Ergodic Theorem.

(1) Definitions:

Def: $X_0, X_1, \cdots$ is said to be stationary seq. if
$(X_0, X_1, \cdots) \sim (X_k, X_{k+1}, \cdots)$. for $\forall k \in \mathbb{Z}^+$.
e.g. (X_k) is i.i.d.

prop. Any stationary seq $(X_k)_{k\geq 0}$ can be embedded in a two-side stationary seq $(Y_n)_{n\in\mathbb{Z}}$.

Pf: Set $P(Y_{-m} \in A_0 \cdots Y_n \in A_{m+n}) = P(X_0 \in A_0 \cdots X_{m+n} \in A_{m+n})$
is finite dimension dist. satisfies consistency
By Kolmogorov. $\exists\ P$ on $(S^{\mathbb{Z}}, \mathcal{S}^{\mathbb{Z}})$ ✓

Thm. For stationary seq $(X_k)_{k\geq 0}$. $g: \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^1$ is measurable. Then: $Y_k = g(X_k, X_{k+1} \cdots)$ is stationary seq.

Pf: For $x \in \mathbb{R}^{\mathbb{N}}$. denote $g_k(x) = g(x_k, x_{k+1} \cdots)$

$$\begin{aligned} P((Y_0 \cdots) \in B) &= P((X_0, \cdots) \in A) \\ &= P((X_k \cdots) \in A) \\ &= P((Y_k \cdots) \in B) \end{aligned}$$

$A = \{ x \in \mathbb{R}^{\mathbb{N}} \mid (g_0(x), g_1(x) \cdots) \in B \}$. $B \in \mathcal{B}_{\mathbb{R}^{\mathbb{N}}}$

Def. i) For prob. space: $(\Omega, \mathcal{F}, P)$. $\varphi: \Omega \to \Omega$. measurable is measure preserving if $P(\varphi^{-1}(A)) = P(A)$. $\forall A \in \mathcal{F}$.

ii) For measure preserving map φ on $(\Omega, \mathcal{F}, P)$. $A \in \mathcal{F}$ is invariant if $\varphi^{-1}(A) = A$. P-a.s.

Rmk: i) $X_n(w) = X(\varphi^n w)$ is a stationary seq. where φ is measure preserving.

Pf: $P(\{w \mid (w_k \cdots w_{k+n}) \in B\}) = P(\varphi^k w \in A)$

$= P(w \in A) = P(\{w \mid (w_0 \cdots w_n) \in B)$

ii) The collection of invariant events w.r.t φ is σ-algebra. denote by $\mathcal{I}$.

Pf: $\varphi^{-1}(\bigcup A_n) = \bigcup \varphi^{-1}(A_n)$. $\varphi^{-1}(A \cap B) = \varphi^{-1}(A) \cap \varphi^{-1}(B)$

prop. $X \in \mathcal{I} \iff X$ is invariant. i.e. $X \circ \varphi = X$. a.s.

Pf: $\varphi^{-1}(\{X \in A\}) = \{X \in A\} \iff \{X \circ \varphi \in A\} = \{X \in A\}$.

for $\forall A \in \mathcal{B}_{\mathbb{R}}$. in a.s. sense. so $X \circ \varphi = X$. a.s.

Def: Measure preserving map φ on $(\Omega, \mathcal{F}, P)$ is ergodic if $\mathcal{I}$ is trivial: $\forall A \in \mathcal{I}$. $P(A) \in \{0, 1\}$.

Rmk: If φ isn't ergodic. Then $\exists A \in \mathcal{F}$. st. $\Omega = A \cup A^c$. $P(A)$. $P(A^c) > 0$. $\varphi(A) = A$. $\varphi(A^c) = A^c$ (not irred)

Pf: $\exists A \in \mathcal{I}$. st. $P(A) \in (0, 1)$. Note: $A^c \in \mathcal{I}$

prop. If φ is shift operator on $(S, \mathcal{S}, P)$. S is countable. Besides all states is recurrent. Then:

φ is ergodic $\Longleftrightarrow$ p is irreducible.

Pf: ($\Rightarrow$) If p isn't irred. Then $S = \bigcup R_i$. decompose.

