t) = \lim_n e^{im_n t - \frac{1}{2}\sigma_n^2 t} = f(t)$. ch.f of X.

$\Rightarrow |f_n| \to |f|$. $\exists\ t_0$. $\sigma_n^2 t_0^2 \to -2\log|f(t_0)| < \infty$.

So $\lim \sigma_n^2 = \sigma^2$ exists. $\Rightarrow m_n \to m$ exists

follows from: $f_n e^{\frac{1}{2}\sigma_n^2 t^2} \to f e^{\frac{1}{2}\sigma^2 t^2}$. So $f = e^{imt - \frac{1}{2}\sigma^2 t^2}$

ii) by i). (σ_n^2). (m_n) are bdd. So:

$\sup_n E|X_n|^q < \infty$. $\forall q \ge 1$. by property of normal dist.

$\Rightarrow (X_n^p)$ is u.i. besides $X_n \xrightarrow{P} X$.

② **Vectors:**

Def: E is d-dim Euclidean space. $\langle\ ,\ \rangle$ is inner product.

$X(\omega)$ take values in E is called Gaussian vector

if $\forall u \in E$. $\langle u, X\rangle$ is Gaussian variable.

Rmk: $\exists\ m_X \in E$. q_X quadratic form on E. st. $\forall u \in E$.

$E\langle u, X\rangle = \langle u, m_X\rangle$. $\mathrm{Var}\langle u, X\rangle = q_X(u) \ge 0$.

So $\langle u, X\rangle \sim N(\langle u, m_X\rangle, q_X(u))$

prop. $(e_i)_1^d$ is o.n.b of E. $X_k = \langle e_i, X\rangle$. Then:

$(X_k)_1^d$ indept $\Leftrightarrow$ $(\mathrm{cov}(X_i, X_k))$ is diagonal

i.e. q_X is diagonal form.

Rmk: For q_X. $\exists$ unique symmetric endomorphism γ_X of E. s.t. $q_X(u) = \langle u, \gamma_X(u) \rangle$ and matrix of γ_X in $(e_i)_1^d$ is $(\mathrm{Cov}(X_i, X_j))_{d\times d}$

Thm. For centered Gaussian vector (i.e. $m_X = 0$)

i) $\forall$ nonnegative symmetric endomorphism γ of E.

Then: $\exists X$ Gaussian vector. s.t. $\gamma_X = \gamma$.

ii) X is centered Gaussian vector. $(\varepsilon_i)_1^d$ is basis of E. s.t. γ_X is diagonal. $\gamma_X \varepsilon_j = \lambda_j \varepsilon_j$. $1 \le j \le d$.

where $\lambda_1 \ge \lambda_2 \ge \cdots \lambda_r > 0 = \lambda_{r+1} \cdots = \lambda_d = 0$. Then:

$X = \sum_1^r Y_i \varepsilon_i$. $(Y_i)_1^r$ indept. $\mathrm{Var}(Y_i) = \lambda_i$. $1 \le i \le r$

So: If $X \sim P_X$. $\mathrm{supp}(P_X) = \mathrm{span}(\varepsilon_i)_1^r$. and

$P_X \ll$ Lebesgue measure of $E \iff r = d$.

pf: i) $(\varepsilon_i)_1^d$ is o.n.b of E. s.t. γ is diagonal.

$\gamma(\varepsilon_i) = \lambda_i \varepsilon_i$. Set Y_i Gaussian variables

$\mathrm{Var}(Y_i) = \lambda_i$. Let $X = \sum_1^d Y_i \varepsilon_i$.

ii) $X = \sum_1^d Y_i \varepsilon_i$. So: $\mathrm{Var}(Y_i) = 0$. $\forall r < i \le d$. $\Rightarrow Y_i = 0$ a.s.

$\Rightarrow X = \sum_1^r Y_i \varepsilon_i$. $\mathrm{Var}(Y_i) = \lambda_i$. $\mathrm{supp}(P_X) = \mathrm{span}(\varepsilon_i)_1^r$.

