Discrete Time Markov Chains

(1) Markov Properties:

① Lemmas:

<u>Thm.</u> (Kolmogorov Extension)

If $\{\mu_\alpha \mid \alpha \subseteq J,\ |\alpha| \neq 0,\ \text{finite}\}$ is family of p.m's.

$P_{Ik} : \mathbb{R}^k \longrightarrow \mathbb{R}^I : x = (x_j)_{j\in k} \longmapsto (x_j)_{j\in I} = x|_I$. for $I \subset k$.

suppose (μ_α) satisfies Kolmogorov consistency condition:

$\mu_\beta(P_{\alpha\beta}^{-1}(A)) = \mu_\alpha(A)$. $\forall\, \alpha \subset \beta \subset J$. finite subsets

and $\forall\, A \in B_{\mathbb{R}^\alpha}$. Then: $\exists$ unique p.m μ on $(\mathbb{R}^J, \sigma(\mathcal{F}))$

where $\sigma(\mathcal{F}) = \sigma(\pi_j \mid j \in J)$ $\pi_j : \mathbb{R}^J \longrightarrow \mathbb{R}^1$. $(x_j)_{j\in J} \longmapsto x_j$

s.t. $\mu(P_{\alpha J}^{-1}(A)) = \mu_\alpha(A)$. $\forall\, \alpha \subset J$. finite nonempty.

and $\forall\, A \in B_{\mathbb{R}^\alpha}$.

<u>Rmk</u>: i) It provides the existence of measure μ on $(\mathbb{R}^J, \sigma(\mathcal{F}))$ for arbitrary index set J with applopriate finite-dimension dist.

ii) $\mathbb{R}^\alpha$ can be replaced by a general space E. which is polish (i.e. separable. complete. metric).

<u>Thm.</u> (Monotone class Thm)

A is π-class. contains Ω. $\mathcal{H} = \{f : \mathbb{R}^1 \to \mathbb{R}^1\}$ s.t.

i) $A \in \mathcal{A} \Rightarrow I_A \in \mathcal{H}$.

ii) $f, g \in \mathcal{H} \Rightarrow f+g,\ cf \in \mathcal{H}.\ \forall c \in \mathbb{R}^1$.

iii) $f_n \geq 0 \uparrow f$ bdd $(f_n) \subseteq \mathcal{H} \Rightarrow f \in \mathcal{H}$.

Then: $\mathcal{H}$ contains all bdd measurable (w.r.t $\sigma(\mathcal{A})$) f.

Pf: ii). iii) implies $\mathcal{G}$ is λ-class. $\mathcal{G} = \{A \mid I_A \in \mathcal{H}\}$

So $\mathcal{G} \supset \sigma(\mathcal{A})$ By MCT.

ii) $\Rightarrow \mathcal{H}$ contains simple func's of $\sigma(\mathcal{A})$.

iii) $\Rightarrow \mathcal{H}$ contains all bdd measurable func's.

② Definitions:

Def: $(S, \mathcal{S})$ is measurable space. $p: S \times \mathcal{S} \rightarrow \mathbb{R}^1$ is said transition probability if:

i) $\forall x \in S.\ A \mapsto p(x, A)$ is p.m. on $(S, \mathcal{S})$.

ii) $\forall A \in \mathcal{S}.\ x \mapsto p(x, A)$ is measurable.

We say (X_n) is Markov Chain w.r.t $\mathcal{F}_n$. with trans. prob. p. if $\exists \widehat{P}$. p.m. $\widetilde{P}(X_{n+1} \in B \mid \mathcal{F}_n) = p(X_n, B)$.

Construction:

Given p (t.p) and μ initial dist on $(S, \mathcal{S})$

Def: $P(X_j \in B_j,\ 0 \leq j \leq n) =: \int_{B_0} \mu(dx_0) \int_{B_1} p(x_0, dx_1) \cdots \int_{B_n} p(x_{n-1}, dx_n)$

By Kolmogrov Extension $(J = N)$. if $(S, \mathcal{S})$ is nice.

$\exists P_\mu$ on seq space $(\Omega_0, \mathcal{F}_0) = (S^N, \mathcal{S}^N)$

So: for $X_n(w) = w_n$. it has desired dist.

<u>Denote:</u> P_x on $(\Omega_0, \mathcal{F}_\infty)$ with $\mu = \delta_x$. p.m.

Then: $P_\mu(A) = \int P_x(A)\mu(dx)$. $\forall A \in \mathcal{F}_\infty$.

<u>Rmk:</u> It's convient to define $\theta_n(w) = (w_n, w_{n+1} \cdots)$ on $(\Omega_0, \mathcal{F}_\infty, P_x)$.

Next, we prove (X_n) is Markov Chain. on $(\mathcal{F}_n^X)$. with p.

<u>Thm.</u> $P_\mu(X_{n+1} \in B \mid \mathcal{F}_n) = p(X_n, B)$. $\forall B \in \mathcal{S}$.

<u>Pf:</u> 1°) For $A = \{X_1 \in B_1 \cdots X_n \in B_n\}$. $B_i, C \in \mathcal{S}$.

$$\int_{B_0} \mu(dx_0) \int_{B_1} p(x_0, dx_1) \cdots \int_{B_n} I_C(x_n)\, p(x_{n-1}, dx_n)$$

$$= \int_A I_C(X_n)\, dP_\mu \text{ holds by def.}$$

2°) By DCT. approxi f bdd measurable by $\{I_C\}$.

Set $f = p(x, B)$. Then:

$$P_\mu(A, X_{n+1} \in B) = \int_A I_{\{X_{n+1} \in B\}}\, dP_\mu$$

$$= \int_{B_0} \mu(dx_0) \cdots \int_{B_n} p(x_n, B)\, p(x_{n-1}, dx_n)$$

$$= \int_A p(X_n, B)\, dP_\mu.$$

3°) By λ-π argue $\Rightarrow$ holds for $\forall A \in \mathcal{F}_n$.

<u>Rmk:</u> Denote $p(X_{n+1} = i \mid X_n = j) = p(j, i)$. Then:

$$p(X_{n+1} = j \mid X_k = i_k,\ 0 \le k \le n) = p(X_{n+1} = j \mid X_n = i_n)$$

$$= p(i_n, j).$$

Thm. $A \in \sigma(X_0, \cdots X_n)$. $B \in \sigma(X_n, \cdots)$. Then:

$$P_m(AB \mid X_n) = P_m(A \mid X_n)\, P(B \mid X_n)$$

Pf: $E_m(I_A I_B \mid X_n) = E_m(I_A E_m(I_B \mid \mathcal{F}_n) \mid X_n)$

$$= P_m(A \mid X_n)\, P_m(B \mid X_n).$$

Rmk: It means: Past and Future are indept condition on Present. They have same role.

