Discrete Martingales.

(1) Conditional Expectation:

① **Def**: In prob. space $(\Omega, \mathcal{F}_0, P)$. $X \in \mathcal{F}_0$. σ-field $\mathcal{G} \subseteq \mathcal{F}_0$. $E(|X|) < \infty$. The conditional expectation of X given $\mathcal{G}$ is $Y =: E(X|\mathcal{G})$. s.t.

i) $Y \in \mathcal{G}$ ii) $\forall A \in \mathcal{G}$. $\int_A X \, dP = \int_A Y \, dP$.

Any Y satisfies i). ii) is said a version of $E(X|\mathcal{G})$

Rmk: Initially we consider $E(X|Y)$. Note for $w_1, w_2 \in \Omega$ but $E(X|Y)$ only depends on value of $Y(w)$ not w. So we only record the partition of Ω by different values of Y. i.e. $\sigma(Y)$. Written in $E(X|\sigma(Y))$. More generally, consider $\mathcal{G} \subseteq \mathcal{F}_0$. σ-subfield.

Lemma. Version of $E(X|\mathcal{G}) = Y$ is integrable.

Pf: $A = \{Y > 0\} \in \mathcal{G}$. Then: by $\int_A Y = \int_A X$

$\int_A Y \, dP \leq \int_A |X| \, dP$

$\Rightarrow E(|Y|) \leq E(|X|)$

$-\int_{A^c} Y \, dP \leq \int_A |X| \, dP$

Lemma. (Uniqueness)

If Y, Y' are versions of $E(X|\mathcal{G})$. Then. $Y = Y'$ a.s

Pf: Set $A = \{Y - Y' \geq \varepsilon > 0\}$.

$$0 = \int_A X - X \, dP = \int_A Y - Y' \, dP \geq \varepsilon P(A)$$

$\therefore P(A) = 0$. $\forall \varepsilon > 0$. By sym $\Rightarrow Y' = Y$. a.s.

Thm. If $X_1 = X_2$ on $B \in \mathcal{G}$. Then: $E(X_1|\mathcal{G}) = E(X_2|\mathcal{G})$ on B

Pf. $A = \{E(X_1|\mathcal{G}) - E(X_2|\mathcal{G}) \geq \varepsilon > 0\}$. $A \cap B \in \mathcal{G}$.

Lemma (Existence)

Version of $E(X|\mathcal{G})$ always exist.

Pf. 1°) For $X \geq 0$:

Set $\nu(A) = \int_A X \, dP$. $\forall A \in \mathcal{G}$. (By DCT. it's measure)

Then: $\nu \ll P$. By R-N Thm: $\frac{d\nu}{dP}$ exists. P-a.s unique.

Set $E(X|\mathcal{G}) = \frac{d\nu}{dP} \geq 0$. integrable.

2°) For general X:

$X = X^+ - X^-$. Set $Y_1 = E(X^+|\mathcal{G})$. $Y_2 = E(X^-|\mathcal{G})$,

$Y_1 - Y_2 \in \mathcal{G}$. intergrable satisfies i). ii)

② Examples:

i) If $X \in \mathcal{G}$. Then: $E(X|\mathcal{G}) = X$. a.s.

Pf: Only need to check condition i).

ii) If X is indept with $\mathcal{G}$. Then: $E(X|\mathcal{G}) = E(X)$.

Pf: $E(X) \equiv$ const $\in \mathcal{G}$. satisfies i)

$$\int_A X dP = E(X I_A) = E(X) E(I_A) = \int_A E(X) dP.$$

iii) If $\Omega = \Sigma \Omega_i$. $P(\Omega_i) > 0$. $\mathcal{G} = \sigma\{\Omega_i\}_{i \in \mathbb{Z}}$. Then:

$$E(X|\mathcal{G}) = \Sigma \frac{E(X I_{\Omega_i})}{P(\Omega_i)} I_{\Omega_i}$$

Pf: It's game of "guess and verify".

Denote: $P(A|\mathcal{G}) = E(I_A|\mathcal{G})$. $P(A|B) = \frac{P(A \cap B)}{P(B)}$

Rmk: So for $\mathcal{G} = \Sigma \Omega_i$. $P(A|\mathcal{G}) = \Sigma \frac{P(A \cap \Omega_i)}{P(\Omega_i)} I_{\Omega_i}$

iv) If $(X, Y) \sim f(x, y)$. $\int f(x, y) dx > 0$. $\forall y$. $E|g(X)| < \infty$.

Then: $E(g(X)|Y) = h(Y) = \int g(x) f(x, Y) dx \Big/ \int f(x, Y) dx$.

Pf: $E(h(Y) I_A) = \int_B \int h(y) f(x, y) dx dy$

$$= \int_B \int g(x) f(x, y) dx dy$$

$$= E(g(X) I_B(Y)) = E(g(X) I_A)$$

where $A \in \sigma(Y)$. $A = \{Y \in B\}$. for $B \in \mathcal{B}_{\mathbb{R}}$.

Note $h(y)$ is measurable $\Rightarrow h(Y) \in \sigma(Y)$.

Rmk: To drop $\int f(x,y)\,dx > 0$. Define h by:

$$h(y) \int f(x,y)\,dy = \int g(x) f(x,y)\,dx.$$

v) X indept with Y. $\varphi(X,Y) \in L^1$. Set $g(x) = E_Y(\varphi(x,Y))$.

Then: $g(X) = E(\varphi(X,Y) \mid X)$.

Pf: For $A \in \sigma(X)$. $A = \{X \in C\}$.

$$\int_A \varphi(X,Y)\,dP = E(\varphi(X,Y) I_C(X))$$

$$= \iint \varphi(x,y) I_C(x)\, \nu(dy)\, \mu(dx)$$

$$= \int I_C(x) g(x)\, \mu(dx) = \int_A g(X)\,dP.$$

③ Properties:

Thm. For $E|X|, E|Y| < \infty$.

i) $E(aX + Y \mid \mathcal{F}) = a E(X \mid \mathcal{F}) + E(Y \mid \mathcal{F})$

ii) If $X \le Y$. Then: $E(X \mid \mathcal{F}) \le E(Y \mid \mathcal{F})$

If $E(X I_A) \le E(Y I_A)$. $\forall A \in \mathcal{F}$. Then: $E(X \mid \mathcal{F}) \le E(Y \mid \mathcal{F})$

iii) $X_n \ge 0$. $X_n \uparrow X$. $E(X) < \infty$. Then: $E(X_n \mid \mathcal{F}) \uparrow E(X \mid \mathcal{F})$

iv) If $\exists \mathcal{G} \subseteq \mathcal{F}$. $E(X \mid \mathcal{G}) = 0$ Then: $E(X) = 0$

Rmk: For iii). descendent case by set $-X_n$.

Fatou's Lemma and DCT can be obtained by iii)

Pf: i) is trivial. Check from RHS.

ii) Set $A = \{ E(X|\mathcal{G}) - E(Y|\mathcal{G}) \geq \varepsilon > 0 \} \in \mathcal{G}$.

Note: $\int_A E(X|\mathcal{G}) = \int_A X \leq \int_A Y = \int_A E(Y|\mathcal{G})$

iii) Set $Y_n = X - X_n$. Show $E(Y_n|\mathcal{G}) \downarrow 0$

Since $Y_n \downarrow 0$. Then $Z_n = E(Y_n|\mathcal{G}) \downarrow$

$\forall A \in \mathcal{G}$. $\int_A Z_n \, dP = \int_A Y_n \, dP$. By DCT on Y_n.

iv) prove: $E(E(X|\mathcal{G})) = E(X)$. i.e.

$\int_\Omega E(X|\mathcal{G}) \, dP = \int_\Omega X \, dP$. Since $\Omega \in \mathcal{G}$.

Cor. i) Hölder Inequality: $E(|XY| \,|\, \mathcal{G}) \leq E(|X|^p|\mathcal{G})^{\frac{1}{p}} E(|Y|^{p'}|\mathcal{G})^{\frac{1}{p'}}$

ii) $P(|X| \geq a \,|\, \mathcal{G}) \leq E(X^2|\mathcal{G})/a^2$. for $a > 0$.