It's easy to check the latter. since Y_i indept.

$P(a_i \le Y_i \le b_i,\ 1 \le i \le k) = \prod P(a_i \le Y_i \le b_i)$

$\Rightarrow r < d$. $P_X \perp M_E$.

Rmk: To obtain p.d.f in ii): $(Y_1 \cdots Y_d) \sim$

$(2\pi)^{-\frac{d}{2}} \sqrt{\lambda_1 \cdots \lambda_d}\, \exp(-\frac{1}{2} \sum y_i^2/\lambda_i)$. For $\forall g$

bdd. conti. $E(g(X)) = E(g(\varphi(Y)))$. $\varphi(\vec{y}) =$

$\sum \varepsilon_i y_i$. So: $X \sim P_X = (2\pi)^{-\frac{d}{2}} |\det \gamma_X|^{-1} e^{-\frac{1}{2}\langle x, \gamma_X^{-1}(x) \rangle}$

(3) Gaussian Space:

Def: i) Gaussian space is closed linear space of $L^2(\Omega, \mathcal{F}, P)$. contains only Gaussian variables.

ii) $(E, \mathcal{E})$ measurable space. A random process with values in E. is collection $(X_t)_{t\in T}$. s.t. $X_t(\omega) \in E$. It's Gaussian process if: Any finite linear combination of $(X_t)_{t\in T}$ is Gaussian variable ($E = \mathbb{R}^1$. $\mathcal{E} = \mathcal{B}_{\mathbb{R}^1}$).

Rmk: $CLS(X_t)_{t\in T}$ is a Gaussian space generated by Gaussian process X. Since L^2 limit of X_t is still Gaussian.

Thm: H is centered Gaussian space. $(H_i)_{i\in I}$ is collection of linear subspace of H. Then: $H_i \perp H_j$. $i \neq j \Leftrightarrow$ $\sigma(H_i)$. $i \in I$. indept.

Pf. ($\Leftarrow$) Indept implies: $\langle X. Y \rangle_{L^2} = \int_\Omega XY\,dP = E(XY) = 0$

($\Rightarrow$) Find o.n.b. $(\xi_i^j)_{i=1}^{n_j}$ for $(H_{i_j})_{j=1}^{p} \subseteq (H_i)_{i\in I}$

$(\xi_1^1 \cdots \xi_{n_1}^1) \cdots (\xi_1^p \cdots \xi_{n_p}^p)$ indept.

Cor. $K \subseteq H$. CLS. For $X \in H$. Then $E(X \mid \sigma(K)) = P_K X$

Rmk: For general r.v. X. $E(X \mid \sigma(K)) = P_{L^2(\Omega. \sigma(K). P)}(X)$

K is much smaller than $L^2(\Omega. \sigma(K). P)$

Cor. For $H_i \subseteq H$. $i = 1.2$. If $E(X_1 X_2) = E(P_K(X_1) P_K(X_2))$ for $\forall X_1 \in H_1$. $X_2 \in H_2$. Then: $\sigma(H_1)$. $\sigma(H_2)$ are conditionally indept given $\sigma(K)$.

Pf: For $X_1^1 \cdots X_{m_1}^1 \subseteq H_1$. $X_1^2, \cdots X_{m_2}^2 \subseteq H_2$.

Show: $E(I_{\{X_i^1 \in A_i^1, 1\le i\le m_1\}} I_{\{X_i^2 \in A_i^2 \cdots\}} | \sigma(K))$

$= E(I_{B_1} | \sigma(K)) E(I_{B_2} | \sigma(K)) \cdots (*)$

Replace (X_i^j) by $(Z_i^j)_{i\ge 1}^{A_j}$. o.n.b. of span (X_i^j)

Then use M.C.T. to obtain $E_i \in \sigma(H_i)$

From condition: $E((Z_i^s - P_K Z_i^s)(Z_j^r - P_K Z_j^r)) = 0$

when $r = s$. $i \ne j$. or $r \ne s$. $\forall i, j$.