Thm. $E_n(\prod_0^n f_k(X_k)) = \int f_0(x_0)\mu(dx_0) \cdots \int f_n(x_n) p(x_{n-1}, dx_n)$

for $\forall f_k$. bdd. measurable

Pf: Apply DCT on $E(I_B(X_{n+1}) \mid \mathcal{F}_n) = \int I_B(x_{n+1}) p(x_n, \cdot)$

$\Rightarrow E(f(X_{n+1}) \mid \mathcal{F}_n) = \int f(y) p(X_n, dy)$

Then by induction:

$$E(\prod_0^n f_m(X_m)) = E(\prod_0^{n-1} f_m(X_m)\, E(f_n(X_n) \mid \mathcal{F}_{n-1}))$$

③ Markov Property:

Thm. (Simple Markov)

$Y: \Omega_0 \to \mathbb{R}'$. bdd. measurable. $\Rightarrow E_\mu(Y \circ \theta_m \mid \mathcal{F}_m) = E_{X_m}(Y)$.

Pf: For $A = \bigcap_0^n \{W_i \in A_i\}$. g_i. bdd. measurable.

Set $f_k = \begin{cases} I_{A_k}. & k < m. \\ g_{k-m}. & m < k \leq m+n \end{cases}$ $\quad f_m = g_0 I_{A_m}$.

Consider $E(\prod_0^{m+n} f_k(X_k)) = \int_{A_0} \mu(dx_0) \cdots \int_{A_m} g_0(x_m) p(x_{m-1}, dx_m) \cdots$

Then: $E_\mu(\prod_0^n g_k(X_{m+k}) I_A) = E_\mu(E_{X_m}(\prod_0^n g_k(X_k)) I_A)$

By MCT $\Rightarrow$ it holds for $\forall A \in \mathcal{F}_m$.

Set $\mathcal{H} = \{Y \mid E_\mu(Y \circ \theta_m \mid \mathcal{F}_m) = E_{X_m}(Y)\}$. Note: $\prod I_{A_k} \in \mathcal{H}$.

Check it satisfies the Lemma. So $\mathcal{H}$ contains $Y \in \mathcal{F}_\infty$.

<u>Rmk</u>: Set $Y = I_{\{w_i \in A_i,\ 1 \le i \le k\}}$. it's common form.

<u>Cor.</u> (Chapman - Kolmogorov Equation)

$$p^{m+n}(x, z) = \sum_y p^m(x, y) p^n(y, z)$$

<u>Pf</u>: $LHS = E_x(I_{\{X_n = z\}} \circ \theta_m) = E_x E_x(I_{\{X_n = z\}} \circ \theta_m \mid \mathcal{F}_m))$

$= E_x(P_{X_m}(X_n = z)) = RHS.$

<u>Def</u>: For N stopping time. $\mathcal{F}_N = \{A \mid A \cap \{N = n\} \in \mathcal{F}_n,\ \forall n\}$.

$\theta_N(w) = \theta_n(w)$ on $\{N = n\}$. / Δ on $\{N = \infty\}$. $\forall w \in \Omega_0$.

Δ is extra point add to Ω_0.

<u>Thm.</u> (Strong Markov Property)

If $Y_n = \Omega_0 \longrightarrow \mathbb{R}'$. measurable. $|Y_n| \le M$. $\forall n$. Then:

$E_\mu(Y_N \circ \theta_N \mid \mathcal{F}_N) = E_{X_N}(Y_N)$ on $\{N < \infty\}$.

<u>Pf</u>: $E_\mu(Y_N \circ \theta_N I_{A \cap \{N < \infty\}}) = \sum_{n=0}^\infty E_\mu(Y_n \circ \theta_n I_{A \cap \{N = n\}})$

reduce to simple Markov Property.

④ Arrival Time:

<u>Def.</u> $T_y^0 = 0$. $T_y^k = \inf\{n > T_y^{k-1} \mid X_n = y\}$. $k \geq 1$.

the k^{th} time arrival at y.

<u>Denote:</u> $T_y = T_y^1$. $\rho_{xy} = P_x(T_y < \infty)$

<u>Thm.</u> $P_x(T_y^k < \infty) = \rho_{xy}\rho_{yy}^{k-1}$.

<u>Pf:</u> $k=1$ is trivial. If $k \geq 2$. Set $Y = \begin{cases} 1. \text{ If } w_n = y. \exists n \\ 0. \text{ otherwise} \end{cases}$

$N = T_y^{k-1}$. Then: $Y \circ \theta_N = 1$ if $T_y^k < \infty$.

$\therefore P_x(T_y^k < \infty) = E_x(Y \circ \theta_N I_{\{N<\infty\}})$

$= E_x(E_x(Y \circ \theta_N \mid \mathcal{F}_N) I_{\{N<\infty\}})$

$= E_x(E_{X_N}(Y) I_{\{N<\infty\}}) = \rho_{yy} P_x(T_y^{k-1} < \infty)$

<u>Thm.</u> (First Entrance Decomposition)

$$p^n(x,y) = \sum_{j=1}^{n} P_x(T_y = j) p^{n-j}(y,y)$$

<u>Pf:</u> $p^n(x,y) = \sum P(X_0 = x.\ T_y = j.\ X_n = y)$

<u>Thm.</u> $\sum_{m=0}^{n} p^m(x,x) \geq \sum_{k}^{n+k} p^m(x,x)$. $\forall k \geq 0$.

<u>Pf:</u> Set $T_{(k)}^x = \inf\{n \geq k \mid x_n = x\}$.

$$RHS = \sum_{m=k}^{n+k} \sum_{j=k}^{m} P_x(T_{(k)}^x = j) p^{m-j}(x,x)$$

$$= \sum_{j=k}^{n+k} P_x(T_{(k)}^x = j) \sum_{m=0}^{n-(j-k)} p^m(x,x) \leq LHS$$

Thm. (Reflection Principle)

ξ_k . i.i.d. with dist is symmetric about 0.

let $S_n = \sum_1^n \xi_i$. If $a > 0$. Then: $P(\sup_{m \le n} S_m \ge a) \le 2P(S_n \ge a)$

Rmk: Alike Brownian Motion. It's discrete form.

Pf: $P(Z=0) = a$. if Z is sym at 0 . then $P(Z>0) = \frac{1-a}{2}$

$\Rightarrow P(Z \ge 0) \ge \frac{1}{2}$. Denote $W_n = S_n(\omega)$

Set $Y_m = 1$ if $m \le n$. $W_{n-m} \ge a$. $Y_m = 0$. otherwise.

So that $Y_N \circ \theta_N = 1$ if $W_n \ge a$. $N \le n$.

Set $N = \inf\{m \le n \mid S_m \ge a\}$ $(\inf \emptyset = \infty)$ $\therefore \{N < \infty\} = \{N \le n\}$.

$\Rightarrow E_0(Y_N \circ \theta_N I_{\{N \le n\}} \mid \mathcal{F}_N) = E_0(E_{S_N}(Y_N) I_{\{N \le n\}})$

$= E_0(I_{\{N \le n\}} P_{S_N}(S_{n-N} \ge a)) \ge E_0(I_{\{N \le n\}} P_{S_N}(S_N - S_{n-N} \ge 0))$

$\ge \frac{1}{2} P(n \ge N)$. Since $S_{n-N} - S_N$ is sym at 0 . $S_N \ge a$

(2) Recurrent and Transience:

① Def: i) State y is recurrent if $\rho_{yy} = 1$. and transient if $\rho_{yy} < 1$.

ii) $N(y) = \sum_{n=1}^{\infty} I_{\{X_n = y\}}$. Number of visit at y .