Note: $\{X_0 \in R_i\}$ is invariant. but not trivial.

($\Leftarrow$) For $A \in \mathcal{I}$. Then $I_A \circ \theta_n = I_A$.

So: $E_\mu(I_A \mid \mathcal{F}_n) = E_{X_n}(I_A) =: h(X_n)$

$\Rightarrow h(X_n) \uparrow I_A \in \{0,1\}$. But $X_n = y$. i.o. $\forall y \in S$

To guarantee the converge $\Rightarrow h(x) \equiv 0$ or 1.

So $E_\mu(I_A) = P_\mu(A) = E_\mu(h(X_n)) \in \{0,1\}$.

Rmk: It shows $\mathcal{I}$ and $\mathcal{T}$ may be different:

Set $\mu > 0$. if p is irred. $d > 1$. Then:

$\mathcal{I}$ is trivial. But $\mathcal{T} = \sigma(\{X_0 \in S_r\})$ not!

(2) Birkhoff's Ergodic Thm:

Suppose φ is measure-preserving on $(\Omega, \mathcal{F}, P)$

① Lemma: (Maximum ergodic)

$X_j(\omega) = X(\varphi^j \omega)$. $X \in L^1$. $S_k(\omega) = \sum_0^{k-1} X_i(\omega)$.

$M_k(\omega) = \max\{0, S_1(\omega) \cdots S_k(\omega)\}$. Then: $E(X I_{\{M_k > 0\}}) \geq 0$.

Pf: $M_k(\varphi\omega) \geq S_j(\varphi\omega)$. $\forall j \leq k$. So:

$X(\omega) + M_k(\varphi\omega) \geq X(\omega) + S_j(\varphi\omega) = S_{j+1}(\omega)$.

So $X(\omega) \geq \max_{1 \leq j \leq k} \{S_j(\omega)\} - M_k(\varphi\omega)$

$\Rightarrow E(X I_{\{M_k > 0\}}) \geq \int_{\{M_k > 0\}} M_k(\omega) - M_k(\varphi\omega) \, dP$

$\geq \int M_k - M_k \, dP = 0$ (φ is m.p)

Cor. (Wiener's Maximal Inequality)

Set $A_k = S_k/k$. $D_k = \max_{1\le i\le k} \{A_i\}$. Then: for $\alpha>0$

$P(D_k > \alpha) \le E|X|/\alpha$.

Pf: Set $X^* = X-\alpha \in L^1$. $\{D_k > \alpha\} = \{M_k^* > 0\}$.

$M_k^* = \max_{1\le i\le k} S_i^*/i$. $\{M_k^* > 0\} = \{\max\{0, S_1^*, \cdots S_k^*\} > 0\}$

So $E((X-\alpha) I_{\{D_k > \alpha\}}) \ge 0$.

Thm. For $\forall X \in L^1$. $\frac{1}{n}\sum_0^{n-1} X(\varphi^m w) \to E(X|\mathcal{I})$. a.s and L^1.

Rmk: When φ is ergodic $\Rightarrow E(X|\mathcal{I}) = E(X)$. ($P(A)=0/1$)

Pf: $E(X|\mathcal{I}) \in \mathcal{I}$. So it's invariant. Set $X' = X - E(X|\mathcal{I})$

So: WLOG. set $E(X|\mathcal{I}) = 0$

1°) $S_n/n \to 0$. a.s.

Set $\bar{X} = \overline{\lim}\, S_n/n$. $\varepsilon>0$. $D = \{w \mid \bar{X}(w) > \varepsilon\}$.

Note: $\bar{X}(\varphi w) = \bar{X}(w)$. So $D \in \mathcal{I}$. Prove: $P(D) = 0$.

Denote: $X^*(w) = (X(w)-\varepsilon) I_D(w)$. $S_n^*(w) = \sum_1^n X^*(\varphi^{k-1} w)$

$M_n^*(w) = \max\{0, \cdots S_n^*(w)\}$. $F_n = \{M_n^*(w) > 0\}$.