Thm. (Bayes' Formula)

If $G \in \mathcal{G}$. $P(G|A) = \int_G P(A|\mathcal{G}) \, dP \Big/ \int_\Omega P(A|\mathcal{G}) \, dP$.

Pf: $P(G|A) = \dfrac{P(A \cap G)}{P(A)} = \dfrac{\int_G E(I_A|\mathcal{G})}{\int_\Omega E(I_A|\mathcal{G})} = \dfrac{\int_G P(A|\mathcal{G}) \, dP}{\int_\Omega P(A|\mathcal{G}) \, dP}$

Thm. (Jensen Inequality)

If φ is convex. $E|X|$. $E|\varphi(X)| < \infty$. Then: we have.

$\varphi(E(X|\mathcal{G})) \leq E(\varphi(X)|\mathcal{G})$

Pf: If φ isn't linear. Note: $\varphi(x) = \sup\{ax + b \,|\, (a,b) \in S\}$ (*)

where $S = \{(a,b) \,|\, a, b \in \mathbb{Q}, \ ax + b \leq \varphi(x), \ \forall x\}$.

$\therefore E(\varphi(X)|\mathcal{G}) \geq a E(X|\mathcal{G}) + b$. $\forall (a,b) \in S$. a.s.

Rmk: $a, b \in \mathbb{Q}$ come in because the last inequality hold for (a, b) except a null set $N_{a,b}$.

$(*)$ is from φ'_+, φ'_- exists. $\varphi'_- \le \varphi'_+$.

$\exists\ (a_n^x) \subseteq \mathbb{Q} \to \varphi'_-(x)$.

Thm. Conditional expectation is contraction in L^p, $p \ge 1$.

Pf: $|E(X|\mathcal{F})|^p \le E(|X|^p|\mathcal{F})$ follows from Jesen.

$$\Rightarrow E|E(X|\mathcal{F})|^p \le E(E(|X|^p|\mathcal{F})) = E|X|^p.$$

Thm. If $\mathcal{F} \subset \mathcal{G}$, $E(X|\mathcal{G}) \in \mathcal{F}$. Then $E(X|\mathcal{G}) = E(X|\mathcal{F})$

Pf: $\forall A \in \mathcal{F}$, $\int_A E(X|\mathcal{G}) = \int_A X = \int_A E(X|\mathcal{F})$

Thm. If $\mathcal{F}_1 \subset \mathcal{F}_2$. Then $E(E(X|\mathcal{F}_1)|\mathcal{F}_2) = E(E(X|\mathcal{F}_2)|\mathcal{F}_1)$

$= E(X|\mathcal{F}_1)$.

Rmk: In other words, smaller σ-field always wins.

Pf: $E(X|\mathcal{F}_1) \in \mathcal{F}_1 \subseteq \mathcal{F}_2$. $\int_A E(X|\mathcal{F}_2) = \int_A E(X|\mathcal{F}_1)$. $A \in \mathcal{F}_1$

Rmk: generally, $E(E(X|\mathcal{F}_1)|\mathcal{F}_2) \ne E(E(X|\mathcal{F}_2)|\mathcal{F}_1)$

e.g. $\Omega = \{a, b, c\}$, $\mathcal{F}_1 = \sigma\{a\}$, $\mathcal{F}_2 = \sigma\{c\}$, $X = I_{\{b\}}$.

Thm. If $X \in \mathcal{F}$, $Y, XY \in L^1$. Then: $E(XY|\mathcal{F}) = X E(Y|\mathcal{F})$

Pf: $XE(Y|\mathcal{G}) \in \mathcal{G}$. Then to check condition ii).

Consider to approxi. X by simple func's $\in \mathcal{G}$.

$X = X^+ - X^-$. Note for $X = I_B$, $B \in \mathcal{G}$. ✓.

Cor. For $X \in L^2$. $E(X|\mathcal{G}) = \arg\min_{Y \in \mathcal{G}} E(X-Y)^2$.

i.e. $E(X|\mathcal{G}) = P_{L^2(\mathcal{G})} X$.

Cor. If $\mathcal{g} \subseteq \mathcal{G}$. $X \in L^2$. Then we have:

$$E(X - E(X|\mathcal{G}))^2 + E(E(X|\mathcal{G}) - E(X|\mathcal{g}))^2 = E(X - E(X|\mathcal{g}))^2.$$

Rmk: More information $\Rightarrow$ Smaller MSE.

Cor. Set $\mathcal{g} = \{\phi, \Omega\}$. $Var(X) = E(Var(X|\mathcal{G})) + Var(E(X|\mathcal{G}))$

Thm. $X, Y \in L^2$. If $E(Y|\mathcal{g}) = X$. $E(X^2) = E(Y^2)$. Then: $X = Y$. a.s.

Pf: Check $E(X-Y)^2 = 0$. Note: $E(XY) = E(XE(Y|\mathcal{g})) = EX^2$.

Thm. For $X, Y \in L^1$. If $E(X|Y) = Y$, $E(Y|X) = X$. Then: $X = Y$. a.s.

Pf: $E((X-Y)(\arctan X - \arctan Y)) = 0$

Thm. For $Y \in L^1$. $E(Y|\mathcal{g}) \overset{d}{=} Y$. Then $Y = E(Y|\mathcal{g})$. a.s.

Pf. $E|E(Y|\mathcal{g}) - c| = E|Y-c| = E(E(|Y-c||\mathcal{g})) = E|Y-c|$

$\Rightarrow E|E(Y-c|\mathcal{g})| = E(E(|Y-c||\mathcal{g}))$

$\therefore |E(Y-c|\mathcal{g})| = E(|Y-c||\mathcal{g})$. a.s.

$\therefore E(Y-c|\mathcal{g})\, sgn(E(Y|\mathcal{g}) - c) = E((Y-c)\, sgn(Y-c)|\mathcal{g})$

i.e. $sgn(E(Y|\mathcal{g}) - c) = sgn(Y-c)$. a.s. $\cdots$ (*)

If $P(E(Y|\mathcal{G}) > Y) > 0$. Since $\{E(Y|\mathcal{G}) > Y\} = \bigcup_{q \in \mathbb{Q}} R_q$.

$R_q = \{E(Y|\mathcal{G}) > q > Y\}$. However,

$\exists q_0$. s.t. $P(R_{q_0}) > 0$. Contradict with $(*)$.

④ <u>Regular conditional Probabilities:</u>

<u>Def</u>: $(\Omega, \mathcal{F}, P)$ is a Prob. space. $X: (\Omega, \mathcal{F}) \to (S, \mathcal{S})$ is measurable map. $\mathcal{G} \subseteq \mathcal{F}$. σ-subfield. We say:

$\mu: \Omega \times \mathcal{S} \to [0,1]$ is regular conditional dist. (r.c.d) of X on $\mathcal{G}$ if:

i) $\forall A \in \mathcal{S}$. $w \longmapsto \mu(w, A)$ is version of $P(X \in A|\mathcal{G})$

ii) For a.e. w. $A \longmapsto \mu(w, A)$ is P.m. on $(S, \mathcal{S})$

If $S = \Omega$. $X = I\mathrm{d}$. Then μ is called r.c.P.

<u>e.g.</u> $(X, Y) \sim f(x,y) > 0$. $\mu(y, A) = \int_A f\,dx \Big/ \int f\,dx$

$\mu(Y(w), A)$ is r.c.d of X on $\sigma(Y)$.

<u>Thm.</u> $\mu(w, A)$ is r.c.d of X on $\mathcal{F}$. If $f: (S, \mathcal{S}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}'})$ has $E(|f(x)|) < \infty$. Then we have:

$E(f(x)|\mathcal{F}) = \int f(x)\, \mu(w, dx)$. a.s.

<u>Pf</u>: $f = f^+ - f^-$. Approxi. f by simple functions. by MCT.

Since $f = \mathbb{1}_A$. it holds by def.

<u>Rmk</u>: r.c.d don't always exist. Note: For $\{A_n\}$ disjoint.