Rmk: If y is recurrent. Then: $P_y(T_y^k < \infty)$

$= \rho_{yy}^k = 1$. So: $P_y(X_n = y \text{ i.o.}) = 1$. It's intuitive.

Thm. y is recurrent $\iff E_y(N(y)) = \infty$

Pf. $E_x(N(y)) = \sum_{k=1}^{\infty} P_x(N(y) \geq k)$

$$= \sum_{k=1}^{\infty} P_x(T_y^k < \infty) = \frac{\rho_{xy}}{1-\rho_{yy}}$$

Thm. If state x is recurrent, $\rho_{xy} > 0$. Then:

y is recurrent, and $\rho_{yx} = 1$.

Pf: 1°) Prove: If $\rho_{xy} > 0$, $\rho_{yx} < 1$. Then: $\rho_{xx} < 1$.

$K = \inf\{k \mid p^k(x,y) > 0\}$. Then $\exists (y_i)_1^{K-1}$

$p(x,y_1)p(y_1,y_2)\cdots p(y_{K-1},y) > 0$.

$P_x(T_x = \infty) \geq p(x,y_1)p(y_1,y_2)\cdots p(y_{K-1},y)(1-\rho_{yx}) > 0$

So: $\rho_{yx} = 1$.

2°) Note: $\exists L$, st. $p^L(y,x) > 0$.

$\therefore p^{L+n+K}(y,y) \geq p^L(y,x)\, p^n(x,x)\, p^K(x,y)$

$$\Rightarrow \sum_{n=1}^{\infty} p^{L+n+K}(y,y) \geq c \sum p^n(x,x) = \infty.$$

Rmk: Recurrent is a class property.

Cor. $\rho_{xy} > 0$, $\rho_{yx} = 0 \Rightarrow x$ is transient.

Def: i) C is closed if $\forall x \in C$, $\rho_{xy} > 0 \Rightarrow y \in C$.

i.e. $\forall x \in C$, $P_x(X_n \in C) = 1$, $\forall n$.

ii) y is accessible from x if $\rho_{xy} > 0$.

x, y communicated if x, y accessible each other

iii) D is irreducible if $\forall x, y \in D$. they're communicated.

Lemma. $\rho_{xz} \geq \rho_{xy} \rho_{yz}$.

Pf: $\rho_{xz} \geq P_x(T_z \circ \theta_{T_y} < \infty, T_y < \infty)$

$$= E_x(E_x(T_z \circ \theta_{T_y} \mid \mathcal{F}_{T_y}) I_{\{T_y < \infty\}}) = \rho_{xy} \rho_{yz}$$

Thm. C is finite closed set. Then C contains a recurrent state.

if C is irreducible. then: C is recurrent.

Pf: By contradiction. if $\forall y \in C$. $\rho_{yy} < 1$.

Note: $\infty > \sum_{y \in C} \frac{\rho_{xy}}{1-\rho_{yy}} = \sum_{y \in C} E_x(N(y)) = \sum_{n=1}^{\infty} \sum_{y \in C} p^n(x,y) = \sum_1^{\infty} 1$

Which is a contradiction!

Rmk: If it's aperiodic and irreducible. Then:

it's positive recurrent.

Cor. If $\exists x \in C$. st. $\forall y \in C$. $\rho_{xy} > 0 \Rightarrow \rho_{yx} > 0$. $|C| < \infty$.

Then x is recurrent.

Pf: Set $C_x = \{y \in C \mid \rho_{xy} > 0\}$. $\forall y, w \in C_x$

Then $\rho_{yw} \geq \rho_{yx} \rho_{xw} > 0$. $\therefore C_x$ is irreducible.

If $\rho_{yz} > 0 \Rightarrow \rho_{xz} > 0$ $\therefore z \in C_x$. C_x is close.

Thm. $R = \{x \in S \mid \rho_{xx} = 1\}$. set of recurrent states. Then:

$R = \bigcup_i R_i$. where R_i is close and irreducible.

Rmk: It shows: To study status of recurrent.

We can WLOG consider a closed irred. set.

Pf: Set $C_x = \{y \in S \mid \rho_{xy} > 0\}$ for $x \in R$.

So $C_x \subset R$. since if $y \in C_x$, then $\rho_{yy} = 1$.

Besides, check: $C_x \cap C_y = \emptyset$. or $C_x = C_y$.

$\Rightarrow R = \sum C_x$.

Thm. S is irred. $\varphi \geq 0$. $E_x(\varphi(X_1)) \leq \varphi(x)$ for $x \notin F$.

F is finite set. $\varphi \to \infty$ as $x \to \infty$. Then:

S is recurrent.

Pf: Set $\tau = \inf\{n \mid X_n \in F\}$. $\Rightarrow Y_n = \varphi(X_{n \wedge \tau})$ is supermart

Set $T_m = \inf\{n \mid X_n \in F$ or $\varphi(X_n) > m\}$. $\Rightarrow T_m < \infty$. a.s.

(Note: $\{\varphi(x) < m\}$ is finite. S is irred.)

By: $\varphi(x) \geq E_x(\varphi(X_{T_m})) \geq m P_x(T_m < \tau)$

$P(\tau = \infty) = P(\bigcap_{m \geq 1}^{\infty} \{\tau > T_m\}) = 0$. $\Rightarrow P_x(\tau < \infty) = 1$. $\forall x \notin F$.

$\therefore P_y(X_n \in F$. i.o$) = 1$. $\forall y \overset{\exists z \in F}{\Rightarrow} P_y(X_n = z$. i.o$) = 1$.

Rmk: The ideal is find a recurrent state in the finite set F.

Cor. Replace "$\varphi \xrightarrow{x \to \infty} \infty$" by "$\varphi \xrightarrow{x \to \infty} 0$" and assume $\varphi > 0$. for $\forall x \in F$. Then: S is transient.

Pf: Set $\varepsilon = \min\{\varphi(x) \mid x \in F\} > 0$.

$\varphi(x) \geq E_x(\varphi(X_{n\wedge\tau})) \geq \varepsilon P_x(\tau < n)$. $\forall x \notin F$.

$\exists x$. s.t. $\varphi(x) < \varepsilon$. $\Rightarrow 1 > P_x(\tau < n)$. $\forall n$

Let $n \to \infty$. $\therefore \exists x$. $P_x(\tau = \infty) > 0$.

③ Applications:

i) M/G/1 Queue:

Suppose arrival $\sim$ Poisson(λ). $a_k = P(\xi_i = k-1) = \int_0^\infty \frac{(\lambda t)^k}{k!} e^{-\lambda t} dF(t)$

Each customer need a indept service with time $\sim F(t)$.

ξ_i. i.i.d. think as number of customers to arrive during the i^{th} serive time subtract 1 (who finish service)

Def: X_n is number of customers waiting in the queue at the time n^{th} customer enter service.

$X_{n+1} = (X_n + \xi_n)^+$. (Note $\xi_i \geq -1$). Set $X_0 = x$.

Thm. $\mu = E(\xi_i) = \sum_{k\geq -1} k a_k$. If $\mu > 1$. Then it's transient.

If $\mu \leq 1$. Then, it's recurrent.