$F = \cup F_n = \{\sup_{k\ge1} S_k^*/k > 0\}$

$= \{\sup_{k\ge1} S_k/k > \varepsilon\} \cap D = D$

By Lemma: $0 \le E(X^* I_{F_n}) \uparrow E(X^* I_F)$. ($X^* \in L^1$)

So: $0 \le E(E(X|\mathcal{I})) - \varepsilon P(D)$. Since $D = F$.

2°) For L^1 part:

It's not a good idea to check u.i.

Set: $X_m'(w) = X I_{\{|X|\le m\}}$. $X_m''(w) = X(w) - X_m'(w)$.

By BDT. and i'): $E|\frac{1}{n}\sum_0^{n-1} X_m'(\varphi^k w) - E(X_m'|\mathcal{Z})| \to 0$

With: $E|\frac{1}{n}\sum_0^{n-1} X_m''(\varphi^k w) - E(X_m''|\mathcal{Z})| \le 2E|X_m''| \to 0$.

Rmk: If $X \in L^p$. $p>1$. Then apply Minkowski. inequility. The converge occurs in L^p.

Cor. i) If $g_n(w) \to g(w)$. a.s. $\sup_k |g_k(w)| \in L'$.

Then. $\frac{1}{n}\sum_0^{n-1} g_m(\varphi^m w) \to E(g|\mathcal{Z})$. a.s.

ii) If $g_n(w) \xrightarrow{L'} g(w) \in L'$. Then. we have:

$\frac{1}{n}\sum_0^{n-1} g_m(\varphi^m w) \to E(g|\mathcal{Z})$ in L'.

Pf: i) $h_m(w) = \sup_{n\ge m} |g_m(w) - g(w)| \Rightarrow h_0 \in L'$.

$\therefore \overline{\lim_n} |\frac{1}{n}\sum_0^{n-1} [g_m(\varphi^m w) - g(\varphi^m w)]| \le$

$\overline{\lim_n} \frac{1}{n}\sum_m^{n-1} h_m(\varphi^k w) = E(h_m|\mathcal{Z})$. $\forall m \in \mathbb{Z}^+$.

By DCT $\Rightarrow E(h_m|\mathcal{Z}) \downarrow 0$. $(m \to \infty)$.

ii) $E|\square| \le \frac{1}{n}\sum_0^{n-1} E|g_m(\varphi^m w) - g(\varphi^m w)| + E|\frac{\sum_0^{n-1} g(\varphi^m w)}{n} - E(g|\mathcal{Z})|$

② Equidistribution:

Consider $([0,1), \mathcal{B}_{[0,1)})$, $\varphi: w \to \theta + w \pmod 1$. $\theta \in (0,1)$.

Thm. If θ is irrational. Then φ is ergodic.

Pf: For $f \in L^2[0,1)$. $f = \sum_1^\infty c_k e^{2\pi i k x}$. by F-expansion.

$f\circ\varphi = \sum_1^\infty c_k e^{2\pi i k(\theta+x)} = f \Leftrightarrow c_k(e^{2\pi i k\theta} - 1) = 0$.

$\Leftrightarrow c_k \equiv 0$. $\forall k \ne 0$. since $\theta \notin \mathbb{Q}$. So: $f \equiv$ const.

Set $f = I_A$. $A \in \mathcal{Z}$. $\Rightarrow I_A \in \{0,1\}$. a.s.

Rmk: Note $\mathcal{Z}$ is trivial. Set $X(\omega) \in L^1$.

By ergodic Thm: $\frac{1}{n}\sum_{m=0}^{n-1} I_{\{\varphi^m \omega \in A\}} \to |A|$. a.s.

Thm. If $A = [a,b)$. Then: $\frac{1}{n}\sum I_{\{\varphi^m \omega \in A\}} \longrightarrow |A|$. pointwise.

Pf: Set $A_k = [a+\frac{1}{k}, b-\frac{1}{k})$. $b-a > \frac{2}{k}$. Then:

$\frac{1}{n}\sum I_{A_k}(\varphi^m \omega) \longrightarrow b-a-\frac{2}{k}$. $\forall \omega \in \Lambda_k$. $P(\Lambda_k)=1$.

Set $G = \cap \Lambda_k$. Then: $P(G)=1$. So G is dense.

$\forall x \in [0,1)$. $\exists \omega_k \in G$. $|x-\omega_k| < \frac{1}{k}$. $\forall k$ large enough.