$P(X \in \Sigma A_n|\mathcal{G}) = \Sigma P(X \in A_n|\mathcal{G})$. a.s.

But if $\mathcal{S}$ contains enough countable collections.

then it may pile up.

<u>Def</u>: $(\Lambda, \mathcal{g})$ is nice if $\exists\, \varphi: S \to \mathbb{R}^1$. bijection.

st. φ. φ^{-1} are both measurable.

<u>Thm</u>. r.c.d's exist if $(S, \mathcal{S})$ is nice.

<u>Rmk</u>: e.g. for $(S, \mathcal{S}) = (\mathbb{R}^1, \mathcal{B}_{\mathbb{R}^1})$.

<u>Thm</u>. For X, Y take values in $(S, \mathcal{S})$. nice space. $\mathcal{G} = \sigma(Y)$.

Then $\exists\, \mu: S \times \mathcal{S} \longrightarrow [0,1]$. st.

i) $\forall A$. $\mu(Y(w), A)$ is a version of $P(X \in A \mid \mathcal{G})$

ii) For a.e. w. $\mu(Y(w), A) = \nu(A)$ is P.m on $(S, \mathcal{S})$.

<u>Rmk</u>: It's generalization of $P(X \in A \mid Y)$. $A \in \mathcal{B}_{\mathbb{R}^1}$.

(2) <u>Martingales</u>:

<u>Def</u>: i) $(\mathcal{F}_n)$ is filtration if $\mathcal{F}_n \uparrow$. $\mathcal{F}_n \subseteq \mathcal{F}$.

ii) (X_n) is adapted to $(\mathcal{F}_n)$ if $X_n \in \mathcal{F}_n$. $\forall n$.

iii) Seq (X_n) is martingale w.r.t. $(\mathcal{F}_n)$ if:

(a) $X_n \in L^1$. $\forall n$.

(b) (X_n) is adapted to $(\mathcal{F}_n)$

(c) $E(X_{n+1} \mid \mathcal{F}_n) = X_n$. $\forall n$

Replace (c) "$=$" by "$\leq$". "$\geq$". it's super/sub martingale.

Rmk: If (X_n) is martingale w.r.t $(\mathcal{G}_n)$. Set:

$\mathcal{F}_n = \sigma(X_1 \cdots X_n)$. Then $\mathcal{F}_n \subseteq \mathcal{G}_n$. (X_n) is

martingale w.r.t $(\mathcal{F}_n)$.

Pf: $X_k \in \mathcal{G}_k \subset \mathcal{G}_n$. $\forall 1 \le k \le n$.

① Examples:

i) (linear Mart.)

If $E(\xi_i) = \mu = 0$. Then (S_n) is mart. w.r.t $(\mathcal{F}_n)$.

$S_n = \sum_0^n \xi_i$. $\mathcal{F}_n = \sigma(\xi_1 \cdots \xi_n)$. $\mathcal{F}_0 = \{\emptyset, \Omega\}$. and

ξ_i's are indept. r.v's.

Rmk: If $\mu \le 0$ or $\mu \ge 1$. it's super / sub mart.

Or apply on $\tilde{\xi}_i = \xi_i - \mu$. $S_n - n\mu$ is mart.

ii) (Quadratic Mart.)

If ξ_i's are indept. $E(\xi_i) = \mu = 0$. $\sigma^2 = Var(\xi_i) < \infty$.

Then: $S_n^2 - n\sigma^2$ is martingale. w.r.t $(\mathcal{F}_n)$.

iii) (Exponential Mart.)

If Y_n nonnegative i.i.d. $E(Y_n) = 1$. $\mathcal{F}_n = \sigma(Y_1 \cdots Y_n)$.

Set $M_n = \prod_{k=1}^n Y_k$. Then M_n is mart. w.r.t $\mathcal{F}_n$.

Thm. For (X_n) is submart. $\forall n \ge m$. $E(X_n | \mathcal{F}_m) \ge X_m$

Pf: $E(X_{m+k} | \mathcal{F}_m) = E(E(X_{m+k} | \mathcal{F}_{m+k-1}) | \mathcal{F}_m)$

Cor. (X_n) is submart. Then: $\forall n > m$. $E(X_n) > E(X_m)$

Thm. (X_n) is mart. w.r.t $(\mathcal{F}_n)$. φ is convex. st. $E|\varphi(X_n)| < \infty$ $\forall n$. Then $\varphi(X_n)$ is submart. w.r.t $\mathcal{F}_n$.

Pf: Apply Jensen Inequility.

Rmk: i) For concave $\Rightarrow$ supermart.

ii) If $X_n \in L^p$. Then: $|X_n|^p$ is submart. w.r.t $\mathcal{F}_n$.

Thm. (X_n) is submart w.r.t $(\mathcal{F}_n)$. φ is increasing. convex. with $E(|\varphi(X_n)|) < \infty$. $\forall n$. Then: $\varphi(X_n)$ is submart.

Rmk: e.g. X_n submart. $\Rightarrow$ $(X_n - a)^+$ submart.

X_n supermart. $\Rightarrow$ $X_n \wedge a$ supermart.

Cor. If X_n. Y_n are submart. w.r.t. $(\mathcal{F}_n)$. Then:

So $X_n \vee Y_n$ is.

Pf: $X_n \vee Y_n = X_n I_{\{X_n \geq Y_n\}} + Y_n I_{\{X_n < Y_n\}}$

Def: (H_n) is predictable seq. w.r.t Filtration $(\mathcal{F}_n)$

if $H_n \in \mathcal{F}_{n-1}$. for $\forall n \geq 1$.

Thm. (X_n) is submart. If $H_n \geq 0$. predictable, bdd.

Then: $(H \cdot X)_n = \sum_{m=1}^{n} H_m (X_m - X_{m-1})$ is submart.

Rmk: The same holds for supermart. mart. If X is mart. " $H_n \geq 0$. " is unnecessary.

Def: r.v. N is stopping time w.r.t $(\mathcal{F}_n)$ if $\{N = n\} \in \mathcal{F}_n$. $\forall n \in \mathbb{Z}^{\geq 0}$.

Thm. N is stopping time. (X_n) is submart. Then:

so $X_{n \wedge N}$ is submart.

Pf: Set $H_n = I_{\{N \geq n\}} = I_{\{N \leq n-1\}^c} \in \mathcal{F}_{n-1}$. Then:

$$X_{N \wedge n} = X_0 + (H \cdot X)_n.$$

① Convergent Thm.

For $a < b$. Set $N_0 = -1$. $\begin{cases} N_{2k-1} = \inf\{m > N_{2k-2} \mid X_m \leq a\}. \\ N_{2k} = \inf\{m > N_{2k-1} \mid X_m \geq b\}. \end{cases}$

$\Rightarrow N_j$ is stopping time. $\forall j$.

Note: $\{N_{2k-1} < m \leq N_{2k}\} = \{N_{2k-1} \leq m-1\} \cap \{N_{2k} \leq m-1\}^c \in \mathcal{F}_{m-1}$

$\Rightarrow H_m = I_{\{N_{2k-1} < m \leq N_{2k}.\ \text{for some } k\}}$ is predictable.

Since $X(N_{2k-1}) \leq a$. $X(N_{2k}) \geq b$.

So between $[N_{2k-1}, N_{2k}]$. X_n cross from a to b.

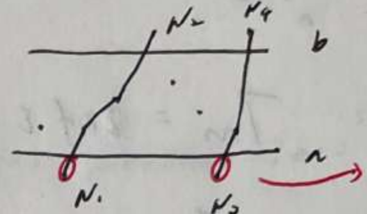
Set: $U_n = \sup\{k \mid N_{2k} \leq n\}$ numbers of upcrossing by time n.

Thm. (Upcrossing Inequality)

(X_n) is submart. Then: $(b-a)E(U_n) \le E(X_n-a)^+ - E(X_0-a)^+$.

Pf: Set $Y_n = a + (X_n - a)^+$. submart.

it crosses same number of times that X_n has.



No contribute to $(H\cdot Y)_n$.

Note: $(b-a)U_n \le (H\cdot Y)_n$

And $Y_n - Y_0 = (H\cdot Y)_n + (K\cdot Y)_n$

where $K_n = 1 - H_n \ge 0$ $(K\cdot Y)_n$ is submart.