Pf: $S_n = \sum_1^n \xi_i$. Set: $N = \inf\{n \mid X_0 + S_n = 0\}$.

Then: $S_{n\wedge N} = X_{n\wedge N}$.

1°) $\mu > 1 \Rightarrow S_n \to \infty$. a.s. $S_n \geq -1$.

So $P_x(N < \infty) < 1$. for x large enough.

2°) $\mu \leq 1 \Rightarrow X_{n\wedge N}$ is supermart. ≥ 0.

Consider $T^m = \inf\{n \mid X_n \geq m\}$. $\tau = T^m \wedge N$.

By $E(X_0) \geq E(X_2) \geq m P_x(T^m \leq N) \Rightarrow P_x(N<\infty)=1$.

Since it's irreducible. 0 is recurrent. ✓

Rmk: It's like a special kind of Branching Process.

ii) Birth and Death Chains on $\{0,1,\cdots\}$:

Let: $p(i,i+1)=p_i$. $p(i,i-1)=q_i$. $p(i,i)=r_i$. $q_0=0$.

Set: $N=\inf\{n \mid X_n=0\}$. $(p_i+q_i+r_i=1)$

Prop. For $\varphi(0)=0$. $\varphi(1)=1$. $\varphi(n)=\sum_{m=0}^{n-1}\prod_{j=1}^{m}\frac{q_j}{p_j}$. $n\geq 2$.

$\varphi(X_{n\wedge N})$ is mart.

Pf: Check: $E(\varphi(X_{n+1})\mid \mathcal{F}_n)=\varphi(X_n)$

If $X_n=k$. Then: $\varphi(k)=p_k\varphi(k+1)+r_k\varphi(k)+q_k\varphi(k-1)$

$\Rightarrow q_k(\varphi(k)-\varphi(k-1))=p_k(\varphi(k+1)-\varphi(k))$.

Thm. If $a<x<b$. Then: $P_x(T_a<T_b)=\frac{\varphi(b)-\varphi(x)}{\varphi(b)-\varphi(a)}$

Pf: $T=T_a\wedge T_b$. Then $\varphi(X_{n\wedge T})$ is bdd mart.

So: $\varphi(x)=E_x(\varphi(X_T))$.

Rmk: Random Walk is special case of B & D. process.

Thm. 0 is recurrent $\Leftrightarrow \varphi(m)\to\infty$. as $m\to\infty$.

Pf: If $\varphi(\infty)<\infty$. Then: $P_x(T_0=\infty)=\frac{\varphi(x)}{\varphi(\infty)}>0$.

(3) Stationary Measure:

① Def. i) A measure μ is stationary measure if:

$$\sum_x \mu(x) P(x,y) = \mu(y), \text{ i.e. } \mu P = \mu.\quad P = (P(i,j))_{S\times S}$$

ii) Stationary measure μ is stationary dist if:

$\mu(S) = 1$. possible equilibrium for some chain

Rmk. i): It means: $P_\mu(X_1 = y) = \mu(y)$. By induction:

$P_\mu(X_n = y) = \mu(y)\ \forall n \geq 1$. follows from Markov Property.

ii) If $\mu(S) < \infty$. Then $\mu' = \mu / \mu(S)$ is a dist.

iii) $A, B \subseteq S$. probability flux from A to B is:

$$\text{flux}(A, B) = \sum_{i \in A} \sum_{j \in B} \mu(i) p(i,j)$$

prop. If μ is stationary dist. Then $\text{flux}(A, A^c)$

$= \text{flux}(A^c, A)$. for $\forall A \subseteq S$.

Pf: 1') $\text{flux}(\{k\}, S) = \text{flux}(S, \{k\})$

$$RHS = \sum_{i \in S} \mu(i) p(i,k) = \mu(k) = \sum_{i \in S} \mu(k) p(k,i) = LHS$$

2') $\sum_{k \in A} \text{flux}(\{k\}, S) = \text{flux}(A, S)$

$$= \sum_{k \in A} \text{flux}(S, \{k\}) = \text{flux}(S, A).$$

3') Substract $\text{flux}(A, A)$ on both sides of 2')

Def. i) (X_n) is time-reversible if $\forall n.\ (X_0, X_1 \cdots X_n) \overset{d}{\sim}$

$(X_n \cdots X_0)$

ii) μ is reversible measure if it satisfies detailed balanced condition if $\mu(x)p(x,y) = \mu(y)p(y,x)$.

<u>prop.</u> μ is reversible measure $\Rightarrow$ stationary measure.

μ is stationary dist. $\Leftrightarrow$ (X_n) is time-reversible.

<u>Pf:</u> 1°) $\sum_x \mu(x)p(x,y) = \mu(y)\sum p(y,x) = \mu(y)$

2°) ($\Leftarrow$) $(X_0, X_1) \sim (X_1, X_0)$ implies : $\mu P = \mu$.

since $\mu_0 P = \mu_1 = \mu_0$ $(X_1 \sim X_0)$

($\Rightarrow$) By induction : begin from $P(X_0=i, X_1=j, X_2=k)$

② <u>Properties:</u>

<u>Thm.</u> μ is stationary dist. $X_0 \sim \mu$. Then : $\forall n$.

$Y_m = X_{n-m}$ $0 \le m \le n$ is Markov chain with initial

measure μ. and transition prob. $q(x,y) = \frac{\mu(y)p(y,x)}{\mu(x)}$

<u>Rmk:</u> q is called dual transition probability.

Note : if μ is reversible. then : $q = p$.

<u>Pf:</u> $P(Y_{m+1}=y \mid Y_m = x) = \frac{P(X_{n-m-1}=y, X_{n-m}=x)}{P(X_{n-m}=x)} = q(x,y)$

<u>Thm</u> (Kolmogorov's cycle condition)

If p is irred. Then : reversible measure μ exists

$\Leftrightarrow$ i) $p(x,y)>0 \Rightarrow p(y,x)>0$ ii) For any loop :

$x_0, x_1 \cdots x_n = x_0$. with $\prod_{1\le i\le n} p(x_i, x_{i-1}) > 0$. We have

$\prod_{i=1}^{n} \frac{p(x_{i-1}, x_i)}{p(x_i, x_{i-1})} = 1$.

Pf: ($\Rightarrow$). $\forall x \in S$. $\mu(x) = \sum \mu(y) p(y,x) > 0$.

By $\mu(x) p(x,y) = \mu(y) p(y,x)$. We prove i).

$\prod \frac{p(x_i, x_{i+1})}{p(x_{i+1}, x_i)} = \prod \frac{\mu(x_i)}{\mu(x_{i+1})} = 1$. We prove ii).

($\Leftarrow$) Fix $a \in S$. Set $\mu(a) = 1$.

$\forall x \in S$. $\exists$ seq: $x_0 = a, x_1 \cdots x_n = x$. st.

$\prod_{1 \le i \le n} p(x_i, x_{i+1}) > 0$ by irreducible of p.

Set $\mu(x) = \prod_1^n \frac{p(x_{i+1}, x_i)}{p(x_i, x_{i+1})}$. Well-def by condition.

Thm. $T = \inf\{n \ge 1 \mid X_n = x\}$. If x is recurrent state.