So: $\varphi^m \omega_k \in A_k \Rightarrow \varphi^m x \in A$. Then:

$\underline{\lim} \frac{1}{n}\sum I_A(\varphi^m x) \geq \lim \frac{1}{n}\sum I_{A_k}(\varphi^m \omega_k) = b-a-\frac{2}{k} \xrightarrow{k\to\infty} b-a$.

Conversely, apply above on $A^c = [0,a), [b,1)$.

(3) Recurrence:

Suppose $X_1 \cdots X_k \cdots$ stationary seq. take values in $\mathbb{R}^d$. φ is shift

$S_k = \sum_1^k X_i$. Set: $A = \{S_k \neq 0, \forall k \geq 1\}$. $R_n = \#\{S_k\}_1^n$. $\mathcal{Z} = \mathcal{Z}(\varphi)$.

Rmk: $R_n = \sum_{k=0}^{m} I_{\{S_i \neq S_k,\ \forall k+1 \leq i \leq n\}} \leq n$. Note: (X_k) is

stationary. Then: $E(R_n) = \sum_{m=1}^{m} g_m$. $g_m = P(S_i \neq 0,\ 1 \leq i \leq m)$

Thm. $R_n/n \xrightarrow{n\to\infty} E(I_A \mid \mathcal{Z})$. a.s. ($\mathcal{Z}$ is w.r.t φ shift operator)

Pf: Consider $W = (W_n) = (X_n(\omega)) \in (\mathbb{R}^d)^{\mathbb{N}}$. φ is its

shift operator. Note: $R_n = \#\{k \mid S_k \neq S_i,\ k+1 \leq i \leq n\}$.

$R_n \geq \sum_{m=1}^{n} I_A(\varphi^m \omega) = \#\{m \mid 1 \leq m \leq n.\ S_l \neq S_m.\ \forall l \geq m+1\}$.

$\Rightarrow \varliminf_n R_n/n \geq E(I_A \mid \mathcal{Z})$. a.s.

Conversely. $A_k = \{S_i \neq 0.\ 1 \leq i \leq k\}$. Then, we obtain:

$$R_n \leq k + \sum_{m=1}^{n-k} I_{A_k}(\varphi^m w) = k + \#\{m \mid 1 \leq m \leq n-k,\ S_l \neq S_m.\ m+1 \leq l \leq m+k\}.$$

$\Rightarrow \varlimsup R_n/n \leq E(I_{A_k} \mid \mathcal{Z}) \downarrow E(I_A \mid \mathcal{Z})$. a.s.

Thm $X_1, X_2 \cdots$ stationary seq take value in $\mathbb{Z}$. $X_k \in L^1$. $\forall k$.

$S_n = \sum_1^n X_k$. $A = \{w \mid S_k \neq 0.\ \forall k \geq 1\}$. Then:

i) $E(X_1 \mid \mathcal{Z}) = 0 \Rightarrow p(A) = 0$

ii) $p(A) = 0 \Rightarrow p(S_n = 0 \text{ i.o}) = 1$.

Rmk: i) It means: Zero mean $\Rightarrow$ recurrence

ii) $E(X_1 \mid \mathcal{Z}) = 0$ is to rule out that has mean 0.

but are combination of seq. with positive and negative means.

Pf. i) $E(X_1 \mid \mathcal{Z}) = 0 \Rightarrow S_n/n \to 0$. a.s.

$$\varlimsup_n \max_{1 \leq k \leq n} (|S_k|/n) \leq \varlimsup_n \max_{K \leq k \leq n} |S_k|/n \leq \max_{k \geq K} |S_k|/k \xrightarrow{K \to \infty} 0$$

So: $\lim_n \left(\max_{1 \leq k \leq n} \frac{|S_k|}{n} \right) = 0$.

Since: $R_n \leq 1 + 2 \max_{1 \leq k \leq n} |S_k| \Rightarrow R_n/n \to 0$. a.s.

$\Rightarrow E(I_A \mid \mathcal{Z}) = 0 \Rightarrow p(A) = 0$.

ii) Set: $F_j = \{S_i \neq 0 \quad 1 \leq i < j.\ S_j = 0\}$. $A^c = \cup F_j. \Rightarrow \sum p(F_j) = 1$.