So: $E(Y_n - Y_0) \ge E((H\cdot Y)_n) + E((K\cdot Y)_n)$

$\ge (b-a)E(U_n)$.

Thm. If (X_n) is submart. st. $\sup E(X_n^+) < \infty$.

Then $\exists X \in L^1$, st. $X_n \xrightarrow{n\to\infty} X$ a.s. and $E(X) \ge E(X_0)$

Pf: Note: $E(U_n) \le \dfrac{|a| + E X_n^+}{b-a} < \infty$. $U_n \uparrow U \in \overline{\mathbb{R}}$.

By Fatou's: $E(U) < \infty$. $\therefore U < \infty$ a.s.

$\Rightarrow P(\bigcup_{a,b\in\mathbb{Q}} \{\underline{\lim} X_n < a < b < \overline{\lim} X_n\}) = 0$.

i.e. $\lim X_n$ exists a.s.

By Fatou's again: $E(X^+) \le \sup E(X_n^+)$.

Besides: $E(X_n^-) = E(X_n^+) - E(X_n) \le E(X_n^+) - E(X_0)$

So: $E(X^-) \le \sup E(X_n^+) - E(X_0)$. i.e. $X \in L^1$.

Rmk: For supermart.: $\sup E(X_n^-) < \infty \Rightarrow$

$\exists X \in L^1$. $X_n \to X$ a.s. $(n\to\infty)$. $E(X) \le E(X_0)$

<u>Cor.</u> (X_n) supermart. $X_n \geq 0$. Then $X_n \to X$ a.s.

with $E(X) \leq E(X_0)$.

<u>Thm.</u> X_n submart. $\sup_n X_n < \infty$. Set $\xi_n = X_n - X_{n-1}$. with

$E(\sup_n \xi_n^+) < \infty$. Then: X_n converges a.s.

<u>Pf:</u> $T_m = \inf\{n \mid X_n^+ > m\}$. stopping time.

$X^+_{T_m \wedge n} \leq M + \xi_n^+$. And $X_{T_m \wedge n}$ is submart.

Note $\sup_n E(X^+_{T_m \wedge n}) \leq m + E(\sup_n \xi_n^+) < \infty$

$\therefore$ X_n converges a.s on $\{T_m = \infty\}$. $\forall m$.

i.e. $X_n I_{\{T_m = \infty\}} \to X I_{\{T_m = \infty\}}$. a.s. (except N_m).

Note: $\Omega / N = \bigcup_{m \in \mathbb{Z}^+} \{T_m = \infty\}$ by: $\sup_n X_n^+ < \infty$. $T_m \uparrow \infty$

$\Rightarrow \forall w \notin N \cup (\cup N_m)$. $X_n(w) \xrightarrow{n \to \infty} X(w)$. $P(N \cup (\cup N_m)) = 0$.

<u>Rmk:</u> The convergence above won't be L^1 probably:

<u>e.g.</u> ξ_i i.i.d. $P(\xi_i = 1) = P(\xi_i = -1) = \frac{1}{2}$. $S_n = \sum_1^n \xi_i$

$N = \inf\{n \mid S_n = 0\}$. $S_0 = 1$. $X_n = S_{N \wedge n} \geq 0$

which is mart $\longrightarrow X_\infty = 0$. But $E(X_n) = 1$.

<u>Thm.</u> $X_n, Y_n \geq 0$. integrable. adapted to $\mathcal{F}_n$. If $E(X_{n+1} \mid \mathcal{F}_n)$

$\leq X_n + Y_n$. $\sum Y_n < \infty$. a.s. Then: $X_n \longrightarrow X$ a.s. which

is a finite limit.

Pf: $X_n - \sum_1^{n-1} Y_k$ is supermart. $N_m = \inf\{k \mid \sum_1^k Y_m > M\}$.

Denote by Z_n. $\therefore Z_{n\wedge N_m} \geq -M$.

Z_n converges to finite limit on $\bigcup\{N_m=\infty\}$ a.s.

Thm (Doob's Decomposition)

Any submart. X_n can be unique written in:

$X_n = M_n + A_n$. M_n is mart. A_n is predictable. $\uparrow$. $A_0 = 0$.

Pf: $E(X_n \mid \mathcal{F}_{n-1}) = E(M_n \mid \mathcal{F}_{n-1}) + E(A_n \mid \mathcal{F}_{n-1})$

$= M_{n-1} + A_n = X_{n-1} - A_{n-1} + A_n$.

$\therefore A_n$ must be: $\sum_{m=1}^n E(X_m - X_{m-1} \mid \mathcal{F}_{m-1})$. Then check condition

(3) Bounded Increments:

Thm. X_n is mart. $|X_{n+1} - X_n| \leq M < \infty$. Set:

$C = \{\lim X_n \text{ exists. finite}\}$. $D = \{\overline{\lim} X_n = +\infty \text{ and } \underline{\lim} X_n = -\infty\}$.

Then: $P(C \cup D) = 1$.

Pf: WLOG. Set $X_0 = 0$. $N_k = \inf\{t \mid X_t \leq -k\}$.

$X_{n\wedge N_k} \geq -k-M$. mart. By Convergent Thm:

So $\lim X_n + k + M$ exists on $\{N_k = \infty\}$.

$\Rightarrow \lim X_n$ exists on $\bigcup\{N_k=\infty\} = \{\underline{\lim} X_n > -\infty\}$.

Apply on $-X_n$. We obtain: $\{\overline{\lim} X_n < \infty\}$.

Cor. (2nd Borel-Cantelli Lemma)

$\mathcal{F}_n$ filtration. $B_n \in \mathcal{F}_n$. Then: We have:

$$\{B_n\ \text{i.o.}\} = \{\sum_{n\geq1}^{\infty} P(B_n|\mathcal{F}_{n-1}) = \infty\}$$

Pf: $X_n = \sum_1^n \mathbb{1}_{B_m}$ submart. By decomposition:

$$A_n = \sum_1^n E(\mathbb{1}_{B_m}|\mathcal{F}_{m-1})$$

$$M_n = \sum_1^n (\mathbb{1}_{B_m} - P(B_m|\mathcal{F}_{m-1})).\ |M_n - M_{n-1}| \leq 1.$$

Apply Thm: Consider on C, D.

Rmk: It's not enough if: $\sup_n |X_{n+1} - X_n| < \infty$ only

e.g. $P(X_{n+1} = -1 \mid X_n = 0) = P(X_{n+1} = -1 \mid X_n = 0) = \frac{1}{2}$

$P(X_{n+1} = 0 \mid X_n \neq 0) = 1 - \frac{1}{n^2}$. $P(X_{n+1} = n^2 X_n \mid X_n \neq 0) = \frac{1}{n^2}$

Then X_n is mart. But $P(X_n = a,\ \text{i.o}) = 1$. $a = \pm1, 0$.

④ **Branching Process:**

ξ_i^n i.i.d. nonnegative integer-valued. Set $Z_0 = 1$.

$$Z_n = \begin{cases} \xi_1^n + \cdots \xi_{Z_{n-1}}^n, & \text{if } Z_{n-1} > 0 \\ 0, & \text{if } Z_{n-1} = 0 \end{cases} \qquad p_k = P(\xi_i^n = k).\ k \in \mathbb{Z}_{\geq 0}.$$

Rmk: i) 0 is absorbing

ii) Z_n means number of individuals in n^{th} generation.

iii) All states are transient. since it's irreducible and "0" happens only once. if $p_0 > 0$. $p_0 + p_1 < 1$.

Then in this case: $Z_n \to \infty$ or get absorbed.

<u>Lemma.</u> $\mathcal{G}_n = \sigma(\xi_i^m \mid i \geq 1,\ 1 \leq m \leq n)$. If $\mu = E(\xi_i^m) \in (0, \infty)$

Then Z_n/μ^n is mart. w.r.t $(\mathcal{G}_n)$.

<u>Pf</u>: On $\{Z_n = k\} \in \mathcal{G}_n$. $E(Z_{n+1} \mid \mathcal{G}_n) = k\mu = Z_n \mu$.