Then: $\mu_x(y) = E_x(\sum_{n=0}^{T-1} I_{\{X_n = y\}})$ defines a stationary measure.

Pf: 1°) $E_x(\sum_{n=0}^{T-1} I_{\{X_n=y\}}) = \sum_{k=1}^{\infty} E_x(\sum_{n=0}^{T-1} I_{\{X_n=y, T=k\}})$ ($T < \infty$ a.s.)

$= \sum_{k=1}^{\infty} \sum_{n=0}^{k-1} P_x(X_n = y, T = k)$

$= \sum_{n=0}^{\infty} P_x(X_n = y, T > n)$

2°) The idea: "cycle Trick":

Note $\mu_x(y)$ is expectation of visit to y

in $[0, \cdots T-1]$. (loop of $x \to x$)

However. $\mu_x p(y) = \sum \mu_x(z) p(z,y)$ is expected

number of visit to y in $\{1, \cdots T\}$. So $= \mu_x(y)$.

3°) Check: Denote $\bar{p}_n(x,y) = P_x(X_n = y, T > n)$

By Fubini: $\sum_y \mu_x(y) p(y,z) = \sum_{n=0}^{\infty} \sum_y \bar{p}_n(x,y) p(y,z)$

Note $p(y,z) = p(X_{n+1} = z \mid X_n = y) = p(X_{n+1} = z \mid X_n = y, T > n)$

$\{T > n\} = \{X_k \ne x, 1 \le k \le n\}$.

i) $z \neq x$. $\Rightarrow \sum\sum \bar{p}_n(x,y)\, p(y,z) = \sum_{n=0} \bar{p}_{n+1}(x,z) = \mu_x(z)$
$\bar{p}_0(x,z)=0$

ii) $z = x$ $\Rightarrow \sum\sum \bar{p}_n(x,y)\, p(y,x) = \sum_{n=0} P_x(T=n+1) = \mu_x(x) = 1$
$P_x(T=0)=0$

Rmk: i) If x is transient. Then $P(T=\infty) > 0$.

So: $\mu_x(z) = \sum P_x(X_n=y.\ T=n) + \sum P_x(X_n=y.\ T>n)$

$\geq \mu_x p(z)$.

ii) $\mu_x(y) < \infty$ holds for $\forall y$. Note:

$\mu_x P = \mu_x \Rightarrow \mu_x P^n = \mu_x.\ \forall n \in \mathbb{Z}^+$.

$\mu_x(x) = 1 = \sum \mu_x(y)\, P^n(y,x) \geq \mu_x(y)\, P^n(y,x)$.

So if $P^n(y,x) > 0.\ \exists n$. Then $\mu_x(y) < \infty$.

We obtain: $\rho_{xy} > 0 \Rightarrow \rho_{yx} = 1 \Rightarrow \mu_x(y) < \infty$

$\rho_{xy} = 0 \Rightarrow \mu_x(y) = 0$

Cor. $w_{xy} = P_x(T_y < T_x)$. Then $\mu_x(y) = \frac{w_{xy}}{w_{yx}}$

Pf: $\mu_x(y) = E_x(N^x(y))$. $N^x(y) = \sum_0^{T-1} I_{\{X_n=y\}}$.

$P_x(N^x(y)=0) = P_x(T_y > T_x) = 1 - w_{xy}$.

$P_x(N^x(y) = k) = P_x(X_{T_y^1} = y.\ \cdots\ X_{T_y^k} = y.\ X_{T_x^1} = x)$

$\overset{\text{m.p.}}{\underset{\text{by induc.}}{=}} P_x(T_y < T_x)\, P_y^{k-1}(T_y < T_x)\, P_y(T_y > T_x)$

Cor. If P is irreducible. recurrent. Then: we have

$\mu_x(y)\,\mu_y(z) = \mu_x(z).\ \forall x.y.z \in S$.

Pf: $\mu_y(z)/\mu_x(z) = \mu_y(y)/\mu_x(y) = 1/\mu_x(y)$

since every stationary measure differs a multiple.

Cor. $w_{xz} w_{zy} w_{yx} = w_{zx} w_{xy} w_{yz}$

Thm. If p is irreducible. recurrent. Then: the measure of stationary is unique up to const. multiple.

Pf: If ν is a stationary measure: for some $a \in S$

$$\nu(z) = \nu(a)p(a,z) + \sum_{y \neq a} \nu(y)p(y,z)$$

$$= \nu(a)p(a,z) + \sum_{y\neq a}\sum_{x\neq a} \nu(x)p(x,y)p(y,z) + \sum_{y\neq a} \nu(a)p(a,y)p(y,z)$$

$$= \nu(a)\left(P_a(X_1=z) + P_a(X_1\neq a, X_2=z)\right) + P_\nu(X_0\neq a, X_1\neq a, X_2=z)$$

$$= \cdots = \nu(a)\sum_1^n P_a(T>m, X_m=z) + P_\nu(T>n, X_n=z)$$

$$\geq \nu(a)\mu_a(z)$$

Note: $\nu(a) = \sum \nu(x)p^n(x,a) \geq \sum \nu(a)\mu_a(x)p^n(x,a) = \nu(a)$, $\forall n$. ($\mu_a(a)=1$)

$\Rightarrow$ "$\geq$" is "$=$" so: $\nu(x) = \nu(a)\mu_a(x)$ for $p^n(x,a)\neq 0$, $\exists n$.

Thm. If $\exists$ stationary dist. π. Then for $y \in S$, we have:

$\pi(y) > 0 \Rightarrow y$ is recurrent.

Pf: By $\pi(y)>0$: $\infty = \sum_{n=1}^{\infty} \pi(y) = \sum_n \sum_x \pi(x)p^n(x,y)$

$$= \sum_x \pi(x) \sum_{n=1}^{\infty} p^n(x,y)$$

$$= \sum_x \pi(x)\frac{\rho_{xy}}{1-\rho_{yy}} \leq \frac{\sum_x \pi(x)}{1-\rho_{yy}} = \frac{1}{1-\rho_{yy}}$$

Thm. If p irreducible, has stationary dist. π.

Then: $\pi(x) = 1/E_x(T_x)$

Pf: Note $\sum_x \pi(x) = 1$. $\therefore \exists x_0 \in S$, $\pi(x_0)>0$

So x_0 recurrent $\Rightarrow S$ is recurrent.

S, $\forall x \in S$. $\exists\ \mu_x(y) = \sum_0^{\infty} P_x(X_n = y, T_x > n)$

$\sum_y \mu_x(y) = \sum_0^{\infty} P_x(T_x > n) = E_x(T_x)$.

$\Rightarrow \pi(y) = \mu_x(y) / E_x(T_x)$ is unique stationary dist.

$\therefore \pi(x) = 1/E_x(T_x)$.

Def: State $x \in S$ is positive recurrent if $E_x(T_x) < \infty$. It's null recurrent if $E_x(T_x) = \infty$.

Rmk: Positive Recurrent $\Rightarrow$ Recurrent.

Thm: If p is irreducible. Then. i). ii). iii) equi.

i) $\exists x \in S$. positive recurrent.

ii) Stationary dist. exists. iii) S is positive recurrent.