Set $Y_i^s = Z_i^s - P_K(Z_i^s)$. replace Z_i^s in $(*)$

Note $P_K(Z_i^s) \in \sigma(K)$. So $(*)$ holds!

Thm. $\Gamma: T \times T \to \mathbb{R}^1$. symmetric. positive type. Then:

there exists prob. space $(\Omega, \mathcal{F}, P)$ and Gaussian process $(X_t)_{t\in T}$. st. covariance function is Γ

Pf: $\forall$ finite subset S of $T \Rightarrow$ exists $(X_i)_{i\in S}$. for $\Gamma|_{S\times S}$

Check the list satisfies consistency condition.

Then apply Kolmogorov Extension Thm.

(4) Gaussian White Noise:

Def. $(E, \mathcal{E})$ measure space. μ is σ-finite measure.

A Gaussian white noise with intensity μ is:

$G: L^2(E, \mathcal{E}, \mu) \xrightarrow{\text{isometry}}$ Gaussian space. $(m_X = 0)$

Rmk: For: $f, g \in L^2(E, \mathcal{E}, \mu)$. $E(G(f) G(g))$

$= \langle G(f), G(g) \rangle_{L^2(\Omega, \mathcal{F}, P)} = \langle f, g \rangle_{L^2(E, \mathcal{E}, \mu)}$

In particular. $f = I_A \Rightarrow$ Denote $G(A) = G(I_A)$

$\sim N(0, \mu(A))$. For (A_i) disjoint. finite measure.

$(G(A_i))$ are indept.

<u>prop.</u> $(E, \mathcal{E})$ measurable space with σ-finite measure

μ. Then there exists prob. space $(\Omega, \mathcal{F}, P)$

a Gaussian white noise with intensity μ on it.

<u>Pf.</u> $(f_i)_{i\in I}$ is o.n.b of $L^2(E, \mathcal{E}, \mu)$

Construct $(X_i)_{i\in I} \sim N(0,1)$. indept. on $(\Omega, \mathcal{F}, P)$

Set $G: f = \sum_{i\in I} \langle f_i, f\rangle f_i \longmapsto \sum_{i\in I} \langle f_i, f\rangle X_i$. isometry.

<u>Rmk:</u> $(\Omega, \mathcal{F}, P)$ should be appropriate since many

prob. space can only contain countable indept r.v.'s.

<u>prop.</u> G. Gaussian white noise with μ on $(E, \mathcal{E})$. $A \in \mathcal{E}$. st.

$\mu(A) < \infty$. If $\exists (A_i^j)_{i=1}^{k_i}$. st. $A = \sum_{i=1}^{k_n} A_i^n$. whose mesh $\to 0$

i.e. $\lim_n \sup_{1\le j\le k_n} \mu(A_j^n) = 0$. Then: $\sum_i^{k_n} G(A_j^n)^2 \xrightarrow{L^2} \mu(A)\ (n\to\infty)$

<u>Pf.</u> For fix n. $(G(A_i^n))_{i=1}^{k_n}$ is indept. $E(G^2(A_j^n)) = \mu(A_j^n)$

Note: $E\left|\sum_i^{k_n} G(A_i^n) - \mu(A)\right|^2 = \sum_i^{k_n} Var(G(A_i^n)^2) = 2\sum_i^{k_n} \mu(A_i^n)^2$

follows from $Var(X^2) = 2\sigma^4$ if $X \sim N(0, \sigma^2)$.

$RHS \le (\sup_j \mu(A_j^n))\, \mu(A) \to 0\ (n\to\infty)$

<u>Rmk:</u> It provides a way to recover $\mu(A)$ by

values of G on atoms of finer partition

of A.