<u>Thm.</u> If $\mu < 1$. Then $Z_n = 0$ for $\forall n$ large enough. So $\frac{Z_n}{\mu^n} \to 0$

<u>Pf</u>: $E(Z_n/\mu^n) = E(Z_0) = 1$. $P(Z_n > 0) \leq E(Z_n) = \mu^n$.

$\therefore P(Z_n > 0 \text{ . i.o}) = 0$ by Borel-Cantelli Thm.

<u>Rmk</u>: If each one on average give birth to less one child. Then the species will die out.

<u>Thm.</u> If $\mu = 1$. $p_1 < 1$. Then $Z_n = 0$ for $\forall n$ large enough.

<u>Pf</u>: $Z_n \geq 0$ is mart. So $Z_n \longrightarrow Z_\infty$ a.s. finite limit.

Note: $P(Z_{n+1} = k \mid Z_n = k) =: \tilde{p} < 1 \therefore P(Z_n = k\ \forall n \geq N) = \tilde{p}^\infty = 0$

$\Rightarrow Z_\infty = 0$. i.e. $\exists \tilde{N}_\varepsilon$. $\forall n > \tilde{N}_\varepsilon$. $Z_n < \varepsilon$. i.e. $Z_n = 0$.

Set: $e = P(Z_n = 0$. for some $n)$. $e = \sum_0^\infty P(Z_n = 0$ for some $n \mid Z_1 = k)\ P(Z_1 = k \mid Z_0 = 1) = \sum e^k p_k =: \varphi(e)$.

Actually $\varphi(s)$ is generating Func. of ξ_i^m.

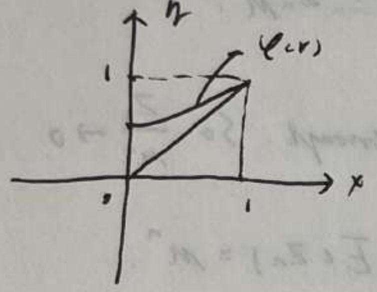
<u>Lemma.</u> (g.f. of Z_n)

$\varphi_n(r) = E(r^{Z_n}) = \sum r^k P(Z_n = k)$. Then: $\varphi_n = \varphi_1 \circ \varphi_{n-1}$

$\underline{Pf}$: $\varphi_n(r) = \sum p(Z_1=i)\, E(r^{Z_n} \mid Z_1=i)$

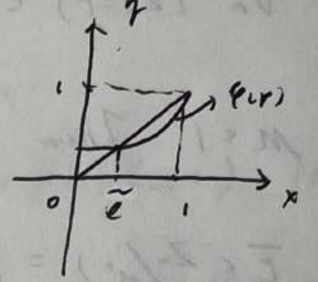
$= \sum p(Z_1=i)\, E^i(r^{Z_{n-1}}) = \varphi_1 \circ \varphi_{n-1}(r)$

Note: $\varphi' > 0$. $\varphi \uparrow$. $m = E(\xi) = \varphi'(1)$. Then:



For $m \le 1$. Then $\varphi(e) = e$ has unique solution 1

$(m \le 1)$



For $m > 1$. $\exists\, \tilde{e} \in (0,1)$ $\varphi(\tilde{e}) = \tilde{e}$

$(m > 1)$

$\underline{Thm}$. If $m > 1$. Then $\tilde{e} = e = p(Z_n = 0$. for some $n)$

is the only solution of $\varphi(r) = r$ in $[0,1)$.

$\underline{Pf}$: 1°) $p_1 = 1$. is trivial case.

2°) $p_1 \in (0,1)$.

Note: $\varphi(1) - \varphi(1-h) \sim mh$ $\left(\frac{\varphi(1) - \varphi(1-h)}{h} \to m > 1 \right)$

$\therefore$ For h small enough. $\varphi(1-h) < 1-h$.

With $\varphi(1) = 1$. $\varphi(0) \geq 0$. So $\tilde{e}$ exists.

For uniqueness:

Denote: e_0 is smallest: $\varphi(e_0) = e_0$.

But $x \in (e_0, 1)$. $\varphi(x) < x$. So e_0 is unique one.

For $e = e_0$: (existence)

Set $\theta_m = p(Z_m = 0) = \sum p(Z_m = 0 \mid Z_1 = k)\, p_k$

$= \sum p_k\, \theta_{m-1}^k = \varphi(\theta_{m-1})$

And $\theta_m \le \theta_{m+1}$ increasing. $\theta_0 = 0 \le e_0$

By induction: $\theta_m = \varphi(\theta_{m-1}) \le \varphi(c_0) = c_0$

With: $\theta_m = \varphi(\theta_{m-1})$. Let $m \to \infty$. $\therefore \theta_\infty = c_0$

Thm. $\lim\limits_{n} \frac{Z_n}{\mu^n}$ isn't $\equiv 0 \iff \sum\limits_{k \ge 1} p_k k \log k < \infty$.

(3) Inequalities:

Thm. X_n is submart. N is stopping time. bdd ($N \le k$)

Then: $E(X_0) \le E(X_N) \le E(X_k)$

Pf: 1°) $X_{N \wedge n}$ is submart. So: $E(X_0) \le E(X_{N \wedge k}) = E(X_N)$

2°) Set $k_n = \mathbb{1}_{\{N < n\}}$ predictable.

So $(k \cdot X)_n$ is submart. $\Rightarrow E(k \cdot X)_k \ge E(k \cdot X)_0$

Cor. X_n is submart. $M \le N$. are stopping times. $N \le k$ a.s.

Then: $E(X_M) \le E(X_N)$.

Pf. Set $Y_n = X_{n \wedge N} \Rightarrow E(Y_M) \le E(Y_k)$

Rmk: Note "$X_M - X_N$". Set $k_n = I_{\{M < n \le N\}}$

can also prove it.

Rmk: For unbdd stopping time, it may not hold.

e.g. $S_n = \sum_1^n \xi_k$. random walk. $T = \inf\{n \mid S_n = 0\}$.

$E(S_0) = 1 > 0 = E(S_T)$

Thm. $M \le N$. stopping time. If $N \le k$ a.s. (M_n) is submart. Then: $X_M \le \mathbb{E}(X_N \mid \mathcal{F}_M)$.

Pf: **Lemma.** $\forall A \in \mathcal{F}_M$. $L = M I_A + N I_{A^c}$ is a stopping time.

Pf: $\{L \le n\} = \{M \le n\} \cap A + \{N \le n\} \cap A^c$

$= \square + \{N \le n\} \cap \{M \le n\} \cap A^c \in \mathcal{F}_n$

$\Rightarrow L \le N$. So: $\mathbb{E}(X_L) \le \mathbb{E}(X_N)$

We have: $\mathbb{E}(X_N I_A) \ge \mathbb{E}(X_M I_A)$. $\forall A \in \mathcal{F}_M$.

Thm. (Doob's Inequility)

X_n is submart. $\bar{X}_n = \max_{0 \le m \le n} X_m^+$. $\lambda > 0$. $A = \{\bar{X}_n \ge \lambda\}$.

Then: i) $\lambda \mathbb{P}(A) \le \mathbb{E}(X_n I_A) \le \mathbb{E}(X_n^+)$

ii) $\lambda \mathbb{P}(\min_{0 \le m \le n} X_m \le -\lambda) \le -\mathbb{E}(X_0) + \mathbb{E}(X_n I_{\{\min_{0 \le m \le n} X_m > -\lambda\}})$

Pf: i) Set $N = \inf\{m \ge 0 \mid X_m \ge \lambda\} \wedge n \le n$.

ii) Set $N = \inf\{m \ge 0 \mid X_m \le -\lambda\} \wedge n \le n$.

$\Rightarrow$ By Thm. above.

Cor. X_n is submart. $\Rightarrow \lambda \mathbb{P}(\max_{0 \le m \le n} |X_m| \ge \lambda) \le 2\mathbb{E}(|X_n|) + \mathbb{E}(|X_0|)$

Cor. X_n is submart. $\Rightarrow \lambda^p \mathbb{P}(\bar{X}_n \ge \lambda) \le \mathbb{E}(|X_n|^p)$. for $p > 1$.