Pf: i) $\Rightarrow$ ii) Define: $\pi(y) = \sum_{n=0}^{\infty} P_x(X_n = y, T_x > n) / E_x(T_x)$

$= \mu_x(y) / E_x(T_x)$.

ii) $\Rightarrow$ iii) $\exists x$, st.

$\pi(x) = \sum_y p^n(y,x)\pi(y) > 0$. $\forall y \in S$.

By irred. $\exists n$. $p^n(y,x) > 0 \Rightarrow \pi(y) > 0$.

Thm: If p is irreducible, positive recurrent.

Then: $E_x(T_y) < \infty$. $\forall x, y \in S$.

Pf: Set $m = \inf\{n > 0 \mid p^n(x,y) > 0\}$. $\Rightarrow m < \infty$. by irred.

From C-K equation. $\exists (z_i)_1^{m-1}$. s.t. $z_i \neq x$. $\forall i$.

$p(x, z_1)\, p(z_1, z_2) \cdots p(z_{m-1}, y) > 0$.

$\Rightarrow E_x(T_x) \geq E_x(T_x \, I_{\{X_1 = z_1, \cdots X_{m-1} = z_{m-1}, X_m = y\}})$

$= E_x(E_x(T_x \circ \theta_m \mid \mathcal{F}_m) I_{\{\cdots\}})$

$= E_y(T_x)\, P_x(X_1 = z_1, \cdots X_m = y)$.

<u>Cor.</u> P is irreducible. Then it's positive recurrent

$\iff E_x(T_y) < \infty$. $\forall x, y \in S$.

<u>Cor.</u> P is irreducible. has stationary measure μ

s.t. $\sum_x \mu(x) = \infty$. Then P is null recurrent.

<u>Pf.</u> By contradiction. Then $\exists \lambda(x)$.

stationary dist. s.t. $\lambda(x) = c\mu(x)$

$\Rightarrow \sum \lambda(x) = \infty$. contradiction!

<u>Thm.</u> If $x, y \in S$. $x \sim y$. i.e. $\rho_{xy} \cdot \rho_{yx} > 0$. $E_x(T_x) < \infty$.

Then: $E_x(T_y)$. $E_y(T_x) < \infty$. So: $E_y(T_y) < \infty$.

<u>Rmk:</u> Positive recurrent is a class property.

<u>Pf:</u> 1°) $\exists n$. the least integer. s.t. $p^n(y, x) > 0$.

Similar to the operation in the Thm above.

$\Rightarrow E_x(T_x) \geq E_y(T_x)\, P_x(X_1 = z_1, \cdots X_m = y)$

2') Introduce a regenerative process:

Set $T_{x,k} = \inf\{n > T_{x,k-1} \mid X_n = x\}$.

$\Delta_k = T_{x,k} - T_{x,k-1}$. i.i.d. $\Delta_1 = T_x$.

$\tau =: \inf\{k \geq 1 \mid T_y < T_{x,k}\}$. stopping time.

$\tau \sim Geo(p)$. $p = P_x(T_y < T_x) > 0$.

By Wald's Equation:

$E_x(T_y) \leq E_x(T_{x,\tau}) = E_x(\sum_1^{\tau} \Delta_k)$

$= E_x(\tau) E_x(\Delta_1) = E_x(\tau) E_x(T_x)$

3') $E_y(T_y) = E_y(T_y I_{\{T_x < T_y\}}) + E_y(T_y I_{\{T_x \geq T_y\}})$

$\leq E_y(E_y(T_y \circ \theta_{T_x} \mid \mathcal{F}_{T_x}) I_{\square}) + E_y(T_x)$

$= E_x(T_y) P(\square) + E_y(T_x) < \infty$.

③ <u>Examples:</u>

i) <u>B & D process:</u>

$\mu(x) = \prod_{k=1}^{x} \frac{p_{k-1}}{q_k}$ is a stationary measure.

So it's positive recurrent $\Leftrightarrow \sum_{x \in S} \prod_1^x p_{k-1}/q_k < \infty$.

ii) <u>M/G/1 Queue:</u>

It's positive recurrent $\Leftrightarrow \mu < 1$.

④ <u>Entropy Method:</u>

<u>Thm.</u> P is irred. has stationary dist. π. Then:

$\forall \mu$. stationary measure. $\mu = c\pi$. c is const.

Rmk: Directly. Stationary dist ⇒ recurrent

⇒ Any stationary measure differs a const.

Pf: For φ concave. Def: entropy of μ.

$$\varepsilon(\mu) = \sum_{\eta \in S} \varphi\left(\frac{\mu(\eta)}{\pi(\eta)}\right) \pi(\eta)$$

Check $\varepsilon(\mu P) \geq \varepsilon(\mu)$. i.e. entropy increases by an application of P. (By $\varphi(\sum \lambda_i x_i) \geq \sum \lambda_i \varphi(x_i)$)

But $\mu P = \mu$. Set $\bar{P}(x,\eta) = \sum_{n=1}^{\infty} 2^{-n} P^n(x,\eta) > 0$

We have $\mu \bar{P} = \mu$ as well.

$\varepsilon(\mu) = \varepsilon(\mu \bar{P}) \geq \varepsilon(\mu)$ implies conclusion.

(4) Asymptotic Behaviors:

Note if η is transient. then: $E_x(N_\eta) = \sum P^n(x,\eta) < \infty$.

So $P^n(x,\eta) \to 0$ as $n \to \infty$.

Denote: $N_n(\eta) = \sum_{m=1}^{n} I_{\{X_m = \eta\}}$.

① Basic limit Thm:

Thm. If η is recurrent. Then $\forall\, x \in S$, we have:

$$N_n(\eta)/n \xrightarrow{n\to\infty} I_{\{T_\eta < \infty\}} / E_\eta(T_\eta) \quad P_x\text{-a.s.}$$

Pf: $R(k) = \min\{n \geq 1 \mid N_n(\eta) = k\}$. start at η.

$t_k = R(k) - R(k-1)$. $R_0 = 0$.

So t_k. i.i.d $R(k)/k \to E_\eta(T_\eta)$ P_η-a.s.

since $R(N_n(y)) \le n < R(N_n(y)+1)$. By Renewal argument:

$n/N_n(y) \longrightarrow E_y(T_y)$. P_y-a.s.

For $x \neq y$. If $P_x(T_y < \infty) < 1$. Then: $\frac{N_n(y)}{n} \to 0$ on $\{T_y = \infty\}$.

By strong Markov Property: $t_2, t_3 \cdots$ i.i.d. $P_x(t_k = n) = P_y(T_y = n)$

$\Rightarrow$ On $\{T_y < \infty\}$. $R(k)/k = t_1/k + \sum_2^k t_i/k \longrightarrow E_y(T_y)$. identical case.

Rmk: It explains positive recurrent: its asymptotic fraction of time spent on x is positive.

Cor. $\frac{1}{n}\sum_{m=1}^{n} p^m(x,y) \longrightarrow \rho_{xy} / E_y(T_y)$. $\forall x, y \in S$.

Pf: $\frac{1}{n}\sum_{m=1}^{n} p^m(x,y) = E_x(N_n(y)/n) \longrightarrow E_x(I_{\{T_y<\infty\}}/E_y(T_y))$

since $0 \le \frac{N_n(y)}{n} \le 1$. by BCT.