$G_{j,k} = \{S_{j+i} - S_j \neq 0.\ 1 \leq i < k.\ S_{j+k} - S_j = 0\} \Rightarrow p(G_{j,k}) = p(F_k)$

Note: $G_{j,k}$ disjoint for fix j. $\Rightarrow \sum_k G_{j,k} = \Omega$. a.s.

$\Rightarrow \sum_k p(F_j \cap G_{j,k}) = p(F_j)$. $\sum_k \sum_j p(F_j \cap G_{j,k}) = 1$.

On $F_j \cap G_{j,k}$. $S_j = S_{j+k} = 0$. $\Rightarrow p(S_n = 0.$ at least twice$) = 1$.

Repeat by replace A^c by $\{S_n = 0.$ at least 2 times$\}$.

Decompose $= \bigcup F_{j,k}$. $F_{j,k} = \{S_i \neq 0. 1 \le i < j. j+1 \le i < k. S_j = S_k = 0\}$.

$\Rightarrow p(S_n = 0$ at least k times$) = 1$. $\forall k$. Let $k \to \infty$.

<u>Cor.</u> Under the condition above. if $p(X_i > 1) = 0$.

$E(X_i) > 0$. (X_n) is ergodic. (i.e. shift on $W = (W_n) \in$

$\Omega = \mathbb{Z}^{\mathbb{Z}}$. is). Then: $p(A) = E(X_i)$.

<u>Pf</u>: $p(X_i > 1) = 0 \Rightarrow \max\limits_{1 \le m \le n} S_m \le R_n \le \max\limits_{1 \le m \le n} S_m - \min\limits_{1 \le m \le n} S_m$.

$\frac{S_n}{n} \to E(X_1)$. a.s. $\Rightarrow S_n \to +\infty$. a.s

So $\min\limits_{m \le n} S_m > -\infty$. a.s. $\min\limits_{m \le n} S_m / n \xrightarrow{n \to \infty} 0$. a.s.

Besides. $\overline{\lim\limits_n} \max\limits_{m \le n} S_m / n \le \max\limits_{k \ge N} \frac{S_k}{k} \to \overline{\lim\limits_n} \frac{S_n}{n}$ $(N \to \infty)$

<u>Thm.</u> (Kac's)

If $X_0, X_1 \cdots$ stationary seq. take value in $(S, \mathcal{S})$.

$A \in \mathcal{S}$. Set. $T_0 = 0$. and $T_n = \inf\{m > T_{n-1} \mid X_m \in A\}$.

St. $p(T_1 < \infty) = 1$. Then: under $p(\cdot \mid X_0 \in A)$. we have:

$t_n = T_n - T_{n-1}$ is stationary seq with $E(T_1 \mid X_0 \in A) = \frac{1}{p(X_0 \in A)}$

<u>Pf.</u> 1°) Show: $p(t_1 = m. t_2 = n \mid X_0 \in A) = p(t_2 = m. t_3 = n \mid X_0 \in A)$

Firstly. extend $\{X_n\}_{n \ge 0}$ to $\{X_n\}_{n \in \mathbb{Z}}$.

$C_k = \{X_i \notin A. 1-k \le i \le -1, X_{-k} \in A\}$.

$(\bigcup\limits_1^m C_k)^c = \{\exists k. 1 \le k \le m. X_{-k} \in A\}^c = \{\forall -m \le k \le -1. X_k \notin A\}$.

has same prob. as $\{\forall 1 \le k \le m. X_k \notin A\}$.

$\Rightarrow$ let $m \to \infty$. $p(\bigcup C_k) = 1$.