Thm. (L^p-maximal Inequi.)

X_n is submart. $1 < p < \infty \Rightarrow \mathbb{E}(\bar{X}_n^p) \le (\frac{p}{p-1})^p \mathbb{E}(|X_n|^p)$

Pf: $E(\bar{X}_n \wedge M)^p = \int_0^\infty p\lambda^{p-1} P(\bar{X}_n \wedge M \geq \lambda)\, d\lambda$

$\leq \int_0^\infty p\lambda^{p-2} \int X_n^+ I_{\{\bar{X}_n \wedge M \geq \lambda\}}\, dP\, d\lambda$

$= \int X_n^+ \int_0^{\bar{X}_n \wedge M} p\lambda^{p-2}\, d\lambda\, dP$

Apply Hölder: $E(|\bar{X}_n \wedge M|^p) \leq \left(\frac{p}{p-1}\right)^p E(X_n^+)^p$.

Set $M \to \infty$. Apply MCT.

Rmk: i) For supermart: $E(\max_{m \leq n} X_m^-)^p \leq \left(\frac{p}{p-1}\right)^p E(X_n^-)^p$.

For mart: $E(\max_{0 \leq m \leq n} |X_m|)^p \leq \left(\frac{p}{p-1}\right)^p E(|X_n|)^p$

ii) There's no L^1 inequility ($p=1$ doesn't hold)

Thm. (L^p convergence Thm)

If X_n is mart. $\sup E(|X_n|^p) < \infty$. for $p > 1$.

Then. $X_n \to X$. a.s. and i L^p.

Pf: $\sup E(X_n^+)^p \leq \sup E|X_n|^p < \infty \Rightarrow \exists X.\ X_n \xrightarrow{a.s} X\ (n \to \infty)$

$E(\sup_{0 \leq m \leq n} |X_m|^p) \leq \left(\frac{p}{p-1}\right)^p E(|X_n|^p) \Rightarrow \sup |X_n| \in L^p$

Note $|X_n - X|^p \leq 2^p \sup |X_n|^p$. Apply DCT.

(4) Square Integrable Martingale:

Next, we suppose: X_n is mart. $X_0 = 0$. $X_n \in L^2$. $\forall n$.

① Decomposition:

Lemma. (Orthogonal of mart. increment)

If $m \le n$. $Y \in \mathcal{F}_m$. $Y \in L^2$. Then: $E((X_n - X_m)Y) = 0$

Cor. $E(X_n - X_m)(X_m - X_l) = 0$. for $n \ge m \ge l$.

Lemma. $X_n, Y_n \in L^2$ mart's. Then, we have equation:

$$E(X_n Y_n) - E(X_0 Y_0) = \sum_1^n E(X_m - X_{m-1})(Y_m - Y_{m-1})$$

Pf. $E(X_n Y_{n-1}) = E(X_{n-1} Y_n) = E(X_{n-1} Y_{n-1})$

Thm. (Conditional Variance Formula)

$$E((X_n - X_m)^2 | \mathcal{F}_m) = E(X_n^2 | \mathcal{F}_m) - X_m^2.$$

Cor. Apply on Branching Process of $\mu > 1$. $X_n = \frac{Z_n}{\mu^n}$.

Then: $E(X_n^2) = 1 + \sigma^2 \sum_2^{n+1} \mu^{-k}$. So: $X_n \to X$ in L^2.

$E(X) = 1$. $X \not\equiv 0$. $\{X > 0\} = \{Z_n > 0, \forall n\}$.

Pf: Note: $\sup_n E(X_n^2) \le 1 + \sum \sigma^2 \mu^{-k} < \infty$.

So $X_n \to X$ in $L^2 \Rightarrow X_n \to X$ in L^1.

For the last: $\{X > 0\} \subseteq \{Z_n > 0, \forall n\}$.

Next show $P(X = 0)$ is solution of $\varphi(r) = r$

So that $P(X > 0) = 1 - \rho$. $\{X > 0\} = \{Z_n > 0, \forall n\}$.

Rmk: Return to (X_n). Note X_n^2 is submart.

By Doob's decomposition: $X_n^2 = M_n + A_n$. where $M_0 = 0$.

$A_n = \sum_1^n E(X_m^2 | \mathcal{F}_{m-1}) - X_{m-1}^2 = \sum_1^n E((X_m - X_{m-1})^2 | \mathcal{F}_{m-1})$.

it can be thought of as path by path measure of variance at time n. Since $A_n \uparrow$. Suppose $A_\infty = \lim_n A_n$. total variance in the path.

③ Convergence:

Thm. $E(\sup_m |X_m|^2) \le 4E(A_\infty)$.

Pf: Note: $E(\sup_{0 \le m \le n} |X_m|^2) \le 4E(X_n^2) = 4E(A_n)$

since $E(M_n) = E(M_0) = 0$. Then by MCT.

Thm. $\lim X_n$ exists and finite a.s on $\{A_\infty < \infty\}$

Pf: $N_a = \inf\{n \mid A_{n+1} > a^2\}$. is stopping time. by $A_n \in \mathcal{F}_{n-1}$.

$\therefore E(\sup_n |X_{N_a \wedge n}|^2) \le 4a^2$. Apply L^2 convergence.

So $X_n \to X$. a.s. on $\cup \{N_a = \infty\} = \{A_\infty < \infty\}$.

Thm. $f \ge c > 0$ and $f \uparrow \infty$. If $\int_0^\infty 1/f(t)^2 \, dt < \infty$. Then:

$X_n / f(A_n) \to 0$. a.s. on $\{A_\infty = \infty\}$.

Pf: $H_m = f(A_m)^{-1}$ is predictable. $Y_n = (H \cdot X)_n$ mart.

Note $B_{n+1} - B_n =: E((Y_{n+1} - Y_n)^2 | \mathcal{F}_n) = (A_{n+1} - A_n)/f(A_n)$

So $B_\infty = \sum \frac{A_{n+1} - A_n}{f(A_n)^2} \le \sum \int_{[A_n, A_{n+1})} f(t)^{-2} dt < \infty$

By Thm. above. $Y_n \to Y_\infty$. finite.

So By Kronecker Lemma. $X_n / f(A_n) \to 0$.

<u>Rmk</u>: We call $A_n = \sum_1^n E((X_m - X_{m-1})^2 | \mathcal{F}_{m-1})$ by increasing process.

<u>Cor</u>. (Strengthened 2^{nd} Borel-Cantelli)

B_n is adapted to $\mathcal{F}_n$. $p_n = P(B_n | \mathcal{F}_{n-1})$.

Then: $\sum_1^n I_{B_m} / \sum_1^n p_m \to 1$. a.s. on $\{\sum_1^\infty p_n = \infty\}$.

<u>Pf</u>: We have had $X_n = \sum_1^n \mathbb{1}_{B_m} - P(B_m | \mathcal{F}_{m-1})$ is martingale.

So: $\sum_1^n \mathbb{1}_{B_m} / \sum_1^n p_m - 1 = X_n / \sum_1^n p_m$

Consider increasing process of X_n: $A_n = \sum_1^n (p_m - p_m^2)$

$A_n - A_{n-1} = p_n - p_n^2 \le p_n$. $\{A_\infty = \infty\} = \{\sum_1^\infty p_n(1-p_n) = \infty\}$.

$\subseteq \{\sum p_n = \infty\}$. Apply on $f(t) = t \vee 1$.

<u>Rmk</u>: The increasing speeds are identical.

<u>Thm</u>. $E(\sup_n |X_n|) \le 3 E(A_\infty^{\frac{1}{2}})$.