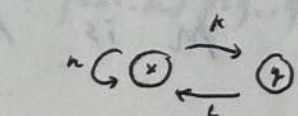
Rmk: $p^n(x,y)$ always converges in Cesáro sense. but itself may not converge. e.g. $p = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ periodic case.

Def: For x is recurrent. $I_x = \{n \ge 1 \mid p^n(x,x) > 0\}$. d_x is g.c.d of I_x. which is called period of x.

Lemma. If $\rho_{xy} > 0$. Then: $d_x = d_y$.

Pf: Note d_x defined on recurrent states. So $\rho_{yx} = 1$. $\exists K, L$. $p^K(x,y) > 0$. $p^L(y,x) > 0$.

$\Rightarrow p^{K+L}(y,y) \ge p^L(y,x)p^K(x,y) > 0$



$p^{K+n+L}(y,y) \ge p^L(y,x)p^n(x,x)p^K(x,y) > 0$. where $p^n(x,x) > 0$

$\therefore d_y \mid (K+L+n) - (K+L) \Rightarrow d_y \mid d_x$.

Def: For irreducible chain. if $\exists x$, $\lambda_x = 1$.

then we call it's aperiodic

Lemma If $\lambda_x = 1$. Then $\exists m_0$. $p^m(x,x) > 0$. $\forall m \geq m_0$.

Pf: 1°) I_x contains two conservcutative integer.

For $(i_k)_1^m \subset I_x$. By Bezout: $\exists (c_i)_1^m$ s.t.

$\sum_1^m c_k i_k = 1$. $\Rightarrow \sum_{c_k > 0} c_k i_k = 1 - \sum_{c_k < 0} c_k i_k$

(Note $\{c_k < 0\}$. $\{c_k > 0\} \neq \emptyset$. Since $i_k \geq 1$)

2°) From k, $k+1$. We have: (for $k \geq 2$)

$2k$. $2k+1$, $2k+2$, $3k$. $3k+1$, $3k+2$, $3k+3$. Then:

$(k-1)k$. $(k-1)k+1$ ---- $(k-1)k + k-1$. ($(k-1)k \sim k^2-1$)

So that we have: $nk \sim nk+n$. $(n \geq 1)$

when $n \geq k$. $\Rightarrow nk+n \geq (n+1)k$. ✓.

Def: Distance of λ. μ. p.m's on $(S, \mathcal{S})$: $\|\lambda - \mu\|_{TV}$

$= \sup_{A \in \mathcal{S}} |\lambda(A) - \mu(A)|$. total variation.

Rmk: So if $\|\lambda - \mu\|_{TV} = 0$. then $\lambda = \mu$ on $\mathcal{S}$.

prop. $\|\lambda - \mu\|_{TV} = \inf_{(X,Y) \sim m} \mathbb{P}(X \neq Y)$. Hamming metric.

m is dist. on S^2 with marginal λ. μ.

Pf: $|\lambda(A) - \mu(A)| \leq E\,|I_{\{X \in A\}} - I_{\{Y \in A\}}|$

$\leq E\,|I_{\{X \neq Y\}}| = \mathbb{P}(X \neq Y)$

For "=" holds. Set $\nu = \lambda + \mu$. $p = \frac{d\lambda}{d\nu}$. $q = \frac{d\mu}{d\nu}$

$\|\lambda - \mu\|_{TV} = (\lambda - \mu)(p > q) = \int (p - p\wedge q)\, d\nu = 1 - \gamma(S)$

Where $\frac{d\gamma}{d\nu} = p \wedge q$: finite. measure. Let $\Delta = \begin{matrix} S \to S^2 \\ x \mapsto (x,x) \end{matrix}$

Then : set $\mu_0 = \frac{1}{1-\gamma(S)} (\lambda - \gamma) \otimes (\mu - \gamma) + \gamma \circ \Delta^{-1}$. $(X, Y) \sim \mu_0$

<u>prop.</u> If S is separable. discrete. Then :

$\|\lambda - \mu\|_{TV} = \frac{1}{2} \sum_{x \in S} |\lambda(x) - \mu(x)| = 1 - \sum_{x \in S} \min\{\lambda(x), \mu(x)\}$.

<u>Pf</u>: 1°) 2^{nd} "=" is from : $\min\{x, y\} = \frac{x+y-|x-y|}{2}$

2°) Set $M = \{x \mid \lambda(x) \geq \mu(x)\}$.

$$1 - \sum \min\{\lambda(x), \mu(x)\} = 1 - \sum_{M} \mu(x) - \sum_{S/M} \lambda(x)$$

$$= \sum_{M} \lambda(x) - \mu(x) = \lambda(M) - \mu(M)$$

$$\leq \sup_A |\lambda(A) - \mu(A)| = \|\lambda - \mu\|_{TV}$$

Conversely. $\|\lambda - \mu\| \leq \varepsilon + \lambda(A) - \mu(A)$ $\exists A \in S$

$$\leq \varepsilon + 1 - \sum_{x \in S} \min\{\lambda(x), \mu(x)\}. \ \forall \varepsilon > 0.$$

<u>Thm.</u> (Convergence Theorem)

If p is irreducible. aperiodic. has stationary dist π.

Then : $p^n(x, y) \xrightarrow{n \to \infty} \pi(y)$.

<u>Pf</u>: The idea is coupling :

Def : $\bar{p}$ on $S \times S$: $\bar{p}((x_1, y_1), (x_2, y_2)) = p(x_1, x_2)\, p(y_1, y_2)$

i.e. each coordinate move indeptly.

1°) $\bar{p}$ is irreducible :

Since $\exists k, L$. $p^k(x_1,x_2), p^L(y_1,y_2) > 0$

Let m large enough $\Rightarrow p^{L+m}(x_2,x_2) \cdot p^{k+m}(y_2,y_2) > 0$.

Then $\bar{p}^{k+L+m}((x_1,y_1),(x_2,y_2)) > p^k(x_1,x_2)\, p^{L+m}(x_2,x_2) \cdots > 0$

2') $\bar{\pi}(a,b) = \pi(a)\pi(b)$ is stationary dist. for $\bar{p}$

3') By 2') (easy to check) $\Rightarrow$ All states for $\bar{p}$ is recurrent.

4') (X_n, Y_n) is chain with $\bar{p}$ on $S\times S$. $T = \inf\{n \geq 1 \mid X_n = Y_n\}$.

Since it's recurrent, irred. $\Rightarrow T_{(x,x)} < \infty$ a.s $\Rightarrow T < \infty$ a.s.

Note: $p(X_n = y, T \leq n) = \sum_{m=1}^{n} \sum_x p(T=m, X_m = x, X_n = y)$

$= \sum_{m=1}^{n} \sum_x p(T=m, X_m = x)\, p^{n-m}(x,y)$

$= p(Y_n = y, T \leq n)$

$$\Rightarrow \begin{cases} p(X_n = y) = p(X_n = y, T \leq n) + p(X_n = y, T > n) \\ p(Y_n = y) = p(Y_n = y, T \leq n) + p(Y_n = y, T > n) \end{cases}$$

$\therefore |p(X_n = y) - p(Y_n = y)| \leq p(X_n = y, T > n) + p(Y_n = y, T > n)$

$\Rightarrow \sum |p(X_n = y) - p(Y_n = y)| \leq 2p(T > n)$. Coupling inequality.