Set $I_{j,k} = \{ i \in [j,k] \mid X_i \in A \}$.

$$p(t_2=m, t_3=n, X_0 \in A) = \sum_{l}^{\infty} p(X_0 \in A, t_1 = l, t_2 = m, t_3 = n)$$

$$= \sum_{l} p(I_{0,l+m+n} = \{0, l, l+m, l+m+n\})$$

$$= \sum p(I_{-l,m+n} = \{-l, 0, m, m+n\})$$

$$= \sum_{l} p(C_l, X_0 \in A, t_1 = m, t_2 = n)$$

$$= p(t_1 = m, t_2 = n, X_0 \in A)$$

2') $E(t_1 \mid X_0 \in A) = \sum_{k=1}^{\infty} p(t_1 \geq k \mid X_0 \in A) = \sum p(t_1 \geq k, X_0 \in A) / p(X_0 \in A)$

$$= \sum p(X_0 \in A, t_1 \leq -k) / p(X_0 \in A) = \frac{\sum p(C_k)}{p(X_0 \in A)} = \frac{1}{p(X_0 \in A)}$$

Rmk: It's generalization of $E_x(T_x) = 1/\pi(x)$, where $A = \{x\}$, X_n is Markov chain. We generalize to $\forall A \in \mathcal{S}$, drop "$X_n$ is Markov Chain".

Cor: If $p(X_n \in A$ at least once$) = 1$, $A \cap B = \emptyset$.

Then: $E(\sum_{1 \leq k \leq T_1} I_{\{X_k \in B\}} \mid X_0 \in A) = p(X_0 \in B) / p(X_0 \in A)$

Pf: $LHS = \sum_{m=1}^{\infty} p(X_0 \in A, X_1 \sim X_{m-1} \notin A, X_m \in B) / p(X_0 \in A)$

$$= \sum p(X_{-m} \in A, X_{1-m} \sim X_{-1} \notin A, X_0 \in B) / p(X_0 \in A)$$

$$= \sum p(C_k, X_0 \in B) / p(X_0 \in A)$$

$$= p(X_0 \in B) / p(X_0 \in A)$$

Rmk: It generalizes the "cyclic Techique" for constructing stationary measure.

Thm (Pioncaré)

$\varphi: \Omega \to \Omega$ is measure perserving. $T_A = \inf\{n \geq 1 \mid \varphi^n(w) \in A\}$.

Then: i) $P(w \in A, T_A = \infty) = 0$

ii) $A \subset \{\varphi^n(w) \in A, \text{ i.o}\}$.

iii) If φ is ergodic, $P(A) > 0$. Then $P(\varphi^n(w) \in A \text{ i.o}) = 1$

Rmk: It checks the hypothesis of Kac's Thm:

by $X_n(w) = X(\varphi^n w)$. Stationary. $A = \{w \mid X(w) \in B\}$.

So it satisfies the condition if start on B.

(i) also implies recurrent of A)

Pf: i) $B = \{w \in A, T_A = \infty\}$. So: $w \in \varphi^{-m}(B) \Leftrightarrow \varphi^m(w) \in A$

and $\varphi^n(w) \notin A$, $\forall n > m$.

$\Rightarrow (\varphi^{-m}(B))$ is pairwise disjoint. But $P(\varphi^{-m}B) = P(B)$

So $P(B) = 0$. Otherwise $\sum_{m=1}^{\infty} P(\varphi^{-m}B) = \infty$.

ii) $\forall k$. φ^k is measure-preserving. by i):

$P(w \in A, \varphi^{nk}(w) \notin A, \forall n \geq 1) = 0 \geq P(w \in A, \varphi^m(w) \notin A, \forall m \geq k)$

iii) $\{\varphi^n(w) \in A, \text{ i.o}\}$ is invariant, and contains A, $P(A) > 0$

(4) Subadditive Ergodic Thm.

Thm (Liggett's)

If $X_{m,n}$, $0 \leq m < n$, satisfies:

i) $X_{0,m} + X_{m,n} \geq X_{0,n}$ ii) $(X_{nk,(n+1)k})_{n \geq 1}$ is stationary, $\forall$ fix k.

iii) dist. of $\{X_{m,m+k}\}_{k \geq 1}$ doesn't depend on m.

iv) $E(X_{0,1}^+) < \infty$. $E(X_{0,n}) \geq \gamma_0 n$, $\forall n$. $\gamma_0 > -\infty$. Then:

i) $\lim_n E(X_{0,n}/n) = \inf_n E(X_{0,m}/m) \equiv \gamma$. ii) $\lim_n X_{0,n}/n = X$ exists a.s. and in L^1. If seq in ii) is ergodic. Then $X = \gamma$ a.s.