<u>Pf</u>: Set $N_a = \inf\{n \mid A_{n+1} > a^2\}$. Stopping time.

$$P(\sup|X_n| \geq \lambda) = P(\sup|X_n| \geq \lambda,\ N < \infty) + P(\square,\ N = \infty)$$

$$\leq P(N < \infty) + P(\sup_m |X_{m \wedge N}| \geq \lambda)$$

$$= P(A_\infty > \lambda^2) + P(\sup_m |X_{m\wedge N}| \geq \lambda)$$

$$P(\sup_{m \leq n} |X_{m\wedge N}|^2 \geq \lambda^2) \leq \frac{1}{\lambda^2} E(X_{N\wedge n}^2) \quad (\text{Doob's inequility})$$

$$= \lambda^{-2} E(A_{N\wedge n}) \leq \lambda^{-2} E(A_\infty \wedge \lambda^2)$$

Then: Since $E(\sup|X_n|) = \int_0^\infty P(\sup|X_n| \geq \lambda)\, d\lambda$

By Fubini Thm. We obtain the estimate

<u>Rmk:</u> $E(A_\infty^{\frac{1}{2}}) < \infty \Rightarrow \exists X \in L^1.\ X_n \xrightarrow{n\to\infty} X.$ a.s. / in L^1

Since $\sup_n |X_n| \in L^1$.

(5) <u>Uniform Integrability:</u>

① <u>Converge for Conditional Expectation:</u>

<u>Thm.</u> In $(\Omega, \mathcal{F}_0, P)$. $X \in L^1$. Then $\{E(X|\mathcal{G}) \mid \mathcal{G}$ is σ-subfield $\subset \mathcal{F}_0\}$ is u.i.

<u>Pf:</u> Choose M large st. $E(|X|)/m \leq \delta$.

Note: $P(E(|X| \mid \mathcal{G}) \geq m) \leq E|X|/m \leq \delta$.

$E(|E(X|\mathcal{G})| I_{\{|E(X|\mathcal{G})| \geq m\}}) \leq$

$E(E(|X| \mid \mathcal{G}) I_{\{E(|X||\mathcal{G}) \geq m\}}) =$

$E(|X| I_{A_m}) \leq \varepsilon.$ Since $P(A_m) \leq \delta$.

Lemma. For submart. i). ii). iii) are equi.

i) (X_n) is u.i. ii) It converges a.s. in L^1.

iii) It converges in L^1.

Pf: u.i. $\Rightarrow \sup_n E|X_n| < \infty$. in $L^1 \Rightarrow$ in pr.

Lemma. If mart. $X_n \xrightarrow{L^1} X$. Then $X_n = E(X|\mathcal{F}_n)$

Pf: $E(X_n I_A) = E(X_m I_A)$ if $n > m$. $A \in \mathcal{F}_m$.

Since $E(X_n|\mathcal{F}_m) = X_m$. Set $n \to \infty$.

By condition $\therefore E(X I_A) = E(X_m I_A)$

$\therefore X_m = E(X|\mathcal{F}_m)$

Cor. For mart (X_n). i). ii). iii). iv). equi.

(*) general: $L^1 \not\Rightarrow$ a.s.
But $L^1 \Rightarrow$ a.s.
in mart. case.

i) It's u.i. ii) It converges. a.s. and in L^1

iii) It converges in L^1 (*) iv). $\exists X \in L^1$. $X_n = E(X|\mathcal{F}_n)$

Thm $\mathcal{F}_n \uparrow \mathcal{F}_\infty$ or $\mathcal{F}_n \downarrow \mathcal{F}_\infty \Rightarrow E(X|\mathcal{F}_n) \xrightarrow{n\to\infty} E(X|\mathcal{F}_\infty)$

a.s and in L^1. for $X \in L^1$.

Pf: $Y_n = E(X|\mathcal{F}_n)$ is u.i. mart.

So $\exists Y_\infty \in L^1$. $Y_n = E(Y_\infty|\mathcal{F}_n) = E(X|\mathcal{F}_n)$

i.e. $\int_A Y \, dP = \int_A X \, dP$. $\forall A \in \mathcal{F}_n$. $\forall n$. $\Rightarrow \forall A \in \mathcal{F}_\infty$

by Monotone Class Thm and DCT. $\Rightarrow Y_\infty = E(X|\mathcal{F}_\infty)$

Cor. If $\mathcal{F}_n \uparrow \mathcal{F}_\infty$. $A \in \mathcal{F}_\infty$. Then: $E(I_A|\mathcal{F}_n) \to I_A$. a.s./L^1

Rmk: set $\mathcal{F}_n = \sigma(X_j . j \geq n)$. (X_k) indept. Then:

$I_A = P(A) \in \{0.1\}$. We have Kolmogrov 0-1 law

Thm. (Dominated Convergence for conditional Expectation)

If $Y_n \to Y$. a.s. $|Y_n| \leq Z$. $\forall n$. $Z \in L^1$. $\mathcal{F}_n \uparrow \mathcal{F}_\infty$ $(n \to \infty)$

Then: $E(Y_n|\mathcal{F}_n) \xrightarrow{n \to \infty} E(Y|\mathcal{F}_\infty)$. a.s.

Pf: Set $W_N = \sup\{|Y_n - Y_m| \; n.m \geq N\} \leq 2Z$

So $\overline{\lim} E(|Y_n - Y| |\mathcal{F}_n) \leq \lim\limits_{n} E(W_N|\mathcal{F}_n) = E(W_N|\mathcal{F}_\infty)$

we have: $\begin{cases} |E(Y_n|\mathcal{F}_n) - E(Y|\mathcal{F}_n)| \leq E(|Y_n - Y| |\mathcal{F}_n) \to 0 \text{ . a.s.} \\ |E(Y|\mathcal{F}_n) - E(Y|\mathcal{F}_\infty)| \to 0 \text{ . a.s} \end{cases}$

Thm. (L^1 convergence)

If $\mathcal{F}_n \uparrow \mathcal{F}_\infty$. $Y_n \xrightarrow{L^1} Y \in L^1$. Then $E(Y_n|\mathcal{F}_n) \xrightarrow{L^1} E(Y|\mathcal{F}_\infty)$

Pf: $E(|E(Y_n|\mathcal{F}_n) - E(Y|\mathcal{F}_\infty)|) \leq$

$E(|E(Y_n|\mathcal{F}_n) - E(Y|\mathcal{F}_n)|) + E(|E(Y|\mathcal{F}_n) - E(Y|\mathcal{F}_\infty)|)$

$\leq \|Y_n - Y\|_{L^1} + o(n) \to 0$.

② Application in Backwards Mart's:

Def. It's mart. (X_n) index by negative integer $n \leq 0$.

Thm. $X_{-\infty} = \lim_{n\to-\infty} X_n$ exists a.s. and in L^1

Pf: By upcrossing Inequility:

$$(b-a)E(U_n) \le E(X_0-a)^+ - E(X_n-a)^+ \le E(X_0-a)^+$$

$$\Rightarrow E(U_\infty) < \infty.\ \exists X_{-\infty}.\ X_{-n} \to X_{-\infty}\ a.s.$$

Note: $X_n = E(X_0 | \mathcal{F}_n)$. $\forall n$. so it's u.i.

$X_n \longrightarrow X_{-\infty}$ in L^1. holds.

Rmk: It holds for backwards sub/supermarts "a.s. case".

Thm. If $\lim_{n\to-\infty} X_n = X_{-\infty}$. $\mathcal{F}_{-\infty} = \cap \mathcal{F}_n$. Then: $X_{-\infty} = E(X_0 | \mathcal{F}_{-\infty})$

Pf: $\forall A \in \mathcal{F}_{-\infty} \subset \mathcal{F}_n$. $\int_A X_n \, dP = \int_A X_0 \, dP$.

So $\int_A X_0 \, dP = \int_A X_{-\infty} \, dP$. as $n \to -\infty$.

Thm. If $\mathcal{F}_n \downarrow \mathcal{F}_{-\infty}$ as $n \downarrow -\infty$. Then: $E(Y|\mathcal{F}_n) \to E(Y|\mathcal{F}_{-\infty})$

as and in L^1. for $Y \in L^1$.

Pf: $Y_n = E(Y|\mathcal{F}_n)$ is backward mart.

$$\Rightarrow Y_n \to Y_{-\infty} = E(Y_0|\mathcal{F}_{-\infty}) = E(E(Y|\mathcal{F}_0)|\mathcal{F}_{-\infty}) = E(Y|\mathcal{F}_{-\infty})$$

Thm. If (X_n) has $X_0 \in L^p$. Then: $X_n \longrightarrow X_{-\infty}$ in L^p.