Set $X_0 = x$, Y_n has initial dist. π.

So: $\sum |p^n(x,y) - \pi(y)| \leq 2p(T > n) \longrightarrow 0$

Rmk: i) The stationary dist π is indept with the initial dist μ of X_n in Thm.

ii) Alternative pf:

Consider (X_n) (Y_n) are indept and with P and stationary dist. π. $X_0 \sim \mu$, $Y_0 \sim \pi$

Set $Y_n^* = \begin{cases} X_n, & T \le n \\ Y_n, & T > n \end{cases}$ Markov chain.

$P_\mu(X_n \in A) - P(Y_n^* \in A) = p^n(x, A) - \pi(A)$ (set $\mu = \delta_x$)

$\le P(T \ge n)$. (Coupling ineq.)

$\Rightarrow \| p^n(x, \cdot) - \pi(\cdot) \| \le P(T \ge n) \to 0$. $(n \to \infty)$

Cor. If S is finite. p irred. aperiodic. Then:

$\exists m$. st. $p^m(x, y) > 0$. for $\forall x, y \in S$.

Pf: Lemma. It's positive recurrent

Pf: $\frac{N_n(x)}{n} \to \frac{I(T_x < \infty)}{E_x(T_x)} = \frac{1}{E_x(T_x)}$. a.s.

So $\sum_{x \in S} \frac{N_n(x)}{n} = 1 \to \sum \frac{1}{E_x(T_x)}$. a.s

It has stationary dist.

$\Rightarrow p^n(x, y) \to \pi(y) > 0$. $\forall y$. (Positive Recurrent is class property) as $n \to \infty$.

Cor. In the Markov Chain above. We have:

$P(T > n) \le C r^n$. $0 < r < 1$. (i.e. it's exponential rapid)

Pf: 1°) For $p(x, y) > 0$. $\forall x, y \in S$.

$P(T > n+1) = P(T > n+1 \mid T > n) P(T > n)$

$P(T \le n+1 \mid T > n) \ge P(T = n+1 \mid T > n)$

$P(T = n+2 \mid X_{n+1} = x, Y_{n+1} = y, T > n+1) =$

$P(X_{n+2} = Y_{n+2} \mid X_{n+1} = x, Y_{n+1} = y) = \sum p(x, z) p(y, z)$

$\ge \varepsilon^2 |S|$. $\Rightarrow$ check $P(T = n+2 \mid T > n) \ge \varepsilon^2 |S|$.

2°) By Cor. above. Consider $P(T > n+m \mid T > n) \le 1 - \varepsilon^2 |S|$.

② Periodic Case:

Lemma. If p is irred. recurrent. All states have period d. Fix $x \in S$. And for each $y \in S$.

$K_y = \{n \geq 1 \mid p^n(x,y) > 0\}$.

i) $\exists r_y \in \{0, 1, \cdots d-1\}$. s.t. $n \in K_y \Rightarrow n \equiv r_y \pmod d$

ii) $S_r = \{y \mid r_y = r\}$. If $y \in S_i$. $z \in S_j$. $p^n(y,z) > 0$.

then: $n \equiv (j-i) \pmod d$.

iii) $(S_i)_0^{d-1}$ are irred. classes w.r.t p.m. p^d.

All states have period 1.

Pf: i) Suppose $p^{m(y)}(y,x) > 0$. For $n \in K_y$.

Then $p^{m(y)+n}(x,x) > 0 \therefore d \mid m(y)+n$.

Set $r_y = -m(y) \pmod d$. $0 \leq r_y < d$.

ii) $p^n(y,z) > 0$. $p^m(x,y) > 0 \Rightarrow m+n \equiv j \pmod d$

Besides: $m \equiv i \pmod d$.

iii) When $y, z \in S_i \Rightarrow n \equiv 0 \pmod d$

Rmk: $\{S_i\}_0^{d-1}$ is called cyclic decomposition.

Thm. If p is irred. has a stationary dist. π. and all states have period d. Suppose that $\{S_i\}_0^{d-1}$ is cyclic decomposition st. $x \in S_0$. Then:

For $y \in S_r$. $\lim_{n \to \infty} p^{nd+r}(x,y) \to d\,\pi(y)$

Pf: Apply Basic Limit Thm on P^d. p.m.: for $y \in S_0$. then:

$\lim_{n\to\infty} p^{nd}(x,y)$ exists. Note: $\sum_1^n p^m(x,y)/n \to \pi(y)$

but $p^m(x,y) = 0$ when $d \nmid m$. So: $\sum_1^n p^{md}(x,y)/nd \to \pi(y)$.

By Stolz: $p^{nd}(x,y) \to d\pi(y)$. Generally for $y \in S_r$:

$p^{md+r}(x,y) = \sum_{z \in S_r} p^r(x,z)\, p^{md}(z,y)$. $p^{md}(z,y) \to d\pi(y)$. by DCT. ✓

③ Tail-σ-algebra:

Denote: $\mathcal{T} = \cap\, \sigma(X_k, k \geq n)$. (X_k) is Markov Chain.

Thm. If p is irreducible, recurrent, all states have period d. Then: $\mathcal{T} = \sigma(\{X_0 \in S_r\}, 0 \leq r \leq d-1)$

Rmk: Precisely. $\forall \mu$ initial dist. For $A \in \mathcal{T}$. Then:

$\exists\, 0 \leq r \leq d-1$. $A = \{X_0 \in S_r\}$. P_μ-a.s. It's intuitive:

Note that $\{X_0 \in S_r\} = \{X_{nd} \in S_r, i.o\} \in \mathcal{T}$. P_μ-a.s.

Thm. If $X_0 \sim \mu$. Then: $h(X_n, n) = E_\mu(Z \mid \mathcal{F}_n)$. $Z = \lim h(X_n, n)$

Set a 1-1 corresponse between bdd $Z \in \mathcal{T}$ and bdd space-time harmonic func.: $h: S \times \mathbb{N} \to \mathbb{R}$. i.e. Satisfies $h(X_n, n)$ is a martingle.

Pf. ($\Rightarrow$) $Z \in \mathcal{T}$. Set $Z = Y_n \circ \theta_n$. $h(x,n) = E_x(Y_n)$.

So: $E_\mu(Z \mid \mathcal{F}_n) = E_{X_n}(Y_n) = h(X_n, n)$ is a mart.

($\Leftarrow$) $\lim h(X_n, n)$ exists. equal Z. bdd. $h(X_n, n) = E(Z \mid \mathcal{F}_n)$

Cor. For d-dimension random walk. $\mathcal{T} = \sigma(\{X_0 \in L_i\}, i = 0, 1)$

$L_0 = \{z \in \mathbb{Z}^d \mid \sum_1^d z^i \text{ is even}\}$. $L_1 = \mathbb{Z}^d / L_0$

Rmk: $S_n \in L_0 \Rightarrow S_{n+1} \in L_1$.