Pf: By Doob's L^p inequility.

Thm. (DCT for Backward Mart.)

$Y_n \to Y_{-\infty}$ a.s. $|Y_n| \le Z$ a.s. $Z \in L^1$. If $\mathcal{F}_n \downarrow \mathcal{F}_{-\infty}$.

Then : $E(Y_n | \mathcal{F}_n) \to E(Y_{-\infty} | \mathcal{F}_{-\infty})$ a.s.

Pf: Set $M_N = \sup\{|Y_n - Y_m| \mid n, m < N\}$. $N \in \mathbb{Z}^-$. identical.

Thm. For backward marts (X_n). $\mathcal{F}_n \downarrow \mathcal{F}_{-\infty}$. Then :

$E(X_n | \mathcal{F}_n) \to E(X_{-\infty} | \mathcal{F}_{-\infty})$ in L^1. automatically holds.

(6) Optional Stopping Thm.

① **Thm.** (X_n) is u.i. submart. For $\forall N$ stopping time, $\Rightarrow$

$X_{N \wedge n}$ is u.i. as well.

Pf: X_n^+ is u.i. submart. And $\sup_n E(X_{N\wedge n}^+) \le \sup_n E(X_n^+) < \infty$

So: $X_{N\wedge n} \to X_N$ a.s. $X_N \in L^1$

Then $E|X_{N\wedge n}| I_{\{|X_{N\wedge n}| > k\}} = E(|X_N| I_{\{N \le n, |X_N| \ge k\}}) +$

$E(|X_n| I_{\{N > n, |X_n| \ge k\}}) \to 0$

follows from (X_n) u.i. and $X_N \in L^1$.

Thm. $X_N \in L^1$. $X_n I_{\{n < N\}}$ is u.i. Then : $X_{N\wedge n}$ is u.i.

So if X_n is submart. then : $E(X_0) \le E(X_N)$

Pf: Argue same as above. Set $n \to \infty$ in $E(X_0) \le E(X_{N\wedge n})$

Cor. If $L \le M$. Stopping times. $Y_{M\wedge n}$ is u.i

Submart. Then: $E(Y_L) \le E(Y_M)$. $Y_L \le E(Y_M | \mathcal{F}_L)$

Pf: i) $X_n = Y_{M\wedge n \wedge L} = Y_{L\wedge n}$ is u.i.

So from: $E(Y_{L\wedge n}) \le E(Y_{M\wedge n})$. Let $n \to \infty$.

2) Set $N = L I_A + M I_{A^c} \le M$. $A \in \mathcal{F}_L$.

Apply the argue i) on $N \le M$.

Rmk: If we can prove: $\forall n$. $Y_{N\wedge n}$ is u.i. The "optional stopping" works!

Thm. If X_n is u.i. submart. Then $\forall$ stopping time N

We have: $E(X_0) \le E(X_N) \le E(X_\infty)$. $X_\infty = \lim_n X_n$.

Pf. Set $n \to \infty$ on $E(X_0) \le E(X_{N\wedge n}) \le E(X_n)$

Thm. If $X_n \ge Y \in L^1$. (X_n) is supermart. N is stopping time.

Then: $E(X_0) \ge E(X_N)$.

Pf. Apply Fatou's Lemma.

Thm. X_n is submart. $E(|X_{n+1} - X_n| \,|\, \mathcal{F}_n) \le B$. a.s. If N is stopping time with $E(N) < \infty$.

Then: $X_{N\wedge n}$ is u.i. $E(X_0) \le E(X_N)$.

Pf: $|X_{N\wedge n}| \le \sum_0^\infty |X_{m+1}-X_m| I_{\{N>m\}}$. Prove: RHS $\in L^1$

$$E(|X_{m+1}-X_m| I_{\{N>m\}}) = E(E(\square|\mathcal{F}_m) I_{\{N>m\}}) \le B\,p(m<N)$$

$$\therefore E(\text{RHS}) \le B \sum p(m<N) = B\,E(N) < \infty.$$

Thm. $X_n \ge 0$. supermart. Then: $p(\sup X_n \ge \lambda) \le E(X_0)/\lambda$

Pf: $T=\inf\{n \mid X_n \ge \lambda\}$. LHS $= p(T<\infty) \le \dfrac{E(X_T I_{\{T<\infty\}})}{\lambda} \le \dfrac{E(X_0)}{\lambda}$

② Application in Random Walk:

Thm. (Wald's Equation)

i) ξ_i. i.i.d. $E(\xi_i)=\mu$. N is stopping time. st. $E(N)<\infty$.

Then: $E(S_N) = \mu E(N)$. $S_n = \sum_1^n \xi_i$

ii) ξ_i indept. $E(\xi_i)=0$. $Var(\xi_i)=\sigma^2$. T is stopping time.

$E(T)<\infty$. Then: $E(S_T^2) = \sigma^2 E(T)$. $S_n = \sum_1^n \xi_i$.

Pf: i) $S_n - n\mu = X_n$ is mart. Check $E(|X_{n+1}-X_n| \mid \mathcal{F}_n) \le B$.

ii) $S_n^2 - n\sigma^2 = X_n$ is mart. Check the same things.

Thm. (Symmetric Simple Random Walk)

For ξ_i. i.i.d. $p(\xi_i=1)=p(\xi_i=-1)=\frac{1}{2}$. $S_0=x$. $N=\min\{n \mid S_n \notin (a,b)\}$.

Then: i) $P_x(S_N=a) = \dfrac{b-x}{b-a}$. $P_x(S_N=b) = \dfrac{x-a}{b-a}$.

ii) $E_0(N) = -ab$. $E_x(N) = (b-x)(x-a)$.

Rmk: set $\begin{cases} T_a = \min\{n \mid S_n = a\} \\ T_b = \min\{n \mid S_n = b\} \end{cases} \Rightarrow \{T_a < T_b\} = \{S_N = a\}$.

Pf: 1) $P(N<\infty)=1$. (It's transient)

2) $P(N > m(b-a)) \leq (1-2^{a-b})^m = \bar{p}^m$ (b-a consecutive +1)

So $E(N) < \infty$.

3) $S_N \in L'$. $S_n I_{\{n<N\}}$ is u.i. (bdd actually)

So $x = E_x(S_N) = a P(S_N=a) + b(1-P(S_N=a))$

4) Consider $X_n = S_n^2 - n$ is mart.

Then: $0 = E_0(S_{N\wedge n}^2) - E_0(N\wedge n)$. Let $n \to \infty$.

Thm. S_n is simple symmetric random walk. $S_0=0$.

$T_1 = \min\{n \mid S_n=1\}$. Then $E(s^{T_1}) = \frac{1-\sqrt{1-s^2}}{s}$

Pf: Set $X_n = e^{\theta S_n}/\phi^n(\theta)$. exponential mart.

$\phi(\theta) = E(e^{\theta \xi_1}) = (e^{\theta}+e^{-\theta})/2$.

Note $X_{T_1\wedge n}$ is bdd $\Rightarrow 1 = E(X_{T_1})$

For convert: Set $\phi(\theta) = \frac{1}{s}$. solve $\theta = \theta(s)$.

Thm. (Asymmetric simple random walk)

For $P(\xi_i=1) = p \neq q = 1-p = P(\xi_i=-1)$. $\varphi(y) = (\frac{1-p}{p})^y$.

i) $\varphi(S_n)$ is mart.

ii) $T_z =: \inf\{n \mid S_n = z\}$. Then: $\forall\ a<x<b$.

$P_x(T_a < T_b) = \frac{\varphi(b)-\varphi(x)}{\varphi(b)-\varphi(a)}$ $\quad P_x(T_a > T_b) = \frac{\varphi(x)-\varphi(a)}{\varphi(b)-\varphi(b)}$

Pf: $T = T_a \wedge T_b$. $\Rightarrow \varphi(S_{N\wedge n})$ is bdd. $\therefore \varphi(x) = E(\varphi(S_N))$